Towards Realistic Spectra from the $\mathbb{Z}_2 \times \mathbb{Z}_4$ Orbifold Model

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A mi madre y mi abuela,
el principio y el fin último de todos mis pasos.

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Chapter 1

Introduction

"Oh, a sleeping drunkard
up in Central Park,
and a lion-hunter
in the jungle dark,
and a Chinese dentist,
and a British queen
all fit together
in the same machine."

Kurt Vonnegut, *Cat's Cradle*

The history of physics is essentially a story of unification. The most outstanding transformations in the perception of our reality, have all been driven by ideas that attempt to explain phenomena which were thought to have entirely different origins, within the same unifying framework.

It is worth noticing the long *devenir* of unifying ideas in a quest for describing all objects we experience in terms of their elementary constituents and interactions. In the microscopic regime, where the main role is played by the electro-weak and strong interactions, quantum field theory arrived at its greatest glory after the formulation of the *standard model* of particle physics (SM), in which the interactions are understood in terms of the exchange of the spin 1 carriers of the gauge symmetry $G_{SM} = \mathrm{SU}(3)_C \times \mathrm{SU}(2)_L \times \mathrm{U}(1)_Y$. Quantum chromodynamics is described by $\mathrm{SU}(3)_C$, whereas $\mathrm{SU}(2)_L \times \mathrm{U}(1)_Y$ accounts for the electroweak interactions. In the standard model, quarks and leptons from the three known *families* are incorporated as chiral fermions transforming in irreducible representations of G_{SM}. The masses for these particles are generated by the VEV of a scalar *Higgs doublet* after the spontaneous breaking of $\mathrm{SU}(2)_L \times \mathrm{U}(1)_Y$ down to $\mathrm{U}(1)_{EM}$.

Despite its predictive power, there is a need for physics beyond the standard model. From the theoretical point of view, the Higgs mass is not protected towards the cutoff scale of the theory, which is known as the *hierarchy problem*. On the observational side, neutrino masses, dark matter and the baryon asymmetry of the universe are still calling for an explanation.

Early measurements of the coupling constants hinted at the possibility that they could coincide at energy scales of the order of $10^{14} - 10^{15}$ GeV. In that case, it seems possible to unify the interactions, so that at higher energies they are described by only one coupling constant and one gauge group factor. This is the underlying idea of *grand unified theories* (GUTs) [1]. These models do not only allow for the unification of the interactions, but also permit to incorporate the fields of the (MS)SM in representations of the unifying group. Among the most widely discussed GUTs, SU(5) is the simplest alternative. There, each family of quarks and leptons is contained in a **10**- and a $\overline{\mathbf{5}}$-plet. One step further, when SO(10) is considered to be the unifying group, one observes that a **16**-plet serves to accommodate one complete SM family in addition to an extra singlet [2]. This extra field can be used to generate neutrino masses via the *see-saw mechanism*. The general problem of such models is, that, when considering the Higgs to arise from a higher reprsentation, one also obtains some color-triplets which lead to fast proton decay. In addition, precision measurements of the coupling constants at the electro-weak scale made less and less likely that they meet at a certain point. Among the positive features of GUT schemes, we have a prediction for the weak mixing angle θ_W and an explanation for the quantization of the electromagnetic charge.

Supersymmetry (SUSY) assigns a new field (*superpartner*) to each particle of the standard model, so that the spectrum is made fermion-boson symmetric [3]. The contributions of the superpartners can make the quadratic corrections to the Higgs mass vanish. This constitutes an elegant solution to the hierarchy problem. The *minimal supersymmetric extension of the standard model* (MSSM), is the most widely studied candidate among supersymmetric theories. To prevent this model from being a phenomenological catastrophe, one is forced to introduce a $\mathbb{Z}_2$ R-symmetry [4] in order to forbid dangerous proton decay operators. This symmetry predicts the stability of the *lightest supersymmetric particle* (LSP). Such a particle is a promising candidate for dark matter. Another nice feature of the MSSM is that gauge coupling unification can be achieved at energy scales of the order of 10^{16} GeV [5]. This revives the possibility for GUT scenarios to be realized in nature.

Gravity is still a missing piece in the game of unification. All attempts to arrive at a quantum version of Einstein's theory of gravity have turned out to be unsatisfactory. *String theory* seems the most promising framework to incorporate gravity as a quantum theory. It differs form the standard field theory approach in the sense that particles are no longer treated as point-like but as extended objects (strings) [6] embedded into some D-dimensional space-time (*target space*).

Consistent tachyon-free string theories can be achieved by imposing worldsheet supersymmetry. In order to prevent Lorentz invariance of the model from being spoiled by quantum effects, one has to fix the dimension of the target space to be $D = 10$. There are five consistent string theories. They are related by a network of dualities which makes them to be thought of as different limits of an underlying 11-dimensional M-theory. A realistic stringy description has to give a target space which is manifestly four dimensional. This can be done by considering the target space to decompose a direct product of Minkowski space and some compact compact six dimensional manifold.

In this work we concentrate in those stringy scenarios originating from the $\mathrm{E}_8 \times \mathrm{E}_8$ heterotic string [7, 8], this is a theory with $\mathcal{N} = 1$ SUSY in ten dimensions. In the spirit realizing some of the very special features of the (MS)SM, many compactification schemes have been implemented. Among those, orbifolds [9, 10] correspond to the simplest case. Despite their simplicity, these models have proven to be very fruitful in the search for realistic vacua, as seen e.g. in the case of the $\mathbb{Z}_{6\text{-II}}$ mini-landscape [11, 12] and the $\mathbb{Z}_2 \times \mathbb{Z}_2$ orbifold model [13]. It was found that the models of physical relevance exhibit some statistical preference for a μ term of the order of the order of the gravitino mass, intermediate scale SUSY breaking and top-Yukawa unification [14].

As pointed out previously, the MSSM requires of a $\mathbb{Z}_2$ R-symmtery which forbids dangerous dimension four operators. Still, proton decay can proceed via operators of dimension five, unless one imposes proton hexality [15], this R-symmetry seems problematic because the charge assignment is not compatible with grand unification. A novel proposal to circumvent this problem is to look for an anomaly-free R-symmetry which is in agreement with grand unification, forbids dimension four and five proton decay operators and in addition forbids the μ term [16]. This symmetry has to be broken by quantum effects to a remnant which looks like the standard $\mathbb{Z}_2$. For the case of SO(10) GUTs, it was found that there is only one possible $\mathbb{Z}_4^R$-symmetry with the previous properties [16]. A symmetry with this features was found in the context of the $\mathbb{Z}_2 \times \mathbb{Z}_2$ orbifold [18]. The appearance of candidates for the $\mathbb{Z}_4^R$ is related to the presence of $\mathbb{Z}_2$ symmetries in the orbifold. In this spirit, we attempt to study the possibilities to achieve realistic models in the $\mathbb{Z}_2 \times \mathbb{Z}_4$ orbifold, which seems to be a suitable ground for stringy realizations of the $\mathbb{Z}_4^R$ [16].

Summary

This work is structured in the following fashion: In the second chapter we discuss the construction of the ten dimensional heterotic string as well as the construction of the massless string spectrum in orbifold compactifications. In the spirit of finding a systematic procedure to determine the R-symmetries present in a given orbifold model, chapter three is devoted to a review of some relevant results from orbifold CFT´s and to a study of discrete remnants of the Lorentz group for some factorizable and non factorizable $\mathbb{Z}_N$ orbifolds. We will investigate which of these remnants introduce R-symmetries in the model. In chapter four we focus on the $\mathbb{Z}_2 \times \mathbb{Z}_4$ orbifold. First we discuss some of its geometrical features, then we describe the algorithm implemented to look for all inequivalent embeddings allowed by the point group. Then we focus on those models in which an SO(10) GUT scheme can be accommodated.

Chapter 2

Heterotic String Theory on Orbifold Backgrounds

"Soubéssemo-las nós e as coisas do céu teriam outros nomes"

José Saramago, *Memorial do Convento*

This chapter is devoted to the construction of the heterotic string and the compactification procedures one can follow in order to obtain an effective model in four dimensions. The excessive amount of supersymmetry at which one arrives when considering purely toroidal compactifications makes them uninteresting for phenomenological reasons. A chiral spectrum can only be obtained by introducing more sophisticated backgrounds, among which orbifolds seem to be the simplest alternative. The first sections involve the construction of all geometrical quantities required for the proceeding discussions. Further we establish the mode expansions, quantization conditions and massless spectrum for the 10D heterotic theory and their four dimensional versions derived from toroidal and orbifold compactifications.

2.1 Lattices

In general, a lattice Γ of rank d can be defined as a free abelian group $\mathbb{Z}^d$ described by a finite number of generators $\mathbf{e}_\alpha$ $(\alpha = 1, ..., d)$ [19], such that:

$$\Gamma = \left\{ \sum_{\alpha=1}^{d} n_\alpha \mathbf{e}_\alpha \mid n_\alpha \in \mathbb{Z} \right\}$$

Each such lattice can be seen as a discrete set of points in $\mathbb{R}^d$, from which the lattice inherits a metric $g_{\alpha\beta} = (\mathbf{e}_\alpha, \mathbf{e}_\beta)$. Provided such a bilinear form, the dual lattice of Γ is defined as

$$\Gamma^* := \left\{ \xi \in \mathbb{R}^d \mid (\xi, \lambda) \in \mathbb{Z};\ \ \forall \lambda \in \Gamma \right\}$$

If $g_{\alpha\beta} \in \mathbb{Z}$ for all α, β the lattice is called *integral*. For integral lattices it is obvious that $\Gamma \subset \Gamma^*$. The lattice is said to be *even* if all of its elements are even i.e. $\lambda^2 = (\lambda, \lambda) \in 2\mathbb{Z}\ \forall \lambda \in \Gamma$. A *self dual* lattice is defined such that it fulfills $\Gamma = \Gamma^*$.

A d-dimensional torus $\mathbb{T}^d$ can be understood as $\mathbb{R}^d/\Gamma$, meaning that any two points in the d-dimensional space are identified if they differ by any vector in a lattice Γ.

2.2 Orbifolds and further useful constructions

Given a finite group[1] $P \subset \mathrm{Aut}(\Gamma)$, one can construct a d dimensional orbifold by dividing P out of the torus. Such quotient manifolds define an *orbifold* [20]:

$$\mathbb{O} \equiv \mathbb{T}^d/P = \mathbb{R}^d/S,$$

where the space group S is defined as $S = P \ltimes \Gamma$. An element $g = (\theta, \lambda) \in S$, $\theta \in P$, $\lambda \in \Gamma$ acts on $\mathbb{R}^d$ via:

$$\begin{aligned} g : \mathbb{R}^d &\to \mathbb{R}^d \\ x &\mapsto gx = \theta x + \lambda, \end{aligned}$$

so that x and gx are identified upon orbifolding.

Given the structure of S, its elements can be grouped into classes whose members share the same point group element. These sets are called *sectors*. The *untwisted sector* contains the identity element, whereas the *twisted sector* T_θ is associated to the non-trivial element $\theta \in P$.

Two elements g_1 and g_2 are *conjugate* to each other, $g_1 \sim g_2$, take $g_1 = (\theta, \lambda)$, if the equivalence relation[2]

$$g_2 = h g_1 h^{-1} = (\theta, \omega\lambda + (\mathbb{1} - \theta)\lambda'),\ \text{for some } h = (\omega, \lambda') \in S. \tag{2.1}$$

is satisfied. Provided an element g of S, its corresponding *conjugacy class* $[g]$ is the set of all members of S which are conjugate to g. As an immediate corollary

[1]For our purposes only Euclidean lattices are of relevance so that the automorphism group of Γ is always a subgroup of $O(d)$.

[2]We specialize to the case of Abelian point groups.

one has: $[g_1] = [g_2]$ iff $g_1 \sim g_2$.

The space group does not act freely on $\mathbb{R}^d$. Given a non-trivial element $g = (\theta, \lambda) \in S$, a point z_f which fulfills

$$z_f = g z_f = \theta z_f + \lambda \tag{2.2}$$

is called a *fixed point* of g. If eq. (2.2) is satisfied not only by a single point but by a connected subspace of dimension d', we will refer to it as a *fixed torus*.

If θ has order n and $(1-\theta)$ has full rank, $g = (\theta, \lambda)$, $\lambda \in \Gamma$ has only one fixed point of the form $z_f = \beta/n$ for some $\beta \in \Gamma$. In case g has some fixed line (torus), such subspace will intersect with points given by $1/n$ times a lattice vector.

Consider now two fixed points $z_{f1} = g_1 z_{f1}$ and $z_{f2} = g_2 z_{f2}$, with g_2 related to g_1 via eq. (2.1), then

$$z_{f2} = h z_{f1} = \omega z_{f1} + \lambda' \, . \tag{2.3}$$

The above relation implies that these two fixed points (tori) are *equivalent* under the space group. *Inequivalent fixed points* are thus in a one to one relation with the conjugacy classes whose elements act non-freely on $\mathbb{R}^d$.

Orbifolds are flat spaces with *curvature singularities* situated at the position of the fixed points. This can be easily inferred from the fact that the local holonomy group of the orbifold is trivial almost everywhere, only at the fixed points we can find a discrete non trivial holonomy which in general is a subgroup of P.

2.2.1 Lattice Deformations

As discussed before, only isometries of the lattice can be consistently modded out of $\mathbb{T}^d(\Gamma)$. One can look at all possible lattices on which a certain point group acts crystallographically. For instance, any element $\rho \in \mathrm{Aut}(\Gamma)$ which satisfies $\rho\theta\rho^{-1} \in P$, for all $\theta \in P$, induces a lattice $\rho\Gamma$ for which P is a set of isometries. However, such elements are not of much interest since there is no way to distinguish between $\mathbb{T}^d(\Gamma)/P$ and $\mathbb{T}^d(\rho\Gamma)/P$.

The basis $\{\mathbf{e}_\alpha\}_{\alpha=1,\dots,d}$ is specified by a set of angles and radii which one can set to vary continously while preserving linear independence. Consider a continuous parameter ξ which accounts for one such transformation. We denote the *deformed* lattice spanned by $\{\mathbf{e}_\alpha(\xi)\}_{\alpha=1,\dots,d}$ by $\Gamma(\xi)$, with $\Gamma(0)$ being the lattice generated by the simple roots of a semi-simple Lie algebra $\mathfrak{g}$. The deformed torus metric

$g_{\alpha\beta}(\xi) = (\mathbf{e}_\alpha(\xi), \mathbf{e}_\beta(\xi))$ needs to satisfy

$$\theta^T g(\xi)\theta = g(\xi), \ \forall\theta \in P\,. \tag{2.4}$$

in order to preserve P as a subgroup of $\mathrm{Aut}(\Gamma(\xi))$. Note that in the above equation we have implicitly imposed that the elements of the point group are invariant under the *deformation*. This is indeed the case, because the isometries can not encompass the continuous variation of ξ.

On the orbifold $\mathbb{T}(\Gamma)/P$, one is only allowed to perform lattice deformations which are consistent with eqn. (2.4). Such continuous transformations are in close relation to the presence of moduli fields in the spectrum of the heterotic theory compacified on this orbifold background.

2.3 The Heterotic String

Closed string theories are characterized by having the left- and right-movers decoupled [21]. One is therefore permitted to construct a theory of closed strings in which the left-moving modes are described by the *26D bosonic string theory* and, in order prevent the appearence of tachyonic states and allow for space-time fermions, the right-moving modes are taken as those of the *10D superstring*. The so-called *heterotic string* possesses in turn $(\mathcal{N}_L, \mathcal{N}_R) = (0, 1)$ world-sheet supersymmetries.

At first it may seem not so clear how the target space of this theory looks. In its earliest formulation [8], the left movers were described by an alternative construction which makes use of 10 bosonic and 32 fermionic coordinates to cancel the conformal anomaly of $10+32/2 = 26$. This *fermionic description* makes it obvious that the heterotic theory is in fact ten-dimensional, given that there are only ten coordinates with both left and right moving degrees of freedom [23]. The fact that two world-sheet Majorana-Weyl fermions play the role of one holomorphic boson, allows for the replacement of the fermions by 16 bosonic fields in the left moving sector, recovering then the standard bosonic string theory. In what follows we concentrate on the *bosonic formulation* given the intuitive geometric description it provides.

The world sheet action for the heterotic string has the form:

$$S = \frac{1}{\pi}\int d^2\sigma\left(2\partial_+ X_\mu \partial_- X^\mu + \mathrm{i}\psi_\mu\partial_+\psi^\mu + 2\partial_+ X_I \partial_- X^I\right), \tag{2.5}$$

where the world sheet metric has been fixed in light-cone coordinates $\sigma^\pm = \tau \pm \sigma$. The target space coordinates $X^\mu(\tau,\sigma)$ $\mu = 0, ..., 9$ embed the world sheet into the target space $\mathcal{M}_{10}$. Using the equations of motion, such coordinates can also be decomposed as:

$$X^\mu(\tau,\sigma) = X_L^\mu(\tau+\sigma) + X_R^\mu(\tau-\sigma)\,. \tag{2.6}$$

$X_R^\mu(\tau-\sigma)$ together with the fermionic fields $\psi^\mu(\tau-\sigma)$ correspond to the superstring right movers, while the 26 bosonic coordinates are described by $X_L^\mu(\tau+\sigma)$ and $X^I(\tau+\sigma)$, where $I = 1, ..., 16$.

As discussed before, heterotic strings are subjected to closeness contraints:

$$X^\mu(\tau,\sigma+\pi) = X^\mu(\tau,\sigma), \tag{2.7}$$

so that the space-time coordinates allow for mode expansions of the form

$$X^\mu(\tau,\sigma) = x^\mu + (p_L^\mu + p_R^\mu)\tau + \sum_{\substack{n\in\mathbb{Z}\\ n\neq 0}} \frac{1}{n}\left[\alpha_n^\mu e^{-2\mathrm{i}n(\tau-\sigma)} + \tilde{\alpha}_n^\mu e^{-2\mathrm{i}n(\tau+\sigma)}\right]\,. \tag{2.8}$$

From the boundary condition (2.7), it follows that the left and right moving momenta must match:

$$p_L = p_R \equiv \frac{p}{2}\,. \tag{2.9}$$

For the right moving fermions one has the freedom to impose either periodic (Ramond) or anti-periodic (Neveu-Schwarz) boundary conditions: $\psi^\mu(\tau-(\sigma+\pi)) = \pm\psi^\mu(\tau-\sigma)$. Mode expansions consistent with these periodicities are given by

$$\psi^\mu(\tau-\sigma) = \begin{cases} \sum_{n\in\mathbb{Z}} d_n^\mu e^{-2\mathrm{i}n(\tau-\sigma)} & \text{(R)} \\ \sum_{r\in\mathbb{Z}+\frac{1}{2}} b_r^\mu e^{-2\mathrm{i}r(\tau-\sigma)} & \text{(NS)} \end{cases} \tag{2.10}$$

The extra sixteen bosonic fields X^I in the left moving sector have to be compactified on a torus $\mathbb{T}^{16}$, in order to preserve modular invariance. In this framework strings can close after encircling the torus a certain amount of times

$$X^I(\tau+(\sigma+\pi)) = X^I(\tau+\sigma) + \pi\sum_{\ell=1}^{16} w_\ell \mathbf{e}_\ell^I\,, \quad I = 1, ..., 16\,,$$

where $w_\ell \in \mathbb{Z}$ are the winding numbers and the basis vectors $\mathbf{e}^I$ span some lattice Γ_{16}. Modular invariance requires the lattice to be even and self dual. The only Euclidean structures with such properties are either the root lattice of $\mathrm{E}_8\times\mathrm{E}_8$ or that of $Spin(32)/\mathbb{Z}_2$. By choosing any of these compacification lattices one sets the gauge group to be either $\mathrm{E}_8\times\mathrm{E}_8$ or SO(32). In fact, these are the only alternatives which allow for anomaly cancelation in a ten dimensional supergravity theory [22]. Since the gauge group arises from the compact coordinates X^I they are often called *gauge degrees of freedom*. The mode expansion for these extra left movers reads

$$X^I(\tau+\sigma) = x^I + p^I(\tau+\sigma) + \sum_{\substack{n\in\mathbb{Z}\\ n\neq 0}} \frac{1}{n}\tilde{\alpha}^I_n e^{-2\mathrm{i}n(\tau+\sigma)}, \tag{2.11}$$

where $p \in \Gamma_{16}$ is the quantized *internal momentum* of the string.

2.3.1 Light Cone Quantization

Upon quantization, the only non vanishing (anti-) commutation relations between the modes present in eqs. (2.8), (2.10) and (2.11) are

$$[\alpha^\mu_n, \alpha^\nu_m] = n\delta_{n+m}\eta^{\mu\nu}, \quad [\tilde{\alpha}^\mu_n, \tilde{\alpha}^\nu_m] = n\delta_{n+m}\eta^{\mu\nu}, \quad \left[\tilde{\alpha}^I_n, \tilde{\alpha}^J_m\right] = n\delta_{n+m}\delta^{IJ},$$

$$\{b^\mu_r, b^\nu_s\} = \delta_{r+s}\eta^{\mu\nu}, \quad \{d^\mu_n, d^\nu_m\} = \delta_{n+m}\eta^{\mu\nu}\,.$$

The reality conditions imposed on both bosonic and fermionic coordinates together with the previous equations, make it possible to interpret these modes as creation $(n, r < 0)$ and annihilation operators $(n, r > 0)$. This motivates the definition of the string-Hilbert space $\mathcal{H}$ which is composed of tensor products between left- and right-moving states carrying the same momentum. However conformal and super-conformal invariance of the left and right moving sectors plague the spectrum of states which are not physically meaningful. To overcome this problem one has to require the states in Hilbert space to be in agreement with the quantum version of the following constraints

$$T_{++} = T_{--} = 0, \quad T_F = 0, \tag{2.12}$$

in which $T_{\alpha\beta}$ is the stress-energy tensor and T_F is the world sheet supersymmetry current; the explicit expressions for these quantities are

$$T_{++} = -\partial_+ X^\mu \partial_+ X_\mu - \partial_+ X^I \partial_+ X_I, \tag{2.13}$$

$$T_{--} = -\partial_- X^\mu \partial_- X_\mu - \frac{\mathrm{i}}{2}\psi^\mu \partial_- \psi_\mu, \tag{2.14}$$

$$T_F = \psi^\mu \partial_- X_\mu\,. \tag{2.15}$$

The mode expansions of these operators

$$T_{++} = \sum_{n\in\mathbb{Z}} \tilde{L}_n e^{-2\mathrm{i}n(\tau+\sigma)}, \quad T_{--} = \sum_{n\in\mathbb{Z}} L_n e^{-2\pi\mathrm{i}n(\tau-\sigma)},$$

$$T_F = \begin{cases} \sum_{n\in\mathbb{Z}} F_n e^{-2\mathrm{i}n(\tau-\sigma)} & \text{(R)} \\ \sum_{r\in\mathbb{Z}+\frac{1}{2}} G_r e^{-2\mathrm{i}r(\tau-\sigma)} & \text{(NS)} \end{cases}$$

define the generators of the *super-Virasoro algebra*. From equation (2.12) it follows that a physical state $|\phi\rangle$ needs to satisfy

$$\tilde{L}_n |\phi\rangle = L_n |\phi\rangle \,, \qquad \forall n > 0 \tag{2.16}$$

$$F_n |\phi\rangle = G_r |\phi\rangle = 0, \qquad \forall n, r > 0 \tag{2.17}$$

$$(L_0 - a_R) |\phi\rangle = (\tilde{L}_0 - a_L) |\phi\rangle = 0 \,. \tag{2.18}$$

In the last equation, the coefficients $a_{L/R}$ have been introduced to account for the effects of normal ordering. In the left-moving sector one has $a_L = 1$, while in the right-moving sector a_R equals 0 and $\frac{1}{2}$ for R and NS sectors, respectively.

Yet another possibility to remove the non physical degrees of freedom is by making an additional gauge choice in which the *super-Virasoro constraints* are implicitly satisfied. In light cone coordinates

$$X^{\pm} = \frac{1}{\sqrt{2}}(X^0 \pm X^1),$$

it is known that one has enough freedom to fix the gauge in such way that only the transverse coordinates X^i and ψ^i $i = 2, ..., 9$ remain physical [23]. This gauge choice leaves us with a theory which is manifestly covariant under the transverse SO(8) little group of the Lorentz group SO(9,1).

Now we are able to compute the mass spectrum of the theory. The mass squared operator in the light-cone gauge is

$$M^2 = M_L^2 + M_R^2,$$

with the constraint $M_L^2 = M_R^2$ imposed by eqn. (2.9). The left moving mass operator is given by

$$\frac{M_L^2}{4} = \frac{p^I p_I}{2} + \tilde{N} - 1 \,, \tag{2.19}$$

where

$$\tilde{N} = \sum_{n=1}^{\infty} (\tilde{\alpha}^i_{-n}\tilde{\alpha}^i_n + \tilde{\alpha}^I_{-n}\tilde{\alpha}^I_n) \,,$$

is the number operator in the left moving sector. For the right movers one has

$$\frac{M_R^2}{4} = \begin{cases} \displaystyle\sum_{m=0}^{\infty} m d_{-m}^i d_m^i + N & \text{(R)} \\ \displaystyle\sum_{r=\frac{1}{2}}^{\infty} r b_{-r}^i b_r^i + N - \frac{1}{2} & \text{(NS)} \end{cases} \tag{2.20}$$

with $N = \sum_{n=1}^{\infty} \alpha_{-n}^i \alpha_n^i$. The above equation can be written more compactly after *bosonization* of the fermionic degrees of freedom. This procedure allows for a weight q in the vector or spinor lattice of SO(8), to account for the effects of the fermionic modes when the GSO projection is introduced. In terms of the *right moving momentum* q, the mass operator takes the form

$$\frac{M_R^2}{4} = \frac{q^2}{2} + N - \frac{1}{2}. \tag{2.21}$$

As expected from the bosonic string, the ground state $|0\rangle_L$ in the left moving sector is tachyonic. The massless states from this side are:

(i) $\tilde{\alpha}_{-1}^i |0\rangle_L$ which transform in the vector representation $\mathbf{8}_v$ of SO(8).

(ii) The Lorentz scalars $\tilde{\alpha}_{-1}^I |0\rangle_L$ and $|p\rangle_L$ with $p^2 = 2$. The former relation is only satisfied by the 480 roots of SO(32) or $E_8 \times E_8$ depending on the choice of Γ_{16}.

In the low energy regime massive string excitations remain uninteresting since their masses are of the order of the string scale $M_s \sim 10^{17}$ GeV [24, 25]. The right moving sector lacks negative mass states, so that $|0\rangle_L$ cannot be used to build physical states. From eq. (2.21) we see that massless states are characterized by $N = 0$ and $q^2 = 1$, this condition is only satisfied by the following right moving momenta[3]

(i) $q^v = \left(\underline{\pm 1, 0, 0, 0}\right)$ corresponding to the weights of the vector representation $\mathbf{8}_v$ of SO(8).

(ii) $q^s = \left(\left[\pm\frac{1}{2}, \pm\frac{1}{2}, \pm\frac{1}{2}, \pm\frac{1}{2}\right]\right)$, which give the weights of the spinor representation $\mathbf{8}_s$ of SO(8).

Tensoring the states previously described one finally arrives at the massless spectrum of the ten dimensional heterotic string:

[3]Here $\left(\underline{\pm 1, 0, 0, 0}\right)$ means all possible permutations and $\left(\left[\pm\frac{1}{2}, \pm\frac{1}{2}, \pm\frac{1}{2}, \pm\frac{1}{2}\right]\right)$ describe all the combinations with an even number of minus signs.

(i) $|q^v\rangle \otimes |p\rangle_L$ and $|q^v\rangle \otimes \tilde{\alpha}^I_{-1}|0\rangle_L$ transform in the adjoint representation $(\mathbf{248}, \mathbf{1}) \oplus (\mathbf{1}, \mathbf{248})$ of $E_8 \times E_8$ or $\mathbf{496}$ of SO(32), so that these spin one fields correspond to the gauge bosons of the theory. The states $|q^s\rangle \otimes |p\rangle_L$ and $|q^s\rangle \otimes \tilde{\alpha}^I_{-1}|0\rangle_L$ lead to the gauginos as expected for $\mathcal{N} = 1$ supersymmetry.

(ii) The gauge singlets $|q^v\rangle \otimes \tilde{\alpha}^i_{-1}|0\rangle_L$ and $|q^s\rangle \otimes \tilde{\alpha}^i_{-1}|0\rangle_L$ yield the $\mathcal{N} = 1$ SUGRA multiplet. The states with bosonic right moving momentum decompose into the following irreducible representations:

$$\mathbf{8}_v \otimes \mathbf{8}_v = \mathbf{1} + \mathbf{28} + \mathbf{35}_v,$$

which correspond to the graviton g ($\mathbf{35}_v$), the dilation Φ ($\mathbf{1}$), and the antisymmetric two form B ($\mathbf{28}$). For the remaining 64 fermionic states one has the decomposition

$$\mathbf{8}_s \otimes \mathbf{8}_v = \mathbf{8}_c + \mathbf{56}_c,$$

which give rise to the dilatino λ ($\mathbf{8}_c$) and the gravitino ψ^μ ($\mathbf{56}_c$). The subindex c denotes cospinor representations, which arise in SO(8) as a consequence of the triality of its Dynkin diagram.

2.4 Toroidal Compactifications

The remarkably simple spectrum we have achieved so far constitutes a beautiful setup which encompasses all interactions of a ten dimensional world. There is no doubt of the conceptual relevance of the heterotic theory as a geometric description of fundamental interactions, however, its physical relevance is subjected to the existence of a certain regime in which only the four space-time coordinates are manifest. Such regime can be achieved by confining the extra dimensions to a certain compact space. Let the 10D space $\mathcal{M}_{10}$ decompose into a direct product of Minkowski space-time $\mathcal{M}_{3,1}$ and a six dimensional manifold $\mathcal{M}_6$:

$$\mathcal{M}_{10} = \mathcal{M}_{3,1} \times \mathcal{M}_6\,. \tag{2.22}$$

One can then assume that the volume of $\mathcal{M}_6$ is small enough, making any attempt of detection unfeasible. A six dimensional torus $\mathbb{T}^6$ seems to be the simplest alternative, not only from the topological but also from the string theory side: the fact that the torus is flat everywhere, guarantees the analytic solvability of the equations of motion for the compact coordinates so that the spectrum and interaction can still be computed exactly.

Decompositions of the target space which have the form (2.22) can be consistent with the requirement of Poincaré invariance in 4D. However, the clear distinction between compact and non compact coordinates hints at the breaking of the Lorentz symmetry SO(9,1) of the tangent space, down to a subgroup which does not mix between $\mathcal{M}_{3,1}$ and $\mathcal{M}_6$. Toroidal compactifications induce the minimal breaking

$$\mathrm{SO}(9,1) \rightarrow \mathrm{SO}(3,1) \times \mathrm{SO}(6) \cong \mathrm{SO}(3,1) \times \mathrm{SU}(4)\,. \tag{2.23}$$

For the transverse SO(8) one has

$$\mathrm{SO}(8) \rightarrow \mathrm{SO}(2) \times \mathrm{SO}(6) \cong \mathrm{U}(1) \times \mathrm{SU}(4), \tag{2.24}$$

where SO(2)≅U(1) is the transverse component of SO(3,1) and coincides with the helicity operator for massless states. In four dimensions, the SU(4) factor can only be seen as an internal symmetry treating bosons and fermions in a different manner. The winding numbers in the compact coordinates do not make any contribution to the massless spectrum so that one finds the same states as irreducible representations of the unbroken subgroup of SO(8). The 10D gravitino decomposition

$$\mathbf{56}_c \rightarrow \mathbf{4}_{3/2} \oplus \overline{\mathbf{4}}_{-3/2} \oplus \mathbf{4}_{1/2} \oplus \overline{\mathbf{4}}_{-1/2} \oplus \overline{\mathbf{20}}_{1/2} \oplus \mathbf{20}_{-1/2},$$

permits us to determine the number of space-time supersymmetries since it coincides with the number of 4D gravitini. The representations $\mathbf{4}_{3/2}$ and $\overline{\mathbf{4}}_{-3/2}$ pair up to complete the helicity multiplets of four massless spin 3/2 fermions, and consequently lead to $\mathcal{N}=4$ SUSY in four dimensions.

This last result is far from leading to a realistic description because supersymmetric theories with $\mathcal{N}>1$ do not allow for a chiral spectrum. On the other hand, a heterotic theory lacking any supersymmetry in 4D also remains uninteresting since the SUSY breaking scale would be of the order of the compactification scale and in that framework supersymmetry is unable to solve the hierarchy problem. From these arguments it follows that the requirement of $\mathcal{N}=1$ SUSY in four dimensions is unavoidable for any stringy attempt to make contact with particle physics.

A phenomenologically interesting theory requires of a less trivial manifold. In this sense, one would be interested in the interplay between the number of unbroken supersymmetries and the topological properties of the manifold; in order to determine its physical relevance. Each of the SUSY charges Q_α induces variations in all fields, weighted by an infinitesimal parameter ϵ_α. Unbroken supersymmetries must lead to invariant backgrounds. This statement is trivially fulfilled in the case of bosonic fields at the classical level, so that one only needs to require the variation of the fermionic fields to be zero. For the 10D supergravity of the heterotic string these variations are[4]

$$\delta\psi_\mu = \nabla_\mu \epsilon, \quad \delta\lambda = -\frac{1}{2}\Gamma^\mu \partial_\mu \Phi \epsilon, \quad \delta\chi = 0; \quad \mu, \nu = 0, ..., 9\,,$$

corresponding to the variations of the gravitino, dilatino and gaugino, respectively. The existence of any unbroken supersymmetry requires a constant dilaton $\partial_\mu \Phi = 0$ and a non-trivial solution for the following *Killing spinor equation*:

$$\nabla_\mu \epsilon = 0,$$

hence ϵ does not depend on the 4D space-time coordinates. Due to the structure of target space (see eq. (2.22)), it can also be written as

$$\epsilon(y^n) = \xi \otimes \eta(y^n),$$

where y^n $(n = 4, .., 9)$ are the coordinates of $\mathcal{M}_6$. The Majorana-Weyl supersymmetry parameter ϵ decomposes as $\mathbf{16} \to (\mathbf{2}, \mathbf{4}) \oplus (\bar{\mathbf{2}}, \overline{\mathbf{4}})$ under (2.23), so that ξ is a constant Weyl spinor in four dimensions. Also note that η needs to satisfy

$$\nabla_n \eta = 0,$$

meaning that in four dimensions *each unbroken supersymmetry is associated to a covariantly constant spinor of internal space*. The Bianchi identity in $\mathcal{M}_6$ can be written as:

$$[\nabla_m, \nabla_n]\eta = \frac{1}{4} R_{mnpq} \Gamma^{pq} \eta = 0$$

where Γ^{pq} are the generators of SU(4). For the case of $\mathcal{N} = 1$ SUSY in 4D, the components $R_{mnpq}\Gamma^{pq}$ must be in a subgroup of $SU(4)$ leaving one component of η invariant. Under the breaking

$$\mathrm{SU}(4) \to \mathrm{SU}(3) \times \mathrm{U}(1),$$

a spinor decomposes as $\mathbf{4} \to \mathbf{3}_{-1} \oplus \mathbf{1}_3$, meaning that the presence of one unbroken global SUSY in the low energy effective theory is only possible for compact spaces with SU(3) *holonomy*. This is the defining property of *Calabi-Yau manifolds* [27].

2.5 Six Dimensional Orbifolds

As we already pointed out, orbifolds are almost flat spaces equipped with a set of finite singular points with a non trivial holonomy group $Q \subseteq P$, so that, in

[4]These variations are valid only in the case of vanishing torsion, for more general expressions, see e.g. [26]

principle one can construct a more realistic heterotic theory upon compactification on a six dimensional orbifold for which Q is a subgroup of SU(3) [9].

To define the orbifold we consider a six dimensional lattice

$$\Gamma_6 = \{n_\alpha \mathbf{e}_\alpha;\ n_\alpha \in \mathbb{Z},\ \alpha = 1, ..., 6\},$$

and a subgroup P of its automorphism group. We restrict to the case of abelian point groups. In such cases P is given by a direct product of cyclic factors, so that the action of any element $\vartheta \in P$ can be viewed as a rotation

$$\vartheta = \exp\{2\pi(v^1 J_{45} + v^2 J_{67} + v^3 J_{89})\}\,, \tag{2.25}$$

where the coefficients v^1, v^2 and v^3 serve to define a *twist vector*[5] $v \equiv (0, v^1, v^2, v^3)$. The operators J_{45}, J_{67} and J_{89} are the three Cartan generators of SO(6), so that P consists of three cyclic factors at most, one per generator. Since we are interested in $P \subset$ SU(3), the twists v^1, v^2 and v^3 are not all independent. This means that Abelian point groups which lead to orbifolds with $\mathcal{N} = 1$ SUSY in 4D must be of the form $\mathbb{Z}_N$ or $\mathbb{Z}_N \times \mathbb{Z}_M$.

Provided the general breaking pattern for the transverse SO(8) described in eq. (2.24), one can use the following complexified coordinates for the internal space

$$Z^i = X^{2i+2} + \mathrm{i}X^{2i+3}, \quad Z^{i*} = X^{2i+2} - \mathrm{i}X^{2i+3}, \tag{2.26}$$

for $i = 1, 2, 3$. The compact manifold can then be described by

$$\mathcal{M}_6 = \frac{\mathbb{C}^3}{S},$$

where S is the space group described in section 2.2. These new coordinates make explicit use of the fact that U(3) $\subset$ SO(6), and allow for setting the orbifold action on the internal coordinates to a diagonal form[6]

$$\vartheta = \mathrm{diag}(e^{2\mathrm{i}\pi v^1}, e^{2\mathrm{i}\pi v^2}, e^{2\mathrm{i}\pi v^3}), \tag{2.27}$$

From the the supersymmetry condition $\vartheta \in$ SU(3), one can conveniently choose the twist vector to satisfy

$$v^1 + v^2 + v^3 = 0. \tag{2.28}$$

[5]The first entry is introduced to indicate that there is no rotation along the generator of the transverse component of the Lorentz group SO(3,1).

[6]Note that the coordinates Z^i and Z^{i*} transform under the fundamental and anti-fundamental representations of SU(4), so that an element of the point group which acts as ϑ on Z^i, acts as $\vartheta^\dagger$ on Z^{i*}.

If $P = \mathbb{Z}_N$ one has only one generator θ, associated to a *twisted vector*

$$v_N \equiv (0, v^1, v^2, v^3).$$

The cyclicity condition ($\theta^N = 1$) implies that the twists v_N^i are all of order N i.e. $Nv_N^i = 0 \mod 1$, for $i = 1, 2, 3$. In the case of $P = \mathbb{Z}_N \times \mathbb{Z}_M$ one has two generators θ and ω related to the twists vectors v_N and v_M of order N and M respectively. Each of these twist vectors has to satisfy eq. (2.28). We introduce the notation $T_{(k_1,k_2)}$ for the twisted sector associated to $\theta^{k_1}\omega^{k_2}$. An analogous notation is used for the twisted sectors $T_{(k)}$ in $\mathbb{Z}_N$ orbifolds.

In the complex basis of $\mathbb{C}^3$, where the elements of the point group are diagonal 3×3 matrices, the underlying lattice can be aligned in various ways inside $\mathbb{C}^3$. An orbifold $\mathbb{O}^6$ is called *factorizable* if its compactification lattice Γ_6 can be continuosly deformed[7] to a direct product of three sublattices, each of which lies completely on one complex plane. If that is not the case, each vector of the basis has to be specified by three complex coordinates which are in general non-zero. In such case the lattice is called *non-factorizable*. Note also that for factorizable lattices, the independent twists $\theta_i = \mathrm{diag}(f_{1i}, f_{2i}, f_{3i})$, $f_{ij} = \exp\{2\pi \mathrm{i} \delta_{ij} v^i\}$ are also lattice automorphisms.

2.6 The Orbifolded Superstring

On the orbifold we have more possibilities for stings to close in comparison to toroidal compactifications. For the bosonic coordinates in 4D Minkowski space, the boundary conditions remain identical as those depicted in eq. (2.7) whereas in compact space, strings can close up to the action of a space group element [28]

$$Z^i(\sigma + \pi) = (gZ)^i(\sigma), \quad i = 1, 2, 3; \tag{2.29}$$

where $g \in S$ is the *constructing element* of the string. Clearly, the untwisted sector reproduces the boundary conditions one obtains from toroidal compactifications, but orbifolding provides the necessary mechanisms to remove the unwanted gravitino and gaugino states from the spectrum.

We can consider a $\mathbb{Z}_N \times \mathbb{Z}_M$ orbifold and take some constructing element $g = (\theta^{k_1}\omega^{k_2}, \lambda) \in S$. The twisted boundary conditions are then determined by the *local twist* $v_g \equiv k_1 v_N + k_2 v_M$

$$Z^i(\sigma + \pi) = e^{2\pi \mathrm{i} v_g^i} Z^i(\sigma) + \lambda^i,$$

[7]Note that the only possible deformations permitted for a certain lattice are those which commute with the point group P.

It can be shown that a mode expansion consistent with the above equation requires the string to wrap around the fixed point/torus $z_f = (1-\theta^{k_1}\omega^{k_2})^{-1}\lambda$

$$Z^i(\tau,\sigma) = z_f^i + \frac{\mathrm{i}}{2}\sum_{n\in\mathbb{Z}}\left(\frac{\tilde{\alpha}^i_{n-w^i}}{n-w^i}e^{-2\mathrm{i}(n-w^i)(\tau+\sigma)} + \frac{\alpha^i_{n+w^i}}{n+w^i}e^{-2\pi\mathrm{i}(n+w^i)(\tau-\sigma)}\right),$$

with $w^i = v_g^i \mod 1$ $(0 \leq w^i < 1)$. Note also that the twisted string is free of momentum and winding number. Such an expansion provides a set of holomorphic *twisted bosonic oscillators* $\tilde{\alpha}^i_{n-w^i}$, $\alpha^i_{n+w^i}$ for the right and left parts of the string. Their anti-holomorphic counterparts can be found from the expansion of the conjugate coordinate

$$Z^{i*}(\tau,\sigma) = z_f^{i*} + \frac{\mathrm{i}}{2}\sum_{n\in\mathbb{Z}}\left(\frac{\tilde{\alpha}^{i*}_{n+w^i}}{n+w^i}e^{-2i(n+w^i)(\tau-\sigma)} + \frac{\alpha^{i*}_{n-w^i}}{n-w^i}e^{-2\mathrm{i}(n-w^i)(\tau+\sigma)}\right).$$

These relations are used to quantize the theory in the light cone gauge. As a result one obtains that the commutation relations for the oscillators are all trivial except for:

$$\left[\tilde{\alpha}^i_{n-w^i}, \tilde{\alpha}^{j*}_{-m+w^i}\right] = (n-w^i)\delta^{ij}\delta_{nm}, \quad \left[\alpha^i_{n+w^i}, \alpha^{j*}_{-m-w^i}\right] = (n+w^i)\delta^{ij}\delta_{nm}.$$

In the presence of twisted oscillators the zero point energy receives the extra contribution

$$\delta_c = \frac{1}{2}\sum_{i=1}^{3} w^i(1-w^i), \tag{2.30}$$

which is the same for both, the right- and the left-moving sector.

The fermionic right-movers in internal space can also be rewritten as three holomorphic and three anti-holomorphic fermionic coordinates as in eq. (2.26). The twisted boundary conditions for these fields are similar to those of the standard R and NS sectors, but weighted by the phases introduced by the local twist. These phases will shift the weight of the fermionic operators, and as a consequence, bosonization will describe the effect of these modes in terms of a shifted right moving momentum $q_{sh} = q + v_g$, with q in the vector or spinor lattice of SO(8). The mass equation for the right moving states reads

$$\frac{m_R^2}{4} = \frac{q_{sh}^2}{2} + N - \frac{1}{2} + \delta_c, \tag{2.31}$$

where N, counts the number of bosonic oscillators in the right moving sector.

2.6.1 Space Group Action on Gauge Coordinates

The modular invariance properties of the one-loop partition function are preserved only if the orbifold is allowed to act on the gauge coordinates. These coordinates then transform under a non-trivial *gauge twisting group* G [9], provided an embedding homomorphism

$$S \hookrightarrow G,$$

which acts consistently on both spaces such that, for a constructing element $g \in S$, the boundary conditions in the gauge left movers read

$$X^I(\tau + (\sigma + \pi)) = (G_g X)^I(\tau + \sigma) + \pi \Lambda^I,$$

where $\Lambda \in \Gamma_{16}$ is responsible for the winding modes (see e.g. eq. (2.11)). The gauge twisting group is in general a subgroup of the automorphisms of $\mathrm{E}_8 \times \mathrm{E}_8$ or $Spin(32)/\mathbb{Z}_2$. Although other posibilities may be interesting, here we specialize to the case in which G is specified as a *translational*[8] symmetry

$$(G_g X)^I = X^I + \pi V_g^I,$$

where V_g is some 16 dimensional vector which respects the group structure of S upon lattice identifications. When considering this kind of embedding, the mode expansion for twisted strings differs from the one introduced in eq. (2.11) only by the shifted left moving momentum $p_{sh} = p + V_g$. The embedding of the space group into the gauge coordinates will break the gauge symmetry of the 10D heterotic string down to a smaller subgroup. Further we will see that this kind of breaking is rank-preserving. This is true for all freely acting embeddings one can conceive[9]. From the previous arguments it follows that the mass equation for the twisted left movers reads

$$\frac{m_L^2}{4} = \frac{p_{sh}^2}{2} + \tilde{N} - 1 + \delta_c, \tag{2.32}$$

where the bosonic number operator for the left moving sector is given by

$$\tilde{N} = \sum_{n > -\omega^i} \tilde{\alpha}^{i*}_{-n-\omega^i} \tilde{\alpha}^{i}_{n+\omega^i} + \sum_{n > \omega^i} \tilde{\alpha}^{i}_{-n+\omega^i} \tilde{\alpha}^{i*}_{n-\omega^i} + \sum_{n>0} \tilde{\alpha}^{I}_{-n} \tilde{\alpha}^{I}_{n}. \tag{2.33}$$

For the abelian orbifolds consistent with the requirement of $\mathcal{N} = 1$ supersymmetry in 4D, the translational embeddings can be described by

$$\mathbb{Z}_N : \quad (\theta^k, n_\alpha \mathbf{e}_\alpha) \mapsto k V_N + n_\alpha W_\alpha \tag{2.34}$$

$$\mathbb{Z}_N \times \mathbb{Z}_M : \quad (\theta^{k_1} \omega^{k_2}, n_\alpha \mathbf{e}_\alpha) \mapsto k_1 V_N + k_2 V_M + n_\alpha W_\alpha \tag{2.35}$$

[8]If one chooses G to be in the inner automorphism of the Lie algebra associated to Γ_{16}, its action can be realized as a shift [36].

[9]For examples of embeddings with non trivial stabilizers see e.g. [30].

The simplest case in which one just embeds the point group (by specifying the action of the generators θ and ω) is in agreement with modular invariance of the vacuum-to-vacuum amplitude. The further embedding of the six dimensional lattice is a freedom one has in terms of the *discrete Wilson lines* W_α [29].

In the $\mathbb{Z}_N \times \mathbb{Z}_M$ orbifold the choice of the vectors V_N, V_M and W_α ($\alpha = 1, ..., 6$) is not entirely arbitrary[10]. G must properly reproduce the structure of S while preserving the modular invariance properties of the theory. Since θ^N and ω^M equal the identity element one has to guarantee that their action is trivial on the gauge coordinates. The only trivial translations one has in this space are related to the identifications imposed by the lattice Γ_{16}. From these arguments we arrive at the conditions

$$NV_1, MV_2 \in \Gamma_{16}. \tag{2.36}$$

For each lattice vector $\mathbf{e}_\alpha \in \Gamma$ one has to find an element $\vartheta = \theta^{k_1}\omega^{k_2}$ which leads to the minimum power N_α for which the following relation is satisfied

$$\sum_{k=1}^{N_\alpha} \vartheta^k \mathbf{e}_\alpha = 0.$$

Consider an element $h = (\vartheta, \mathbf{e}_\alpha) \in S$, its corresponding embedding is given by

$$V_h = (k_1 V_1 + k_2 V_2) + W_\alpha.$$

On the other hand, the action of $h^{N_\alpha} = (\vartheta^{N_\alpha}, 0)$ is associated to

$$V_{h^{N_\alpha}} = N_\alpha(k_1 V_1 + k_2 V_2).$$

Consistency with the product structure of S implies that $V_{h^{N_\alpha}}$ and $N_\alpha V_h$ can differ at most by lattice vector from the sixteen dimensional lattice, so that the Wilson line must satisfy

$$N_\alpha W_\alpha \in \Gamma_{16},$$

Due to this property, N_α is referred to as the *order of the Wilson line*. The set of Wilson lines which are *inequivalent* can also be found by a similar procedure. Consider for example, the case in which two basis vectors are related by a point group transformation: $\vartheta \mathbf{e}_\alpha = m_{\alpha\beta}\mathbf{e}_\beta$ for some $m_{\alpha\beta} \in \mathbb{Z}$ for all α, β and for some $\vartheta = \theta^{k_1}\omega^{k_2}$. An element $h = (\vartheta, \mathbf{e}_\alpha)$ is embedded as $V_h = (k_1 V_1 + k_2 V_2) + W_\alpha$. Similarly, the shift for h^2 is given by $V_{h^2} = 2(k_1 V_1 + k_2 V_2) + W_\alpha + m_{\alpha\beta} W_\beta$. Therefore a consistent embedding requires

$$W_\alpha - m_{\alpha\beta} W_\beta \in \Gamma_{16}. \tag{2.37}$$

[10] On $\mathbb{Z}_N$ one also has to impose some consistency constraints, but these can be straighforwardly derived from those we will derive for $\mathbb{Z}_N \times \mathbb{Z}_M$.

This *equivalence* of the Wilson lines is denoted as $W_\alpha \simeq m_{\alpha\beta} W_\beta$.

The constraints at which one arrives by requiring modular invariant amplitudes can be summarized in the following set of equations [31]

$$
\begin{aligned}
N(V_N^2 - v_N^2) &= 0 \mod 2, && (2.38)\\
M(V_M^2 - v_M^2) &= 0 \mod 2, && (2.39)\\
\gcd(N,M)(V_N \cdot V_M - v_N \cdot v_M) &= 0 \mod 2, && (2.40)\\
N_\alpha(W_\alpha \cdot V_i) &= 0 \mod 2, && (2.41)\\
\gcd(N_\alpha, N_\beta)(W_\alpha \cdot W_\alpha) &= 0 \mod 2. && (2.42)
\end{aligned}
$$

2.6.2 Orbifold GSO Projectors

In the previous sections we found the expressions for the mass of left- and right-moving string states. For the particular case when one takes the identity $(\mathbb{1}, 0) \in S$ as constructing element, the mass equations are identical to those we found for the heterotic string compactified on a torus. Thus, in principle, one arrives at the same set of physical states, in which one has phenomenologically uninteresting $\mathcal{N} = 4$ SUSY. As we previously pointed out, orbifolding prevents this catastrophe, but we have not yet established the explicit mechanism responsible for breaking the extra three unwanted supersymmetries. The fact that the point group has been already modded out of the target space is consistent with requiring the physical states to be invariant under the action of S. Each element $h \in S$ has an associate projection operator Δ_h which acts trivially on any physical state, the combined effect of these operators will in the end provide a spectrum with the correct amount of supersymmetry. Since they play a similar role as the GSO projectors one normally introduces in 10D string theories, these operators are called *Orbifold GSO projectors* [32].

In order to describe the actual structure of these projectors, we can look at the transformations for the independent pieces: The states are composed of. In the right moving sector, the presence of bosonic oscillators is forbidden in the massless spectrum, so that the states can only be of the form $|q_{sh}\rangle$. The corresponding transformation rule is given by [33]

$$|q_{sh}\rangle \xrightarrow{h} e^{-2i\pi v_g \cdot q_{sh}} |q_{sh}\rangle \ . \qquad (2.43)$$

In the untwisted sector, the only massless states we can find are the weights of the representations $\mathbf{8}_v$ and $\mathbf{8}_s$ of SO(8), as pointed out in section 2.4. The breaking of this transverse component proceeds according to eq. (2.24), inducing the decompositions

$$\mathbf{8}_v \to \mathbf{1}_1 \oplus \mathbf{6}_0 \otimes \mathbf{1}_{-1},$$
$$\mathbf{8}_s \to \mathbf{4}_{1/2} \oplus \overline{\mathbf{4}}_{-1/2}\,.$$

From the 4D perspective $\mathbf{1}_1$ and $\mathbf{4}_{1/2}$ are interpreted as a vector boson and four fermions with positive helicity, respectively. The weights $\mathbf{1}_{-1}$ and $\overline{\mathbf{4}}_{-1/2}$ correspond to the CPT conjugates of the former ones. From $\mathbf{6}_0$ one gets three complex scalars and their conjugates. We can now focus on the states of positive chirality to exemplify the decomposition induced by eq. (2.43). Using the convention introduced in eq. (2.28), one can immediately see that the weights

$$q^v = (1,0,0,0) \quad \text{and} \quad q^s = (\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2}), \tag{2.44}$$

are invariant under space group transformations. These states of positive chirality transform in the vector and spinor representations of the 4D Lorentz group. Since these states have the same phase transformation they combine to form the $\mathcal{N}=1$ SUSY vector multiplet. The three remaining weights of $\mathbf{4}_{1/2}$ and the scalars of $\mathbf{6}_0$ can be paired up in a similar way (see table 2.1), and lead to the three left chiral multiplets of the untwisted sector.

For the twisted sectors we can immediately see that the presence of δ_c in eq. (2.31)

$q_1^v = (0,1,0,0)$	$q_1^s = (\frac{1}{2},\frac{1}{2},-\frac{1}{2},-\frac{1}{2})$
$q_2^v = (0,0,1,0)$	$q_2^s = (\frac{1}{2},-\frac{1}{2},\frac{1}{2},-\frac{1}{2})$
$q_3^v = (0,0,0,1)$	$q_3^s = (\frac{1}{2},-\frac{1}{2},-\frac{1}{2},\frac{1}{2})$

Table 2.1: SO(8) weights for the three possible left chiral multiplets of the untwisted sector. Under the action of a space group element h, the states q_i^v and q_i^s transform with the same phase $e^{-2\pi i v^i}$.

only permits them to be populated by chiral matter (i.e. no 4D vector bosons). Considering for instance some massless fermion q_{sh}^s of positive chirality, it can be shown that there is a massless boson

$$q_{sh}^v = q_{sh}^s + (-\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2}), \tag{2.45}$$

in the same twisted sector which transforms under the space group in the same way as q_{sh}^s does.

Now we can consider the operators in the left moving sector which lead to massless states. In what follows we specify their transformation under the space group. In the untwisted sector one can make use of $\tilde{\alpha}^{\mu}_{-1}$, $\mu = 1, 2$ to generate the vector representation of the transverse 4D Lorentz group and $\tilde{\alpha}^I_{-1}$ to generate the Cartan generators of the gauge symmetry. These states are all invariant under S.

For the twisted sectors the only bosonic oscillators which can lead to massless states are of the form $\tilde{\alpha}^i_{-w^i}$ or $\tilde{\alpha}^{i*}_{-w^{i*}}$ ($w^{i*} = -w^i \mod 1$, $0 \leq w^{i*} \leq 1$). The corresponding phase transformation is given by

$$\tilde{\alpha}^i_{-w^i} \xrightarrow{h} e^{+2\pi \mathrm{i} v^i_h}\, \tilde{\alpha}^i_{-w^i},$$
$$\tilde{\alpha}^{i*}_{-w^{i*}} \xrightarrow{h} e^{-2\pi \mathrm{i} v^i_h}\, \tilde{\alpha}^{i*}_{-\bar{w}^{i*}}.$$

Note that for the computation of the twisted massless spectrum one can use a simpler expression for the number operator $\tilde{N}$

$$\tilde{N} = w^i \tilde{N}^i + w^{i*} \tilde{N}^{i*},$$

where $\tilde{N}^i$ and $\tilde{N}^{i*}$ are the occurrence numbers for $\tilde{\alpha}^i_{-w^i}$ and $\tilde{\alpha}^{i*}_{-w^{i*}}$ respectively.

The gauge coordinates are subject to the following transformation rule

$$X^I \xrightarrow{h} X^I + \pi V^I_h.$$

Given that the vertex operator for a twisted string contains a piece of the form $e^{2ip^I_{sh} X^I}$, any state $|p_{sh}\rangle_L$ acquires an extra phase under the action h

$$|p_{sh}\rangle_L \xrightarrow{h} e^{2\pi \mathrm{i} p_{sh} \cdot V_h}\, |p_{sh}\rangle_L\,.$$

Yet there is an extra piece we have to introduce when considering the physical states in an orbifold model. This has to do with the fact that twisted strings are associated to vacuum configurations which differ from the untwisted vacuum. This situation will be more exhaustively stressed in section [string selection rules] but for our current purposes we can simply notice that by tensoring left and right moving states to generate a twisted string, we are missing the information regarding its localization in compact space. To specify the string center of mass we can make use of its constructing element g. Naïvely one could think of relating one Hilbert space $\mathcal{H}_g$ to each space group element g, so that the physical states it contains are tensor products of the left and right moving parts together with a piece $|g\rangle$ related to the vacuum associated to such a space. However, the transformation $h : |g\rangle \mapsto |hgh^{-1}\rangle$ implies that $\mathcal{H}_g$ is not closed under the action of the space group, which is not suprising beacuse all elements within the same conjugacy class lead to the same fixed points (tori) on the orbifold. A more suitable Hilbert space we can construct as

$$\mathcal{H}_{[g]} = \bigoplus_{g' \in [g]} \mathcal{H}_{g'}. \tag{2.46}$$

Invoking all arguments introduced along this section, we can infer that any massless physical states has the form

$$|\mathrm{phys}\rangle \sim |q_{sh}\rangle \otimes \prod_{i=1}^{3} \left(\tilde{\alpha}^{i}_{-\omega^i}\right)^{\tilde{N}^i} \left(\tilde{\alpha}^{i*}_{-\omega^{i*}}\right)^{\tilde{N}^{i*}} |p_{sh}\rangle \otimes \left(\sum_{g'\in[g]} e^{2\pi \mathrm{i}\gamma(g')} |g'\rangle\right) , \qquad (2.47)$$

where the phases $\gamma(g')$ have been introduced in order to guarantee that under a space group transformation the Hilbert space element gets invariant up to a phase:

$$\gamma(hgh^{-1}) = \gamma_{[g]}(h) + \gamma(g) \bmod 1, \quad \forall h, q \in S, \qquad (2.48)$$

where $\gamma_{[g]}(h)$ is the so called *gamma phase* [34]. Some consistency checks permit to derive the following properties

$$\gamma_{[g]}(h) = 0 \bmod 1, \qquad \text{if } [h, g] = 0 \qquad (2.49)$$

$$\gamma_{[g]}(h_1 h_2) = \gamma_{[g]}(h_1) + \gamma_{[g]}(h_2) \bmod 1, \qquad (2.50)$$

$$N_h \gamma_{[g]}(h) = 0 \bmod 1, \qquad \text{if } h^{N_h} = (\mathbb{1}, 0) \qquad (2.51)$$

Finally, we are in a position to describe the effect of the orbifold GSO projectors on the massless spectrum

$$\Delta_h |\mathrm{phys}\rangle = \exp\left\{2\pi i \left[\left(p_{sh} - \frac{1}{2}V_g\right) \cdot V_h - \left(R - \frac{1}{2}v_g\right) \cdot v_h + \gamma_{[g]}(h)\right]\right\} |\mathrm{phys}\rangle , \qquad (2.52)$$

In which we have included an extra term $\exp\{-\pi i(V_g \cdot V_h - v_g \cdot v_h)\}$ required by consistency of the projector with a modular invariant partition function [38]. Due to its origin this term is referred to as the vacuum phase. The contribution of q_{sh} and the bosonic left moving oscillators have been conveniently written in terms of the so-called *R-charges*

$$R^i = q^i_{sh} - \tilde{N}^i + \tilde{N}^{i*}. \qquad (2.53)$$

As was shown in ref. [33], one can take advantage of eq. (2.51) and choose $\gamma_{[g]}(h)$ to make $|\mathrm{phys}\rangle$ invariant under Δ_h, however, this is not the case if one can find an element $g' \in [g]$ for which $[h, g'] = 0$, in that situation h is called a *commuting element of* $[g]$ and its gamma phase is trivial (see eq. (2.49)). From the above arguments it follows that the only relevant projections which apply on the states of $\mathcal{H}_{[g]}$ are those induced by the commuting elements[11] of $[g]$.

[11] The projections induced by the commuting elements are not all inequivalent, one can take a representative g' of $[g]$ and look for all the elements of S which commute with it, these elements give rise to all inequivalent projection conditions on $\mathcal{H}_{[g]}$.

2.6.3 Massless Spectrum

So far we have discussed the operators we are allowed to use in order to generate a massless state as well as the projection phases which determine whether such state is physical or not, in contrast to the 10 dimensional theory, we have a wide panorama of geometries and embeddings from which a diverse landscape of gauge groups and matter fields is obtained. Since these properties are very model dependent, it is very hard to describe how these spectra look in general. That is the reason for which, instead of attempting any actual computation, this section is intended more to describe some fields common to all models, and briefly sketch the manner in which the chiral matter and the gauge group are obtained.

The identity element $e = (\mathbb{1}, 0) \in S$ is the constructing element for the massless states of the untwisted sector. As we pointed out already, we obtain the untwisted spectrum by looking at the states which survive the orbifold projection, among the states found in the context of toroidal compactifications.

(i) $|q^v\rangle \otimes \tilde{\alpha}^{\mu}_{-1} |0\rangle_L \otimes |e\rangle$, $|q^s\rangle \otimes \tilde{\alpha}^{\mu}_{-1} |0\rangle_L \otimes |e\rangle$, $\mu = 2, 3$ together with their CPT conjugates, give rise to the supergravity multiplet and a chiral multiplet in four dimensions, q^v and q^s have been defined in eq. (2.44). The decomposition of the bosonic part leads to a real spin 2 particle which we identify as the 4D graviton and a complex scalar corresponding to the axion-dilaton field. The combinations of the $|q^s\rangle \otimes \tilde{\alpha}^{\mu}_{-1} |0\rangle_L \otimes |e\rangle$ provide the gravitino and the dilatino required by the condition of $\mathcal{N} = 1$ supersymmetry.

(ii) The internal components of the 10D SUGRA multiplet which are not projected out we denote as

$$N_{i\bar{j}} = |q^v_i\rangle \otimes \tilde{\alpha}^{j*}_{-1} |0\rangle_L \otimes |e\rangle \,, \quad N_{ij} = |i\rangle \otimes \tilde{\alpha}^{j}_{-1} |0\rangle_L \otimes |e\rangle \,,$$

with q^v_i as defined in table 2.1. These *moduli* fields are related to geometric variations of the internal manifold, $N_{i\bar{j}}$ are real states related to the Kähler properties, whereas N_{ij} are fields which account for the variations of the complex structure. Note that the moduli $N_{1\bar{1}}$, $N_{2\bar{2}}$ and $N_{3\bar{3}}$ are always part of the spectrum. The presence of additional moduli depend exclusively on the structure of the point group, however, their interpretation in terms of lattice deformations requires the compactification lattice to be specified [35].

(iii) The states of the form $|q^v\rangle \otimes |p\rangle_L \otimes |e\rangle$, which satisfy $p \in \Gamma_{16}$, $p^2 = 2$ and

$$p \cdot V_1 = 0 \mod 1, \quad p \cdot V_2 = 0 \mod 1, \quad p \cdot W_\alpha = 0 \mod 1 \quad \forall\, W_\alpha, \quad (2.54)$$

correspond to the gauge bosons of the four dimensional theory. Since the emdedding is rank preserving, the Cartan elements $|q^v\rangle \otimes \tilde{\alpha}^{I}_{-1} |0\rangle_L \otimes |e\rangle$ are all present in the spectrum.

(iv) In many cases can also find some leftover gauge fields $\left|q_i^b\right\rangle \otimes |p\rangle \otimes |e\rangle$ of the 10D theory, which satisfy

$$p\cdot V_1 - v_1^i = 0 \mod 1, \quad p\cdot V_2 - v_2^i = 0 \mod 1, \quad p\cdot W_\alpha = 0 \mod 1 \quad \forall\, W_\alpha. \tag{2.55}$$

Such states, together with their fermion companions, give rise to chiral multiplets in the untwisted sector.

In the twisted sectors, one first has to look at all inequivalent fixed points and their corresponding generators. Given the shift in the zero point energy, one has to look at the massless right movers q_{sh} which are permitted in each sector, then one can choose a maximal set of inequivalent constructing elements and for each of them, look for all massless solutions of eq. (2.32). Then one can tensor massless left and right movers to attain states similar as those depicted in eq. (2.47). Finally one looks for the effective GSO projectos associated to the constructing element, and if the state under consideration survives, then it is incorporated to the twisted spectrum. A more detailed description of this procedure is described in section 4.5 for the specific case of $\mathbb{Z}_2 \times \mathbb{Z}_4$ on the lattice of $\mathrm{SU}(2)^2 \times \mathrm{SO}(4)^2$.

Chapter 3

String Selection Rules

"Si el espacio es infinito estamos en cualquier punto del espacio.
Si el tiempo es infinito estamos en cualquier punto del tiempo"

Jorge Luis Borges, *El libro de arena*

In the previous section we discussed the set of constraints leading to the spectrum of physical states for a given heterotic orbifold model. At this stage one could also ask for the allowed interactions between such fields and look at the correlation functions leading to the holomorphic terms in the superpotential.

Since orbifold compactifications correspond to free CFT´s, the string couplings -among other quantities- can be exactly computed. However, instead of calculating the actual value of each possible coupling, we will follow the approach of the previous literature [40, 41, 42] to look at the symmetries of the correlation functions which set some couplings to vanish. The structure of this chapter is close to the recent review by Kobayashi *et. al.* [32] with few extensions to non factorizable $\mathbb{Z}_N$ orbifolds and some special emphasis on the arguments conjecturing the presence of R-symmetries [39] in the superpotential of the effective supergravity theory. The results can be straightforwardly extended to orbifolds of the type $\mathbb{Z}_N \times \mathbb{Z}_M$

In order to incorporate the non trivial boundary condition (2.29) we first shift to complex world-sheet coordinates $z = e^{2(\tau + \mathrm{i}\sigma)}$twisted fields $\hat{\sigma}_{[g]}(z, \bar{z})$ ($g = (\theta^k, \lambda)$) are defined such that in the neighborhood of a twist field located at the origin, the *monodromy* for the coordinate Z^i:

$$\partial Z^i(e^{2\pi i} z, e^{-2\pi i} \bar{z}) = e^{2\pi i w^i} \partial Z^i(z, \bar{z}),$$

(with w^i as defined in section 2.6) is implemented [41]. The twist field $\hat{\sigma}_{[g]}$ also serve to create a twisted ground state out of an untwisted vacuum $\left|\sigma_{[g]}\right\rangle = \hat{\sigma}_{[g]}(0,0)\left|0\right\rangle$.

Similarly as in section 2.6.2, we will think of such twist fields as appropriate linear combinations of twists creating ground states at different fixed for which their corresponding constructing elements belong all to $[g]$

$$\hat{\sigma}_{[g]} \sim \sum_{g' \in [g]} e^{2\pi i \gamma(g')} \hat{\sigma}_{g'}. \tag{3.1}$$

The most singular parts in the operator product expansions for the free fields are given by [32]

$$\begin{aligned}
\partial Z(z,\bar{z})\hat{\sigma}_{[g]}(w,\bar{w}) &\sim (z-w)^{-(1-\frac{k}{N})}\tau_{[g]}(w,\bar{w}) + \ldots \\
\partial Z^*(z,\bar{z})\hat{\sigma}_{[g]}(w,\bar{w}) &\sim (z-w)^{-\frac{k}{N}}\tau'_{[g]}(w,\bar{w}) + \ldots \\
\bar{\partial} Z(z,\bar{z})\hat{\sigma}_{[g]}(w,\bar{w}) &\sim (\bar{z}-\bar{w})^{-\frac{k}{N}}\tilde{\tau}'_{[g]}(w,\bar{w}) + \ldots \\
\bar{\partial} Z^*(z,\bar{z})\hat{\sigma}_{[g]}(w,\bar{w}) &\sim (\bar{z}-\bar{w})^{-(1-\frac{k}{N})}\tilde{\tau}_{[g]}(w,\bar{w}) + \ldots
\end{aligned} \tag{3.2}$$

The above equations define four excited twisted fields $\tau_{[g]}$, $\tau'_{[g]}$ and their tilded companions which are related to the former ones by conjugation on the world sheet [41].

Twisted fields also allow us to write vertex operators for twisted strings, for instance

$$V_{-1} = e^{-\phi} \left(\prod_{i=1}^{3} (\partial Z^i)^{\tilde{N}^i} (\partial Z^{i*})^{\tilde{N}^{i*}} \right) e^{i q^v_{sh} \cdot H} e^{i p^I_{sh} X^I} \hat{\sigma}_{[g]} \tag{3.3}$$

corresponds to the emission vertex of a bosonic field, where ϕ is the superconformal ghost, ∂Z^i and ∂Z^{i*} $(i = 1,2,3)$ represent left moving bosonic oscillators with the powers N^i and N^{i*} analogously as in section 2.6.2. H is a four dimensional vector whose entries correspond to bosonized free fields representing the right moving fermions and q^v_{sh} are the shifted bosonic right moving momenta. X^I $(I = 1, ..., 16)$ are the gauge coordinates and p_{sh} is the gauge momentum. The fermionic analog of the above vertex is

$$V_{-1/2} = e^{-\phi/2} \left(\prod_{i=1}^{3} (\partial Z^i)^{\tilde{N}^i} (\partial Z^{i*})^{\tilde{N}^{i*}} \right) e^{i q^s_{sh} \cdot H} e^{i p^I_{sh} X^I} \hat{\sigma}_{[g]}, \tag{3.4}$$

where q^s_{sh} is the shifted fermionic right moving momentum. States within the same chiral multiplet are related by eq. (2.45). The subindices in the vertices denote the conformal charge. It is also useful to define the summed H-momentum as the sum of the internal components of q_{sh}, note that in the case $\sum_i v_i = 0$ the summed H-momenta for V_{-1} and $V_{-1/2}$ are equal to 1 and $-1/2$ respectively.

It is important to recall that the vertices (3.3) and (3.4) are expressed in the zero 4D momentum limit which is enough for the matters we are interested in. In those expressions the cocycle factors [43] arising from consistency with fermionic anti-commutation properties as well as the correct normalization factors [32] have been omitted for simplicity.

The untwisted fermionic and bosonic vertex operators are identical to those presented before but with the shifted right movers replaced by the bosonic or fermionic weights of SO(8) in the absence of the twist fields. The physical states in the Hilbert space are equivalent to operators in the conformal field theory, by this reason tree level correlation functions can be used to investigate the couplings between massless fields. In particular, a vanishing correlation function for $\psi\psi\phi^{L-2}$ will imply the absence of a coupling of the form Φ^L in the superpotential.

The emission vertices have to be accomodated in the correlator in a way which makes it possible to cancel the background charge of two on the sphere. This makes it necessary to write some of them in a different ghost picture, such operation can be done by using the internal part of the right-moving world sheet supersymmetry current [28]

$$T_F = \left(\bar{\partial} x^\mu\right) \psi_\mu + \bar{\partial} Z^i \bar{\psi}^i + \bar{\partial} Z^{i*} \psi^i \tag{3.5}$$

where $\psi^j = \exp\{-iq_j^v \cdot H\}$ with q_j^v as in table 2.1. Note that the summed H-momentum for q_j^v is equal to 1. This picture changing operation allows to write some bosonic vertices with zero conformal charge

$$V_0 = e^{\phi} V_{-1} \sum_{j=1}^{3} (e^{\mathrm{i} q_j^v \cdot H} \bar{\partial} Z^j + e^{-\mathrm{i} q_j^v \cdot H} \bar{\partial} Z^{j*}). \tag{3.6}$$

at the price of introducing the right moving oscillators $\bar{\partial} Z^j$ and $\bar{\partial} Z^{j*}$. The correlation function can then be expressed as

$$\mathcal{F} = \left\langle V_{-1/2}(z_1, \bar{z}_1) V_{-1/2}(z_2, \bar{z}_2) V_{-1}(z_3, \bar{z}_3) V_0(z_4, \bar{z}_4) \ldots V_0(z_L, \bar{z}_L) \right\rangle \tag{3.7}$$

In the following we study each of the components of this quantity to determine the conditions that set it to vanish.

3.1 Space Group Selection Rule

Among the pieces which appear in the correlation function there is a product of twist fields each labeled with a given conjugacy class of the space group. Note that after integrating out all the compounds of (3.7) we will be left with the expectation value of such product in the presence of the background of the untwisted vacuum, which will vanish except for the case

$$(\mathbb{1}, 0) \in \prod_{\alpha=1}^{L} [g_\alpha] \tag{3.8}$$

which defines the *space group selection rule*. Given the structure of the space group ($g_\alpha = (\theta^{k_\alpha}, \lambda_\alpha)$), it is possible to split the last condition into its twist and lattice part. It implies that the product of twists must be equal to the identity i.e.

$$\sum_{\alpha=1}^{L} k_\alpha = 0 \mod N \tag{3.9}$$

leading to the so called *point group selection rule* which can be seen as a discrete $\mathbb{Z}_N$ symmetry from the perspective of the effective field theory. For $\mathbb{Z}_N \times \mathbb{Z}_M$ this selection rule provides two discrete symmetries one per each factor. Equation (3.8) also implies that there exists a set of lattice vectors $\tau_\alpha \in \Gamma_6$ and some suitable numbers $j_\alpha \in \mathbb{Z}$ which fulfill

$$\sum_{\alpha=1}^{L} \left(\prod_{\beta=\alpha+1}^{L} \theta^{k_\beta} \right) [\theta^{j_\alpha} \lambda_\alpha + (\mathbb{1} - \theta^{k_\alpha}) \tau_\alpha] = 0, \tag{3.10}$$

which can be rewritten in terms of the fixed points as:

$$\sum_{\alpha=1}^{L} \left(\prod_{\beta=\alpha+1}^{L} \theta^{k_\beta} \right) (\mathbb{1} - \theta^{k_\alpha}) [\theta^{j_\alpha} z_{f_\alpha} + \tau_\alpha] = 0. \tag{3.11}$$

This *fixed point selectivity* determines the configuration of fixed points in which fields can sit in order to give a non vanishing coupling.

3.2 Gauge group invariance

Now we want to look at the gauge part of the correlation function. The gauge coordinates X^I are sixteen independent and arbitrary functions, for this reason any non trivial dependence on such fields will set the correlator to vanish. Non zero couplings are then characterized by

$$\sum_{\alpha=1}^{L} p^I_{sh\ \alpha} = 0, \tag{3.12}$$

which reproduces the requirement of invariance under gauge transformations for each of the couplings present in the superpotential.

3.3 Lorentz Invariance in Compact Space

A similar argument as the one used to show the invariance under gauge transformations, can be applied in this context: the arbitrariness of the bosonized fermions leads to the conclusion that H-momentum must be conserved plane by plane [45]. Note first that for the bosonic fields in the zero picture (see eq. (3.6)) the term arising from the supersymmetry current T_F corresponds to a sum over the complex coordinates in compact spac. Expanding such terms in the correlation function (3.7), we can observe that for the coupling not to vanish, at least one of the following equations needs to be fulfilled

$$q_{sh\ 1}^{s\ i} + q_{sh\ 2}^{s\ i} + q_{sh\ 3}^{v\ i} + \sum_{\alpha=4}^{L}(q_{sh\ \alpha}^{v\ i} + (-1)^{n_\alpha} q_j^{v\ i}) = 0, \tag{3.13}$$

the sign $(-1)^{n_\alpha}$ depends on which term -either $\bar\partial Z^j \bar\psi^j$ or its conjugate- is present in the piece under consideration. The occurrence numbers of such right movers we note as N^j and N^{j*} respectively. The structure of T_F. From these arguments it follows

$$(L-3) + \sum_j N^j - \sum_j N^{j*} = 0.$$

Since there are only $L-3$ bosonic fields in the zero picture, the only non vanishing contributions to the correlation function arise from pieces which satisfy $N^j = 0$, $j = 1, 2, 3$. This result allows to write the *H-momentum conservation condition* as

$$\sum_{\alpha=1}^{L} q_{sh\ \alpha}^{v\ i} = 1 + N^{i*}. \tag{3.14}$$

For trilinear couplings there are no fields in the zero picture $N^{i*} = 0$. Note that the N^{i*}´s correspond to a specific property of the correlation function. They differ from the left moving oscillator numbers $\tilde N^i$ and $\tilde N^{i*}$ in the sense that it does not have a counterpart as quantum numbers for the physical states. All eqn. (3.14) tells us is that for a nonzero amplitude, the sum of the H-momenta in each plane should give positive integers. If we write $q_{sh\ \alpha}^{v\ i} = q_\alpha + k_\alpha v^i$, for some weight $q_\alpha \in \Gamma(\mathbf{8}_v)$ with positive chirality and k_α labeling the twisted sector, the left hand side of (3.14) becomes

$$\sum_{\alpha=1}^{L} q_{sh\ \alpha}^{v\ i} = \sum_{\alpha=1}^{L} q_\alpha^i + \left(\sum_{\alpha=1}^{L} k_\alpha\right) v^i.$$

Consistency with eq. (3.9) sets this quantity to be always integer[1] and hence all what sets H-momentum conservation far from being trivial is the requirement of

[1]One might also worry about the fact that the summed H-momentum has to be equal to L, but from our early assumption: $\sum_i v_i = 0$, it is straightforward to show that this condition always holds.

positiveness:

$$\sum_{\alpha=1}^{L} q_{sh\ \alpha}^{v\ i} \geq 1. \tag{3.15}$$

From table 3.1 we see that in all the models, the right movers with positive chirality are characterized by weights with non negative entries. Any coupling made out of left chiral superfields will have positive H-momentum. All what is left for us to check is that the sum of the H-momentum is non zero in all of its components, and this is not explicit for non prime orbifolds, because there one finds that weights in certain twisted sector happen to have vanishing components in certain complex planes. To exemplify this, we can look at $\mathbb{Z}_{6-\mathrm{II}}$, a coupling of the form $\theta^2\theta^2\theta^2$ is allowed by the point group, but the H momenta in the third plane sum up to 0, so that it is forbidden by H-momentum. In the $\mathbb{Z}_{8-\mathrm{I}}$ orbifold, couplings of the form $\theta^2\theta^2\theta^4$ are permitted, however in $\mathbb{Z}_{8-\mathrm{II}}$ such coupling is forbidden by H-momentum in the third complex plane.

The above result shows that the effectiveness of H-momentum conservation as a selection rule is very model dependent, but still imposes non-trivial conditions in certain situations. In what follows we will see that H-momentum serves as a powerful tool for writing further selection rules in terms of the quantum numbers of the physical states in the Hilbert space formulation.

In a certain manner one can think of H-momentum conservation as a consequence of the invariance of the couplings under some discrete remnant of the Lorentz group which survives compactification and orbifolding of the internal dimensions. We expect symmetries of this kind, to leave all physical quantities of the 4D effective theory invariant, so that allowed couplings should remain invariant under such discrete transformations. In what follows we attempt to develop a procedure to find the relevant discrete transformations of a given model and describe the constraints that they impose on non vanishing correlators.

The selection rules we have presented so far simplify the correlation function (3.7) to

$$\mathcal{F} = \left\langle \prod_{\alpha=1}^{L} \left(\prod_{i=1}^{3} (\partial Z^i)^{\tilde{N}_\alpha^i} (\partial Z^{i*})^{\tilde{N}_\alpha^{i*}} (\bar{\partial} Z^{i*})^{N_\alpha^{i*}} \right) \hat{\sigma}_{[g_\alpha]} \right\rangle, \tag{3.16}$$

First we can focus on the space group part. Among the symmetries of the internal space, we consider those which leave the compactification lattice invariant. By looking at the space group part of $\mathcal{F}$ we see that certain restrictions need to be imposed in order to guarantee that the structure of the conjugacy classes is

	1	2	3	4	5	6
$\mathbb{Z}_3$	$\frac{1}{3}(0,1,1,1)$					
$\mathbb{Z}_4$	$\frac{1}{4}(0,1,1,2)$	$\frac{1}{2}(0,1,1,0)$				
$\mathbb{Z}_{6-\mathrm{I}}$	$\frac{1}{6}(0,1,1,4)$	$\frac{1}{3}(0,1,1,1)$	$\frac{1}{2}(0,1,1,0)$			
$\mathbb{Z}_{6-\mathrm{II}}$	$\frac{1}{6}(0,1,2,3)$	$\frac{1}{3}(0,1,2,0)$	$\frac{1}{2}(0,1,0,1)$	$\frac{1}{3}(0,2,1,0)$		
$\mathbb{Z}_7$	$\frac{1}{7}(0,1,2,4)$	$\frac{1}{7}(0,2,4,1)$		$\frac{1}{7}(0,4,1,2)$		
$\mathbb{Z}_{8-\mathrm{I}}$	$\frac{1}{8}(0,2,1,5)$	$\frac{1}{4}(0,2,1,1)$		$\frac{1}{2}(0,0,1,1)$	$\frac{1}{8}(0,2,5,1)$	
$\mathbb{Z}_{8-\mathrm{II}}$	$\frac{1}{8}(0,1,3,4)$	$\frac{1}{4}(0,1,3,0)$	$\frac{1}{8}(0,3,1,4)$	$\frac{1}{2}(0,1,1,0)$		$\frac{1}{4}(0,3,1,0)$
$\mathbb{Z}_{12-\mathrm{I}}$	$\frac{1}{12}(0,4,1,7)$	$\frac{1}{6}(0,4,1,1)$	$\frac{1}{4}(0,0,1,3)$	$\frac{1}{3}(0,1,1,1)$		$\frac{1}{2}(0,0,1,1)$
			$-\frac{1}{4}(0,0,3,1)$		$-\frac{1}{12}(0,4,7,1)$	
$\mathbb{Z}_{12-\mathrm{II}}$	$\frac{1}{12}(0,1,5,6)$	$\frac{1}{6}(0,1,5,0)$	$\frac{1}{4}(0,1,1,2)$	$\frac{1}{3}(0,1,2,0)$	$\frac{1}{12}(0,5,1,6)$	$\frac{1}{2}(0,1,1,0)$
		$-\frac{1}{6}(0,5,1,0)$		$-\frac{1}{3}(0,2,1,0)$		

Table 3.1: Massless right movers for some $\mathbb{Z}_N$ orbifolds with positive chirality. The CPT conjugates of the negative weights in the second rows of $\mathbb{Z}_{12-\mathrm{I}}$ and $\mathbb{Z}_{12-\mathrm{II}}$ correspond to states with positive chirality in the inverse twisted sector.

preserved. But despite of that, we can look to a more reduced class

$$W = \{\varrho \in \mathrm{Aut}(\Gamma_6)|\ \varrho(h) = (\varrho\theta\varrho^{-1}, \varrho\lambda) \in [h],\ \forall h = (\theta, \lambda) \in S\},$$

where $h = (\theta, \lambda)$, $\theta \in P$ and $\lambda \in \Gamma_6$, such that ϱ commutes with the orbifold action ensuring that it maps fixed points to fixed points with the same network of identifications. For instance consider two fixed points z_1 and z_2 with the identification $\theta^k z_1 = z_2 + \lambda'$, this condition also holds for ϱz_1 and ϱz_2. This property also implies that ϱ maps fixed points to fixed points, from which it follows that all conjugacy classes are recovered after applying such transformation given that $\varrho \in \mathrm{O}(6)$ has full rank.

We can study now the simplest case in which the conjugacy classes of S are all invariant under ϱ i.e. $\varrho z_f = \theta^k z_f + \lambda$, $\forall z_f$ and for some $k \in \mathbb{Z}$ and some $\lambda \in \Gamma_6$ and consider the situation in which ϱ can be expressed in terms of the Cartan generators[2] of SO(6): $\varrho = \mathrm{diag}(\rho_1, \rho_2, \rho_3)$ with $\rho_k = e^{2\pi i \varsigma^k}$. The group generated for such

[2]These Cartan generators correspond to the same which were used to write the elements of the point group P of the orbifold.

symmetries we denote from now on as $\mathcal{G}_u$. The action of ϱ on the derivatives reads

$$\begin{aligned} \partial Z^i &\xrightarrow{\varrho} e^{2\pi i \varsigma^i} \partial Z^i, \qquad & \partial Z^{i*} &\xrightarrow{\varrho} e^{-2\pi i \varsigma^i} \partial Z^{i*}, \\ \bar{\partial} Z^i &\xrightarrow{\varrho} e^{2\pi i \varsigma^i} \bar{\partial} Z^i, \qquad & \bar{\partial} Z^{i*} &\xrightarrow{\varrho} e^{-2\pi i \varsigma^i} \bar{\partial} Z^{i*}. \end{aligned} \tag{3.17}$$

Since ϱ arises from the Lorentz group, it treats bosons and fermions in a different way, so that the supersymmetry generator is not an invariant quantity. This cannot be the case for the worldsheet supersymmetry current [44] and that is why the above set of equations forces the action on the bosonized fermions to be $H^i \xrightarrow{\varrho} H^i + 2\pi i \varsigma^i$. The correlation function must also be an invariant quantity. By looking at the transformation of (3.16) we see from the extra phase one obtains that it is only non-trivial for the case

$$\sum_i \varsigma^i \left[\sum_{\alpha=1}^{L} (-\tilde{N}_\alpha^i + \tilde{N}_\alpha^{i*} + N_\alpha^{i*}) \right] = 0 \mod 1, \tag{3.18}$$

which with the help of eq. (3.14) can be rewritten as:

$$\sum_{\alpha=1}^{L} \left(\sum_i \varsigma^i R_\alpha^i \right) = \sum_i \varsigma^i \mod 1, \tag{3.19}$$

where R_α^i is the R-charge of a physical state α in the i-th complex plane (see eq. (2.53)). Note that the orbifold action is a symmetry which happens to fulfill the conditions which lead us to eq. (3.19), thus we find the relation

$$\sum_{\alpha=1}^{L} \left(\sum_i v_h^i R_\alpha^i \right) = 0 \mod 1. \tag{3.20}$$

Note that this leads to a non-R discrete symmetry which in general is not equivalent to the one we found in the context of the point group selection rule.

In order to investigate the possible discrete symmetries that can be present in a certain model and their corresponding effects, we have constructed the automorphism group for the lattices listed in tables 3.2 and 3.3. First we present our findings for the lattices which factorize along the complex coordinates, which are the most broadly studied in the literature and then we specialize to the non-factorizable ones.

3.3.1 Factorizable Lattices

In factorizable lattices one can decompose the orbifold action $\theta = \theta_1\theta_2\theta_3$ into three twists θ_i acting independently on each of the complex planes (see section 2.5).

These independent twists have the convenient feature of being lattice automorphisms which commute with the orbifold action. All we need to incorporate these independent twists as R-symmetries of the theory is to guarantee that all fixed points are left invariant up to identifications[3]. Factorizable lattices also make it possible to write the fixed points as a direct sum of the fixed points in three complex planes: $z_f = (z_1, z_2, z_3)$, so that $(\theta_i)^k z_i = z_i + \lambda^i$.

A twist θ_i is said to be *prime* if its order is a prime number; from the number of prime twists present in a given orbifold we can distinguish among the following situations:

(i) Assume one of the twists θ_j is prime. This implies that any z_j of $\mathbb{T}^2_j$ which is a fixed point of $(\theta_j)^k$ is also fixed point of θ_j. This result makes $\theta_j z_f = z_f + \lambda$; since the fixed points are invariant, θ_j corresponds to an R-symmetry of the supergravity theory.

(ii) If there is only one prime twist θ_j, from the above argument it follows that

$$\theta(\theta_j)^{-1} z_f = \theta z_f + \lambda,$$

for some $\lambda \in \Gamma_6$, so that the fixed points are left invariant by the the simultaneous action of the non prime orbifold twists. No general conclusions can be drawn for the cases when the twists act independently.

(iii) If the orbifold action is composed only of one non-prime twist θ_q, then one can write $\theta_q z_f = \theta(\theta_q \theta^{-1} z_f)$. From our assumption $\theta_q \theta^{-1}$ is the product of two prime twists, so that $\theta_q \theta^{-1} z_f = z_f + \lambda'$ and hence

$$\theta_q z_f = \theta z_f + \theta \lambda',$$

which implies that $\theta_q z_f$ and z_f are identified under the orbifold action. In this case each independent twist composing the orbifold can be incorporated as an R-symmetry.

Note that in a model for which one of the above conditions is satisfied, the R-symmetries reproduce the invariance condition (3.20). A comprehensive classification of $\mathbb{Z}_N$ orbifolds can be found in ref. [47] from which we examined the factorizable examples listed in table 3.2. One can see that any of these models belongs to one of the categories listed before. Conditions (i)-(ii) are very helpful in order to get some general intuitions on the expected panorama of R-symmetries present in a given model, but we can also go to the basic geometric features of

[3] This is equivalent to ask the conjugacy classes of S to remain unaffected under the action of θ_i.

	Lattice	Shift	N. F. Points		
$\mathbb{Z}_3$	SU(3)⊗SU(3)⊗SU(3)	$\frac{1}{3}(1,1,-2)$	27	27	
$\mathbb{Z}_4$	SO(4)⊗SO(4)⊗SU(2)^2	$\frac{1}{4}(1,1,-2)$	16	10	
$\mathbb{Z}_{6-\mathrm{I}}$	$G_2 \otimes G_2$⊗SU(3)	$\frac{1}{6}(1,1,-2)$	3	15	6
$\mathbb{Z}_{6-\mathrm{II}}$	G_2⊗SU(3)⊗SO(4)	$\frac{1}{6}(1,2,-3)$	12	6	8

Table 3.2: Factorizable $\mathbb{Z}_N$ orbifolds under our consideration, the lattice, the shift and the number of inequivalent fixed points for each twisted sector are given.

the compactification lattice to construct the subgroup W of automorphisms which commute with the point group and then look at the set $\mathcal{G}_u$ of symmetries preserving the fixed points; our findings are described in the proceeding paragraph.

(i) $\mathbb{Z}_3$ has three prime twists composing the orbifold action, the constraints they impose on the L point couplings [48] are of the form:

$$\sum_{\alpha=1}^{L} R_\alpha^i = 1 \bmod 3; \quad i = 1, 2, 3,$$

these twists are enough to generate the group $\mathcal{G}_u$ of the lattice.

(ii) $\mathbb{Z}_4$ contains only one prime twist so that only $\theta_1\theta_2$ and θ_3 leave the fixed points invariant. From the side of the automorphism group one can find that the symmetry $(\theta_2)^2$ also fulfill this property, such that one has the following selection rules

$$\sum_{\alpha=1}^{L} (R_\alpha^1 + R_\alpha^2) = 2 \bmod 4, \quad \sum_{\alpha=1}^{L} R_\alpha^2 = 1 \bmod 2, \quad \sum_{\alpha=1}^{L} R_\alpha^3 = 1 \bmod 2\,. \tag{3.21}$$

(iii) $\mathbb{Z}_{6-\mathrm{I}}$ also contains only one twist which is prime, but there only the R-symmetries deduced in (ii) are present:

$$\sum_{\alpha=1}^{L} (R_\alpha^1 + R_\alpha^2) = 2 \bmod 6, \qquad \sum_{\alpha=1}^{L} R_\alpha^3 = 1 \bmod 6\,. \tag{3.22}$$

(iv) For the $\mathbb{Z}_{6-\mathrm{II}}$ orbifold the presence of two prime twists permit to recover the independent action of all the three twists leading to the symmetries found in ref. [39]

$$\sum_{\alpha=1}^{L} R_\alpha^1 = 1 \bmod 6, \qquad \sum_{\alpha=1}^{L} R_\alpha^2 = 1 \bmod 3, \qquad \sum_{\alpha=1}^{L} R_\alpha^3 = 1 \bmod 2\,. \tag{3.23}$$

This corresponds to the only set of independent constraints one can impose from the automorphism group of $G_2 \otimes \mathrm{SU}(3) \otimes \mathrm{SO}(4)$.

3.3.2 R-symmetries and discrete Wilson Lines

Previously we saw that the equivalence relations for the space group elements are preserved under transformations which commute with the orbifold action. This fact raises the relevance of the symmetries which map between inequivalent fixed points with the same physical states. It can very well be that such symmetries allow for consistent maps between the states of the model and in some special cases, they can even be able to restrict some couplings in the superpotential. We attempt to look at the set of inequivalent transformations which commute with the orbifold action but map between some conjugacy classes of S. In ensuring the matter content at two different points to be the same, the Wilson lines play a key role. A more explicit example of the effects of the symmetries we just mentioned can be found in section 4.6 for the case of $\mathbb{Z}_2 \times \mathbb{Z}_4$.

3.3.3 Non Factorizable Lattices

	Lattice	Shift	N. F. Points					
$\mathbb{Z}_4$	$\mathrm{SU}(4)\otimes\mathrm{SU}(4)$	$\frac{1}{4}(1,1,-2)$	16	4				
$\mathbb{Z}_{6-\mathrm{II}}$	$\mathrm{SU}(6)\otimes\mathrm{SU}(2)$	$\frac{1}{6}(1,2,-3)$	12	3	4			
$\mathbb{Z}_7$	$\mathrm{SU}(7)$	$\frac{1}{7}(1,2,-3)$	7	7	7	7		
$\mathbb{Z}_{8-\mathrm{I}}$	$\mathrm{SO}(5)\otimes\mathrm{SO}(9)$	$\frac{1}{8}(2,1,-3)$	4	10	4	6		
$\mathbb{Z}_{8-\mathrm{II}}$	$\mathrm{SO}(8)\otimes\mathrm{SO}(4)$	$\frac{1}{8}(1,3,-4)$	8	3	8	6		
$\mathbb{Z}_{12-\mathrm{I}}$	$\mathrm{SU}(3)\otimes F_4$	$\frac{1}{12}(4,1,-5)$	3	3	2	9	3	4
$\mathbb{Z}_{12-\mathrm{II}}$	$F_4\otimes\mathrm{SO}(4)$	$\frac{1}{12}(1,5,-6)$	4	1	8	3	4	4

Table 3.3: Some basic information concerning the non-factorizable $\mathbb{Z}_N$ orbifolds we investigated [46].

Contrary to the approach we followed in sect. 3.3.1, we will study some non-factorizable orbifolds (see table 3.3) in a more model dependent fashion[4].

[4]Given that in such cases the set of fixed points do not factorize along the complex planes, it is hard to tell wether or not the twists of the orbifold can be incorporated as global R-symmetries.

(i) In the case of $\mathbb{Z}_4$, the automorphism group for the lattice of SU(4)⊗SU(4) is generated by considering all inequivalent products of automorphisms each factor with the operator which exchanges between them. From this group only 128 elements commute with the orbifold action and only 16 preserve the structure of the conjugacy classes. Any element of the former group can be written as a product of the point group elements with powers of $(\theta_1)^2$. Invariance of the couplings under this element implies

$$\sum_{\alpha=1}^{L} R_\alpha^1 = 1 \bmod 2\,. \tag{3.24}$$

This is the only global R-symmetry one has in this model.

(ii) As found in ref. [37] the automorphism group of SU(6)⊗SU(2) does not give rise to any R-symmetry for the non-factorizable $\mathbb{Z}_{6-\mathrm{II}}$.

(iii) This also the case of $\mathbb{Z}_7$ where only 14 elements were found to commute with the point group, among those, the only transformations leaving the fixed points untouched are found to belong to the point group.

(iv) For the $\mathbb{Z}_{8-\mathrm{I}}$ orbifold one finds that there is only one constraint:

$$\sum_{\alpha=1}^{L} R_\alpha^1 = 1 \bmod 2\,. \tag{3.25}$$

(v) While in $\mathbb{Z}_{8-\mathrm{II}}$ the twist θ_3 leaves the fixed points invariant, so that the following constraint is present:

$$\sum_{\alpha=1}^{L} R_\alpha^3 = 1 \bmod 2\,. \tag{3.26}$$

This is the only symmetry we found in addition to the orbifold action whose corresponding constraint is given by eq. (3.20).

(vi) A similar situation is observed for $\mathbb{Z}_{12-\mathrm{I}}$, where all R-symmetries are equivalent to impose (3.20) and

$$\sum_{\alpha=1}^{L} R_\alpha^1 = 1 \bmod 3\,. \tag{3.27}$$

(vii) In $\mathbb{Z}_{12-\mathrm{II}}$ the only constraint one has is given by

$$\sum_{\alpha=1}^{L} R_\alpha^3 = 1 \bmod 2\,. \tag{3.28}$$

This is due to the $\mathbb{Z}_2$ symmetry of the sublattice in the third complex plane.

Chapter 4

MSSM Searches in the $\mathbb{Z}_2 \times \mathbb{Z}_4$ Orbifold Model

"You've got to jump off the cliff all the time and build your wings on the way down."

Kurt Vonnegut, *Hocus Pocus*

In this chapter we study a particular orbifold example. Provided the machinery we developed in chapter 3, we attempt to construct the physical models which characterize the geometry of $\mathbb{Z}_2 \times \mathbb{Z}_4$ in the lattice of $\mathrm{SU}(2)^2 \times \mathrm{SO}(4)^2$. First we describe some geometrical properties, then we search for all translational embeddings which are permitted by modular invariance and the breaking patterns they induce. Among those gauge structures, we will focus on those which, upon the introduction of Wilson Lines, incorporate the gauge group of the standard model as a factor. We choose the lattice of $\mathrm{E}_8 \times \mathrm{E}_8$ to compactify the gauge coordinates, due to its nice phenomenological features [33, 37]

A suitable breaking will be our first filter in the search for model with realistic properties. Then we specialize to those containing SO(10) as a unifying gauge group, a model in which three effective families can be achieved is our final goal.

4.1 Generalities

test
The $\mathbb{Z}_2 \times \mathbb{Z}_4$ orbifold of our interest is characterized by the following properties: we consider a factorizable lattice Γ_6 as the tensor product of the root lattices of $\mathrm{SU}(2)^2$ and $\mathrm{SO}(4)^2$. Note that the first lattice is basically arbitrary and is invariant under rotations by 180 degrees whereas the two SO(4) factors are invariant under

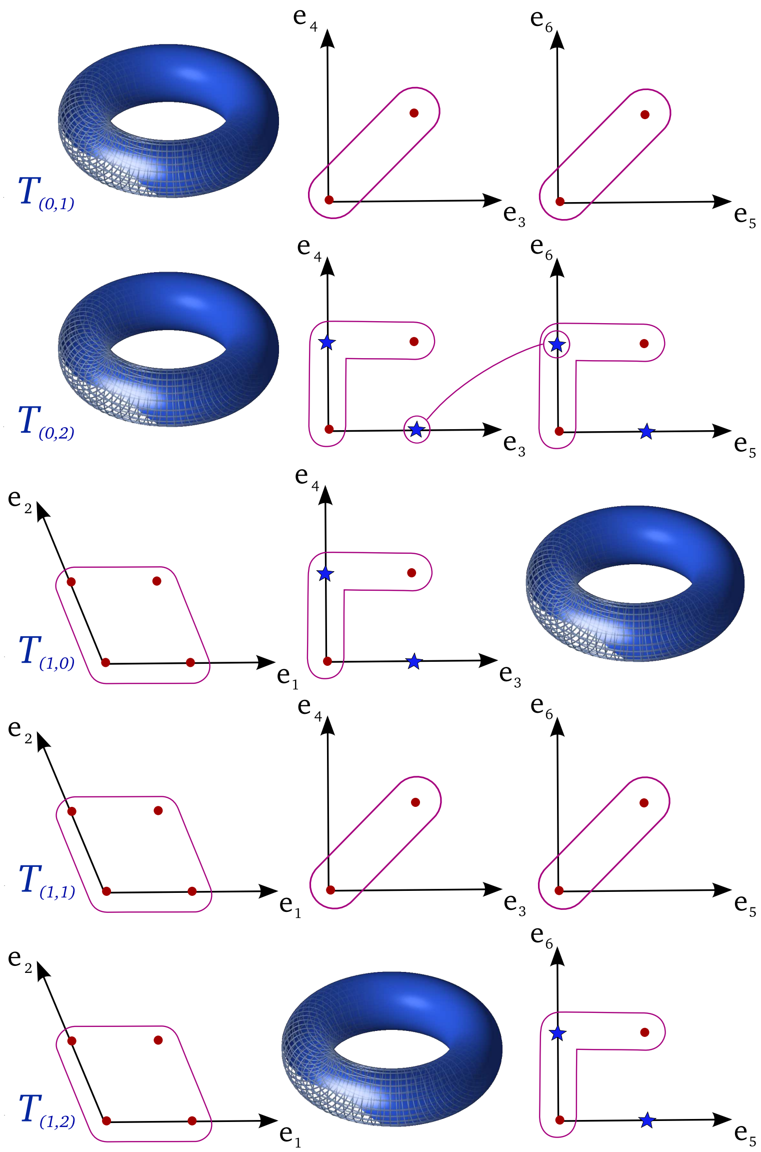

Figure 4.1: Inequivalent fixed points for each twisted of the $\mathbb{Z}_2 \times \mathbb{Z}_4$ orbifold, comments on the fixed point structure.

rotations by 90 degrees, so that one is permitted to construct a point group out of the following twist vectors

$$v_2 = (0, \frac{1}{2}, -\frac{1}{2}, 0), \quad v_4 = (0, 0, \frac{1}{4}, -\frac{1}{4}),$$

which clearly satisfy the $\mathcal{N} = 1$ SUSY condition. The fixed points in each twisted sector $T_{(k_1,k_2)}$ are constructed in the following manner: For each complex plane we take the set of all lattice elements $\lambda^{(i)} \in \Gamma_6$ so that the vector $f^{(i)} = \lambda^{(i)}/q$ is inside the fundamental domain, with q being the order of the element $\theta^{k_1}\omega^{k_2}$. We combine such vectors to a six dimensional one $z = f^{(1)} \otimes f^{(2)} \otimes f^{(3)}$ and check whether z is a fixed point or not.

This algorithm provides the fixed points within the fundamental domain. In order to look for the equivalence relations between them, one can consider all possible pairs of fixed points z_{f1}, z_{f2} within the same twisted sector, and for each of those pairs check wether there is any element $\vartheta \in P$, for which $z_{f1} - \vartheta z_{f2} \in \Gamma_6$. The inequivalent fixed points are depicted in fig 4.4.

4.2 Gauge Group Diversity

A naïve look at eq. (2.36) might suggest that there are infinitely many possibilities to realize the gauge embedding. Fortunately, among all those possibilities there is only a finite set which actually leads to models which are *physically inequivalent*. Two models are called physically equivalent if they have the same physical spectrum, i.e. they share the same gauge group and the same matter content[1].

In order to construct the machinery required to derive all physical possibilities permitted in a certain model, we can first focus on the case in which all Wilson lines are switched off, so that we only need to specify the embedding for the generators of the point group. We first discuss the general case of a $\mathbb{Z}_N \times \mathbb{Z}_N$ orbifold, and then Draw some conclusions for the model of our interest. Consider a certain embedding which we denote by $[V_N; V_M]$, meaning that V_N and V_M reproduce the effects of the point group generators on the gauge space. The role this embedding plays on the spectrum can be seen from the mass equation (2.32). Assume that for the $T_{(k_1,k_2)}$ twisted sector one can find a weight $p \in \Gamma_{16}$, such that up to a certain oscillator combination; $p_{sh} = p + k_1 V_N + k_2 V_M$ permits constructing a massless left moving state. As an immediate consequence one can find that for the embedding $[V_N + \lambda_N; V_M + \lambda_M]$ with $\lambda_N, \lambda_M \in \Gamma_{16}$, the weight p_{sh} is also a solution for the massless equation, with the same oscillator configuration as before. On the other hand, the embedding $[\sigma V_N; \sigma V_M]$ for $\sigma \in \mathrm{O}(16)$ is only consistent if σ is an automorphism of Γ_{16}. In such a case $\sigma(p + k_1 V_N + k_2 V_M)$ also gives a left moving massless state. From these simple arguments it follows that the embeddings which differ by lattice vectors as well as those which are related via automorphisms acting

[1]This definition of equivalence requires of course that each physical state in one model, has a counterpart in the other with the same quantum numbers.

simultaneously on both V_N and V_M contain the same massless left moving states[2].

The fact that two models have the same left moving massless states is, however, not sufficient to guarantee that these models will have the same physical states. For that one also has to ensure that the orbifold projectors have the same effects in both models. Assume now that, for some fixed point z_f of $T_{(k_1,k_2)}$, the space group element $h = (\theta^{n_1}\omega^{n_2}, \lambda)$ commutes with its generating element. The gauge part of the phase projection induced by h we denote by

$$\delta^{(k_1,k_1)}(n_2, n_2) = \exp\left\{2\pi i \left[p_{sh} - \tfrac{1}{2}(k_1 V_N + k_2 V_M)\right] \cdot (n_1 V_N + n_2 V_M)\right\} , \quad (4.1)$$

for some weight p_{sh} which, combined with a certain oscillator configuration, leads to some massless left moving state, consider $[\sigma(V_N + \lambda_M); \sigma(V_M + \Lambda_M)]$ to be an embedding, for some $\sigma \in \mathrm{Aut}(\Gamma)$ and for some $\lambda_N, \lambda_M \in \Gamma_{16}$. Previously we noticed that under this choice, σp_{sh} also leads to a left moving state with the same properties as the one in the previous case. It can be shown that its corresponding phase transformation under h is given by

$$\delta'^{(k_1,k_1)}(n_2, n_2) = \exp\{2\pi \mathrm{i}(k_1 n_2 - k_2 n_1)\Phi\}\delta^{(k_1,k_1)}(n_2, n_2), \quad (4.2)$$

with

$$\Phi = \frac{1}{2}\left(V_M \cdot \lambda_N - V_M \cdot \lambda_N + \lambda_N \cdot \lambda_M\right) . \quad (4.3)$$

Note that the above equation is actually independent of the automorphism σ. This is a very powerful result since we can now mod the automorphism group out of the set of all possible embeddings. Nevertheless, for the case in which one shifts the embedding by lattice vectors, one only arrives at an equivalent model when the *brother phase* Φ [31] is an integer number. Note also that from eqs. (2.54) and (2.55) and from the fact that Γ is integral, gauge embeddings which differ by lattice vectors have the same untwisted matter content. This is the reason why we left the effects of lattice shifts to be studied at the point where we discuss the construction of the twisted spectra.

Now we specify to the particular case of $\mathbb{Z}_2 \times \mathbb{Z}_4$. Given that we decided to compactify the gauge coordinates on the lattice of $\mathrm{E}_8 \times \mathrm{E}_8$, we can decompose the embedding vectors as

$$V_4 = A_4 \oplus B_4, \quad V_2 = A_2 \oplus B_2, \quad (4.4)$$

[2]When comparing the weights p_{sh} and σp_{sh} which lead to massless left movers under the embeddings $[V_N; V_M]$ and $[\sigma V_N; \sigma V_M]$ respectively, one can observe that σ also maps between the simple roots of the gauge groups induced by these embeddings, meaning that p_{sh} and σp_{sh} have the same Dynkin label.

where $4A_4$, $4B_4$, $2A_2$ and $2B_2$ are vectors from the lattice of one single E_8. The decomposition takes advantage of the fact that the isometries of Γ_{16} which are of physical interest, can all be written as the direct product of inner automorphisms of each E_8 root lattice[3]. This implies the equivalence of the embeddings $[A_4 \oplus B_4; A_2 \oplus B_2]$ and $[\sigma_1 A_4 \oplus \sigma_2 B_4; \sigma_1 A_2 \oplus \sigma_2 B_2]$, where $\sigma_1, \sigma_2 \in \mathrm{Aut}(\mathfrak{e}_8)$.

The above arguments permit us to consider just one E_8 factor. At this stage, we are not interested in the complete spectrum, but only in models which share the same untwisted fields. Because of that, we can focus on the vectors from the following minimal set

$$A_4 = \frac{1}{4} \sum_{k=1}^{8} a_k \alpha_k, \quad a_k = 0, 1, 2, 3, \tag{4.5}$$

$$A_2 = \frac{1}{2} \sum_{k=1}^{8} b_k \alpha_k, \quad b_k = 0, 1, \tag{4.6}$$

to construct all combinations (A_4, A_2) and apply all E_8 automorphisms to find all those which are inequivalent. After that we can construct all possible embeddings as pairs of such combinations which fulfill the modular invariance conditions (2.38), (2.39) and (2.40).

Though this procedure seems relatively straightforward, one faces some technical difficulties, as the Weyl group of E_8 is composed of 696.729.600 elements. Such a big number makes its construction unfeasible with current computer resources. We can still follow a more reliable approach by considering the inequivalent E_8 vectors found for the $\mathbb{Z}_4$ orbifold [50]. These correspond to the A_4 vectors (see appendix **??**), so that one only needs to look for the equivalence relations of the $\mathbb{Z}_2$ vectors A_2.

By completely modding all isometries and lattice shifts out of the $\mathbb{Z}_4$ vectors, we have to modify the the equivalence relations for their $\mathbb{Z}_2$ companions. For a given A_4, two combinations (A_4, A_2) and (A_4, A_2') are equivalent only if one can find an element σ, from the subgroup

$$\mathrm{Aut}(\mathfrak{e}_8|A_4) = \{\sigma \in \mathrm{Aut}(\mathfrak{e}_8) \mid \sigma A_4 = A_4\} \ , \tag{4.7}$$

such that $\sigma A_2 = A_2'$.

Still, generating the subgroups $\mathrm{Aut}(\mathfrak{e}_8|A_4)$ is far from feasible. To avoid such

[3]There is an outer automorphism which maps the E_8 factors to each other. However, it is not of physical interest since it will just lead to the same group structure in a different order.

exhaustive construction we can appeal to the fact that any element in a Weyl Coxeter group can be written as a product of Weyl reflections. For each inequivalent A_4 vector we will look at the set of Weyl reflections which leave it invariant

$$\mathcal{W}(A_4) = \left\{ \sigma_\alpha = \mathbb{1} - \frac{\alpha\alpha^T}{\langle \alpha, \alpha \rangle} \mid \alpha \in \mathfrak{e}_8,\ \sigma_\alpha A_4 = A_4 \right\} . \tag{4.8}$$

In order to maximize the number of equivalence relations one can obtain by using this set of transformations, we exploit the spherical symmetry which characterizes the isometries and consider all the vectors $A_2 = \lambda/2$ where λ is a vector in the lattice of E_8 which fulfills $\lambda^2 \leq 8$. This set is clearly non-minimal in comparison to that introduced in eq. (4.6). For this reason one has to take care of possible lattice translations which may arise. The equivalence is checked in the following manner: for each vector A_2 we consider all the transformations induced by $\mathcal{W}(A_4)$. For each of them we check whether there is another vector A_2' which satisfies

$$(\sigma_\alpha A_2 - A_2') \in \Gamma(\mathfrak{e}_8) \, . \tag{4.9}$$

If this is the case, then A_2 and A_2' are equivalent. The algorithm implemented[4] for this purposes permits checking for the equivalence relations induced by all Weyl Coxeter elements which can be written as products of the Weyl reflections of $\mathcal{W}(A_4)$. Under some circumstances, some automorphisms which can not be written as a product of elements in $\mathcal{W}(A_4)$ act trivially on A_4, so that the equivalence relations induced by them are not considered by the program. Still, the output is manageable enough so than one just needs to take special care of those combinations (A_4, A_2) and (A_4, A_2') which lead to the same gauge structure. At the point in which we generate the complete spectrum we can decide whether they are equivalent or not.

The untwisted sector will be studied in more detail in the proceeding section. Clearly, at the stage when one has all inequivalent combinations (A_4, A_2) it is possible to determine the group decomposition of the corresponding E_8 factor as well as its contribution to the untwisted matter. Here we focus only on the gauge group we obtain when considering a certain combination, The roots α of E_8 which satisfy

$$\alpha \cdot A_4 = 0 \mod 1, \quad \alpha \cdot A_2 = 0 \mod 1 \tag{4.10}$$

correspond to the roots of the decomposition. To find the non abelian factors we split the roots in sets which are orthogonal to each other and from each set we look for the positive roots. Then we compute the Cartan matrix to identify the semisimple Lie algebra associated to it. The amount of U(1) factors is determined

[4]The detailed construction of this algorithm is given in ref. [49]

from the mismatch between the rank of E_8 and the summed rank of the non abelian structures.

The final results are presented in appendix A, The combinations which are equivalent up to automorphisms which cannot be reached by our program are not presented. The gauge groups associated to each combination are also shown. All possible breaking structures expected from point groups of order eight are obtained [36]. From them we can actually see the diversity of this model and how rich it is in terms of GUT structures. Among the relevant decompositions one can find $E_6 \times U(1)^2$, $E_6 \times SU(2) \times U(1)$, $SO(10) \times SU(4)$, $SO(10) \times SU(2) \times SU(2) \times U(1)$, $SO(10) \times U(1)^3$ and $SU(5) \times SU(3) \times U(1)^2$ which seem to be fertile grounds for phenomenology[5] when the accompanying non Abelian factors are broken by suitable configurations of Wilson lines.

The combinations we previously found have to be paired in order to define the actual embedding. Not all pairings are allowed by modular invariance, notice first that in some cases these pairs can only be consistent upon addition of some lattice vectors. Consider two vectors $\lambda_4, \lambda_2 \in \Gamma_{16}$ and the embedding $[V_4 + \lambda_4; V_2 + \lambda_2]$, induced by pairing (A_4, A_2) and (A'_4, A'_2), such that $V_4 = A_4 \otimes A'_4$ and $V_2 = A_2 \otimes A'_2$ and $V_4 = A_4 \otimes A'_4$. This embedding is only valid if the following conditions hold

$$2\left(V_2^2 - \frac{1}{2}\right) = 0 \mod 2, \tag{4.11}$$

$$4\left(V_4^2 - \frac{1}{8}\right) = 0 \mod 2, \tag{4.12}$$

$$2\left(V_2 \cdot V_4 + V_4 \cdot \lambda_2 + V_2 \cdot \lambda_4 + \frac{1}{8}\right) = 0 \mod 2. \tag{4.13}$$

The above equations are the version of (2.38), (2.39) and (2.40) for the case of $\mathbb{Z}_2 \times \mathbb{Z}_4$. Note that the first two equations, in contrast to the third, do not receive any contribution from the lattice shifts. These conditions are called strong since they can be used to decide which combinations (A_4, A_2) and (A'_4, A'_2) can be paired together in a consistent way. In many cases one can satisfy the third modularity condition by introducing some lattice vectors. Nevertheless, the properties of Γ_{16} only make this possible if

$$4\left(V_2 \cdot V_4 + \frac{1}{8}\right) = 0 \mod 1. \tag{4.14}$$

[5]Decompositions such as $SU(4)^2 \times SU(2) \times U(1)$ and $SU(4) \times SU(2)^2 \times U(1)$ are also found, which are suitable to accommodate Pati-Salam GUT schemes.

This weak condition permits us to avoid the search for lattice vectors, especially at this stage in which we are only interested in the gauge group decomposition introduced by each embedding. Clearly there is not only one single choice of lattice vectors for which the consistency conditions for modular invariance are satisfied. In order to determine all models we can obtain just by adding different lattice vectors, we have to look at all possible brother phases which modify the projectors in a non-trivial manner. One can deduce from eqs. (4.3) and (4.13), that for each valid embedding it is possible to achieve two inequivalent models by introducing lattice vectors.

The 144 valid embedding we found to satisfy eqs. (4.11), (4.12) and (4.14), as well as the corresponding brother phases required to fulfill (4.13), can be found in appendix B.

4.3 Massless States

Now we are equipped with all possible rank preserving embeddings for the $\mathbb{Z}_2 \times \mathbb{Z}_4$ orbifold. The next step involves the computation of the spectrum of physical states allowed by each such embedding while considering all Wilson lines to be swtiched off. In this section we discuss the algorithms implemented for such calculations with special emphasis on the construction of the twisted states.

The 4D SUGRA multiplet arises in the untwisted sector as described in section 2.6.3. There we also pointed out that, depending on the point group, one will finds some moduli fields accounting for variations of the torus geometry. For our model we find the three standard Kähler moduli

$$N_{i\bar{i}} = |q_i^v\rangle \otimes \tilde{\alpha}_{-1}^{i*} |0\rangle_L \otimes |e\rangle\,, \quad i = 1, 2, 3\,, \tag{4.15}$$

which are related to variations of the volume of each two dimensional torus $\mathbb{T}^2$. In addition one has the modulus

$$N_{11} = |q_1^v\rangle \otimes \tilde{\alpha}_{-1}^{1} |0\rangle_L \otimes |e\rangle\,, \tag{4.16}$$

related to the freedom one has to choose the complex structure since it is always consistent with the $\mathbb{Z}_2$ symmetry we modded out of the torus in the first complex plane.

The untwisted matter we compute in a similar fashion as for the case of the gauge fields, by combining the roots p of $E_8 \times E_8$ with each of the three left chiral multiplets $q = q_i^v, q_i^s$ $i = 1, 2, 3$ available from the right mover´s side. Those combinations which survive the projections

$$p \cdot V_2 + q \cdot v_2 = 0 \mod 1, \quad p \cdot V_4 + q \cdot v_4 = 0 \mod 1, \tag{4.17}$$

are the physical states of the untwisted sector which are charged under the surviving gauge group of the model. We use the simple roots of the gauge group to compute the Dynkin label of each state and then we look for those Dnykin labels with non-negative entries. From them we can deduce the representations under which the matter fields transform [2].

For obtaining the matter content in the untwisted sector we start by constructing the massless right movers. Note first that the mass equation (2.31) contains a modification δ_c in the normal ordering constant in comparison to the untwisted sector. This modification was computed using eq. (2.30). One can now consider the corresponding mass equations for each twisted sector and compute the solutions with positive chirality[6]. This is done by bounding a region in the vector and spinorial lattices of E_8 where all possible solutions are included. Consider for instance the $T_{(k_1,k_2)}$ twisted sector and assume that that there is some lattice vector q for which

$$q_{sh}^2 = (q + k_1 v_2 + k_2 v_4)^2 = 1 - 2\delta_c \,. \tag{4.18}$$

This means that each component q^i has to satisfy

$$-k_1 v_2 - k_2 v_4 - \sqrt{1 - 2\delta_c} \leq q^i \leq -k_1 v_2 - k_2 v_4 + \sqrt{1 - 2\delta_c} \,. \tag{4.19}$$

Since q is a lattice vector, q^i can only attain half integer values. This property together with the above equation serves to reduce our search to a finite number of possibilities. The massless right movers of our model are shown in table 4.1. Note

(k_1, k_2)	(0,1)	(0,2)	(0,3)	(1,0)	(1,1)	(1,2)	(1,3)
δ_c	$\frac{3}{16}$	$\frac{1}{4}$	$\frac{3}{16}$	$\frac{1}{4}$	$\frac{5}{16}$	$\frac{1}{4}$	$\frac{5}{16}$
q_{sh}^v	$\frac{1}{4}(0,0,1,3)$	$\frac{1}{2}(0,0,1,1)$	$\frac{1}{4}(0,0,3,1)$	$\frac{1}{2}(0,1,1,0)$		$\frac{1}{2}(0,1,0,1)$	$\frac{1}{4}(0,2,1,1)$

Table 4.1: Modifications to the normal ordering constant and massless right movers of positive chirality for each twisted sector of $\mathbb{Z}_2 \times \mathbb{Z}_4$.

from the table that $T_{(1,1)}$ is empty of right moving states. This means that this twisted sector does not contain physical states with positive chirality[7].

[6]Negative chirality states serve to construct CPT conjugates from states with positive chirality in the inverse twisted sector. This makes their computation redundant for our purposes.

[7]Note however, that the CPT conjugates of $T_{(1,3)}$ are present in $T_{(1,1)}$.

In the left moving sector one has more alternatives to construct massless states. The condition:

$$\frac{p_{sh}^2}{2} + \tilde{N} - 1 + \delta_c = 0, \tag{4.20}$$

involves combinations of oscillators as well as vectors from the gauge lattice. Note that we are equipped with at most[8] six oscillators $\tilde{\alpha}^i_{-w^i}$, $\tilde{\alpha}^{i*}_{-w^{i*}}$ $i = 1, 2, 3$, which we can combine according to

$$\mathcal{O}(N^i, N^{i*}) = \prod_{i=1}^{3} \left(\tilde{\alpha}^i_{-\omega^i}\right)^{N^i} \left(\tilde{\alpha}^{i*}_{-\omega^{i*}}\right)^{N^{i*}}, \tag{4.21}$$

where N^i and N^{i*} are properly chosen so that the corresponding $\tilde{N}$ does not exceed the value of $1 - \delta_c$. From these combinations we can achieve certain oscillator numbers $\tilde{N}_1$,..., $\tilde{N}_p$, which we can split in sets with the same $\tilde{N}$

$$\mathcal{C}_k = \left\{ \mathcal{O}(N^i, N^{i*}) \mid w^i N^i + w^{i*} N^{i*} = \tilde{N}_k \leq 1 - \delta_c \right\}, \quad k = 1, ..., p\,. \tag{4.22}$$

Now we consider the contribution of each $\tilde{N}_k$ to the mass equation in order to find all possible gauge momenta leading to massless states. Note that in the absence of background fields, all fixed points within the same twisted sector are *degenerate*, in the sense that all of them possess the same massless left movers

$$p_{sh} = p + k_1 V_1 + k_2 V_2.$$

Note also that we have avoided to add lattice vectors to our gauge embeddings in order to make them completely modular invariant. As discussed in the previous section, adding lattice vectors does not affect the spectrum of left moving states since such lattice vectors can be reabsorbed in p. This means that the vectors shown in appendix B can be used to construct the massless left movers. Similar to the case of the right movers, we can restrict to a finite set of possibilities by considering all vectors $p \in \Gamma_{16}$ whose entries satisfy

$$-k_1 V_2^\alpha - k_2 V_4^\alpha - \sqrt{m_k} \leq p^\alpha \leq -k_1 V_2^\alpha - k_2 V_4^\alpha + \sqrt{m_k}, \quad \alpha = 1, ..., 16\,, \tag{4.23}$$

where $m_k = 2(1 - \tilde{N}_k - \delta_c)$. For each p_{sh} obtained by considering a given oscillator number $\tilde{N}_k$, we can construct massless left movers acting with any oscillator product $\mathcal{O}(N^i, N^{i*})$ from $\mathcal{C}_k$ on $|p_s h\rangle_L$. These left movers we can tensor with the massless right mover of the twisted sector and the constructing element g of a certain fixed point. In order for this state to be physical we have to guarantee that

[8] In the case of a twisted sector with a torus along the i-th complex plane, the oscillators $\tilde{\alpha}^i_{-1}$, $\tilde{\alpha}^{i*}_{-1}$ cannot be used to construct massless states.

the orbifold GSO projectors act trivially on it. These projectors are induced by the space group elements which commute with g. Since many of our embeddings are not completely modular invariant we need to modify the phase transformation (2.52) in order to account for the brother phase. A commuting element $h = (\theta^{n_1}\omega^{n_2}, \lambda)$ projects out all states[9] which do not satisfy the condition

$$\left(p_{sh} - \frac{1}{2}V_g\right) \cdot V_h - \left(R - \frac{1}{2}v_g\right) \cdot v_h + (k_1 n_2 - k_2 n_1)\Phi = 0 \mod 1. \qquad (4.24)$$

The commuting elements for each twisted sector can be found in appendix ... From there we note that in all twisted sectors each generating element commutes with at least one member of the space group whose point group part is given by the $\mathbb{Z}_2$ generator. This is not the case for the $\mathbb{Z}_4$ generator, since in the twisted sectors $T_{(0,2)}$, $T_{(1,0)}$ and $T_{(1,0)}$ there are some special fixed points whose generators commute only with space group elements of the form (ϑ, λ) where ϑ is of order two.

The previous projection conditions are the only inequivalent ones one can find for the case in which all Wilson lines are switched off. The fact that within the same twisted sector we have different projection conditions on states sitting at different fixed points, implies that the point group breaks the degeneracy of the fixed points as it is depicted in table 4.2. These degeneracies are very useful to determine the amount of chiral fields one obtains in a certain model.

So far we have not made clear how we are going to proceed in our quest for

(k_1, k_2)	(0,1)	(0,2)	(0,3)	(1,0)	(1,1)	(1,2)	(1,3)
N	4	4	4	8	16	8	16
N_s		6		4		4	

Table 4.2: Splitting in the fixed point degeneracies by the orbifold GSO projectors in the absence of Wilson Lines, here N labels the number of fixed points where one can apply projections by using both $\mathbb{Z}_2$ and $\mathbb{Z}_4$ generators. N_s is the number of fixed points where only projections induced by point group elements of order two can be applied.

models with a realistic phenomenology, this will become more transparent in the following section. Just for now, let us remark that we will be interested in a framework in which the families of the standard model arise from representations of a higher grand unifying group. That is the reason why we only explore the models which already contain a suitable GUT structure in the gauge group. We computed

[9] Clearly this projection only applies for states of the $T_{(k_1,k_2)}$ twisted sector which have g as generating element.

the complete spectra for models which incorporate[10] E_6, SO(10) and SU(5). The results for SO(10) can be found in appendix C.

4.4 Wilson Lines and Local GUT Scenarios

The models we found in the previous section will be the starting point of our heterotic road. Given that in GUT schemes one has the fields of the standard model in more compact representations, the search for realistic models in such scenarios is less tedious than just trying to complete families out of states distributed all over the twisted sectors. Still, such GUT structures should only be manifest at certain fixed points while globally, the gauge topography looks like that of the standard model[11]. This is the idea of local grand unification [51] and we can already see how it works just by looking at the spectra we found previously. Recall that there we found some special fixed points in the $T_{(0,2)}$, $T_{(1,0)}$ and $T_{(1,2)}$.For these fixed points the generators do not commute with any space group element whose point group part is of order four, meaning that there are no projection conditions induced by V_4 but only by $2V_4$. This projection is less strict than that one imposes on the twisted states sitting at ordinary fixed points. For this reason the gauge group at the special fixed points looks enhanced in comparison to that we see in the bulk. As an example, consider the model 67 of appendix B. The gauge group we found for this model is $\mathrm{SO}(10) \times \mathrm{U}(1)^3 | \mathrm{SO}(10) \times \mathrm{SU}(4)$ (the bar is used to separate the subgroups of each E_8 factor). It can be shown that at the special fixed points the matter fields come in irreducible representations of $E_6 \times \mathrm{U}(1)^2 | \mathrm{SO}(16)$. This distribution of gauge structures all over the orbifold will be even more accentuated when introducing Wilson lines, since these quantities will introduce a bigger variety of projection conditions and contributions to the left-moving mass equations depending on the fixed point under consideration.

The gauge groups we have achieved by embedding the point group into the gauge space are still at an intermediate stage in the sense that none of them contains a factor which looks like that of the standard model. In the previous section we focused especially on those for which the gauge group contains a factor which can be thought of as a grand unifying structure. This brings some systematics onto our exploration since we attempt to use the freedom we have to introduce Wilson

[10] Models with Pati-Salam symmetry are not considered since in those situations it is very hard to control the spectrum in order to achieve a certain family structure. Such Pati-Salam models are somehow, more easy to deal with when they arise from an underlying SO(10) GUT.

[11] The standard model gauge group is normally accompanied of some U(1) factors and possibly some other non-abelian gauge groups. This is not problematic for phenomenology as long as those extra factors are not manifest at low energy.

lines in order to break such factor down to some group which looks like that of the standard model. The nice advantage of choosing Γ_{16} as the lattice spanned by the simple roots of $E_8 \times E_8$ is precisely the direct product structure. Given that most of the matter states are charged under one E_8 only, the E_8 factor from which the GUT group is originated, is interpreted as an observable sector where we expect to find the fields of the standard model, while the fields charged under the other E_8 belong to some hidden sector which provide a fruitful scenario for SUSY breaking in the low energy regime. In some cases one can find matter states charged under both E_8 factors. Such fields will mediate interactions between observable and hidden sector fields [52, 53]. In order for them not to spoil the phenomenology one has to make them massive enough. The mass of those messenger fields will set the scale at which the hidden sector dynamics becomes observable and in the case of gauge mediation [54] it will also provide the scale for supersymmetry breaking.

Our approach in the quest for realistic models will be based on many of the model building strategies implemented for the $\mathbb{Z}_{6\text{-II}}$ mini-landscape [14]. At this stage we are equipped with the spectra of models with suitable group factors. These spectra can be used to engineer configurations of Wilson lines which break the GUT factor in a proper manner and lead to three families.

First we have to look for all inequivalent Wilson lines permitted for the $\mathbb{Z}_2 \times \mathbb{Z}_4$ orbifold, so that we need to look at the identifications induced by the point group.

(i) In the first complex plane one has two Wilson lines W_1 and W_2, because the orbifold action does not relate the lattice vectors $\mathbf{e}_1$ and $\mathbf{e}_2$. These lines are of order two since

$$\omega \mathbf{e}_\alpha + \mathbf{e}_\alpha = 0, \quad \alpha = 1, 2.$$

(ii) In the second an third planes we have the identifications:

$$\theta \mathbf{e}_3 = \mathbf{e}_4, \quad \theta^3 \mathbf{e}_5 = \mathbf{e}_6, \tag{4.25}$$

so that we only have one inequivalent Wilson line per complex plane. These we denote by W_3 and W_4. Due to the conditions

$$\theta^2 \mathbf{e}_3 + \mathbf{e}_3 = 0, \quad \text{and} \quad \theta^2 \mathbf{e}_5 + \mathbf{e}_5 = 0, \tag{4.26}$$

these Wilson lines are of order two as well.

In addition to the previous constraints one has to satisfy the following requirements in order to ensure a modular invariant vacuum-to-vacuum amplitude:

$$2V_i \cdot W_\alpha = 0 \mod 2, \quad i = 1, 2,\ \alpha = 1, 2, 3, 4\,, \tag{4.27}$$

$$2W_\alpha \cdot W_\beta = 0 \mod 2, \quad \alpha, \beta = 1, 2, 3, 4\,. \tag{4.28}$$

Now we want to explore the role these Wilson lines play with regards to the spectrum. Previously we saw that within the same twisted sector there is one copy of the fields sitting at each fixed points except for those sectors where one has special fixed points. Two fixed points which have the same content we call *degenerate*. Note that in the sectors $T_{(0,2)}$, $T_{(1,0)}$ and $T_{(1,2)}$ the degeneracies have been already broken due to the presence of the special fixed points. These degeneracies are broken even further when we switch on the Wilson lines. Note first that in the mass equation for the left-movers, the gauge momentum is shifted by the embedding of the generating element under consideration. On the other hand, the orbifold GSO projectors are induced by centralizers of the generating element, so that these operators are also sensitive to the effects of the Wilson lines. We want to find out how these Wilson lines split the degeneracy of the fixed points within the same twisted sector, in order to get the correct multiplicities for all matter fields in our model. The results of our analysis are depicted in figure 4.2. There we consider all configurations of Wilson lines one can have. There are sixteen possibilities ranging from the case in which there are none to that in which all four are switched on. For each of these situations we compute the embedding of the generators and their centralizers. Then we look at all those fixed points with the same corrections to the mass equation which share the same projectors. Those points will be degenerated. Each fixed point corresponds to a box in the figure and degenerate fixed points (within the same twisted sector) are represented by boxes with the same color.

From our previous study we can now focus on those fixed points whose matter

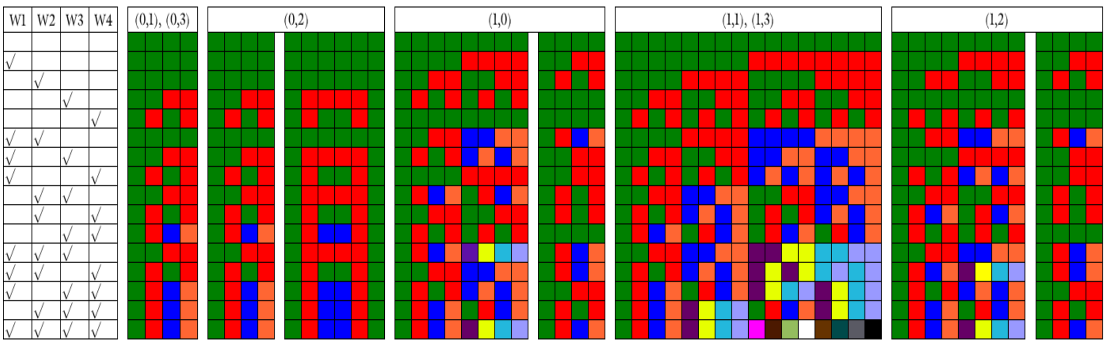

Figure 4.2: Constraints for the physical states for different configurations of Wilson lines in $\mathbb{Z}_2 \times \mathbb{Z}_4$. Within the same twisted sector, boxes with the same color share the same shift in the left-moving mass equation and the same set or GSO projectors apply. This means that these points share the same set of physical states. For the case of $T_{(0,2)}$, $T_{(1,0)}$ and $T_{(1,2)}$ we have split between ordinary and special fixed points.

content is unaffected by the Wilson line configuration. Basically we are interested in those fixed points where the Wilson lines keep the GUT structure unbroken. In those points, the representations we obtained in the previous section survive completely. Such protected fixed points are depicted in in figure 4.3.

Now we can look back to the spectra we found in the previous section. We will

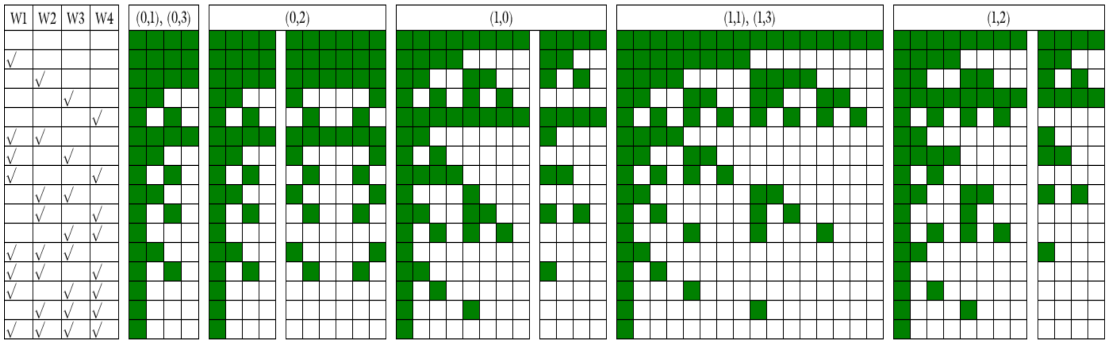

Figure 4.3: Protected fixed points under different Wilson line configurations, the boxes colored in green represent those fixed points which are unaffected under the action of the Wilson lines, for any of the 144 embeddings discussed in the previous section. The matter representations we found in the absence of Wilson lines will completely survive, when sitting at those fixed points.

mainly focus on those models where an E_6 or SO(10) GUT scheme seems possible. The reason for not considering SU(5) or Pati-Salam structures comes from the fact that there the families arise from more than one representation, for instance, in SU(5) one will have to control not only the number of **10**-plets but also the number of $\overline{\mathbf{5}}$-plets in the model. In addition we find that SU(5) stuctures are normally accompanied by SU(3) factors which we will have to break. The search for promising candidates is performed in the following manner: taking into account the matter in the untwisted sector we will go over the sixteen possible configurations of Wilson lines and count the number of **27** ($\overline{\mathbf{27}}$) for the case of E_6 or the number of **16** ($\overline{\mathbf{16}}$) for SO(10). Considering the multiplicities of each configuration we will search for those situations in which the effective number of protected **27**´s or **16**´s is equal to three. We will also have to require that the number of Wilson lines is enough to break the observable sector down to $SU(3) \times SU(2) \times U(1)^5$.

4.5 An Explicit Example

From the promising candidates obtained in the previous section we see that the simplest case is that of model 67_1 when the brother phase is taken to be $\Phi = 0$ mod 1, the gauge group of this model, at the stage when all Wilson lines are off is $\mathrm{SO}(10) \times \mathrm{U}(1)^3 | \mathrm{SO}(10) \times \mathrm{SU}(4)$. The SO(10) factor of our interest is that originated from the first E_8. There we only need to switch on two Wilson lines in order to accommodate tree complete families coming from twisted **16**-plets sitting at the protected fixed points in $T_{(1,0)}$. This model has the nice feature that the number of Wilson lines is the minimum required to break the observable sector down to $\mathrm{SU}(3) \times \mathrm{SU}(2) \times \mathrm{U}(1)^5$. We specialize to those particular cases in which the hypercharge generator is the one would obtain from SU(5) grand unification.

The previous arguments permit us to focus on all Wilson lines for which the following breaking is obtained:

$$\mathrm{SO}(10) \xrightarrow{W_1} \mathrm{SU}(5) \times \mathrm{U}(1) \xrightarrow{W_2} \mathrm{SU}(3) \times \mathrm{SU}(2) \times \mathrm{U}(1)^2 .$$

Firstly we want to find an embedding for the point group which is fully modular invariant and leads to the brother model of our interest. This is done by adding lattice vectors to the original embedding and it can be shown that

$$V_2 = \frac{1}{2}(-11, 2, 27, 22, 8, 12, 18, -6) \oplus \frac{1}{2}(1, -3, 3, -9, 5, -1, -1, -1),$$

$$V_4 = \frac{1}{4}(4, -5, -9, 14, -6, 2, -10, -2) \oplus \frac{1}{2}(1, -3, 9, -8, 6, -2, 6, -2),$$

is an allowed solution. Using these vectors we now have to look for the Wilson lines. Note that we can follow an approach similar to the one we used to compute the embedding vectors for the point group. We can concentrate in one E_8 factor and look for all possible lattice vectors which are of order two pu to lattice vectors:

$$W^k = \frac{1}{2} \sum_{m=1}^{8} n_m^k \alpha_m , \quad 0 \leq n_m^k \leq 1 , \tag{4.29}$$

with α_m being a simple root. There are 256 possibilities. Since weights for generators of SO(10) are only non zero in the components of the first E_8, its breaking is determined only by the vector W^k. Thus we can look at all those vectors for which the breaking to SU(5) is achieved and one finds 120 of them for which this condition is satisfied. We denote them as W_1^k. For each such vectors we now have to look for all $W_2^{j_k}$ which break the SU(5) to $\mathrm{SU}(3) \times \mathrm{SU}(2) \times \mathrm{U}(1)$. For each W_1^k we have 128 alternatives.

We can then construct the Wilson lines by tensoring these vectors with the candidates shown in equation (4.29)

$$W_1 = W_1^k \oplus W^p,$$
$$W_2 = W_1^{j_k} \oplus W^q. \tag{4.30}$$

Given that we still have the freedom to introduce lattice vectors, the modular invariance conditions can be relaxed to:

$$4W_\alpha \cdot V_4 = 0 \mod 1, \quad 2W_\alpha \cdot V_2 = 0 \mod 1; \quad \alpha = 1,2\,, \tag{4.31}$$

$$W_\alpha^2 = 0 \mod 1, \quad 2W_\alpha \cdot W_\beta = 0 \mod 1; \quad \alpha \neq \beta, \quad \alpha,\beta = 1,2. \tag{4.32}$$

Following this procedure one finds 8365 different alternatives. All configurations which are not modular invariant require of introducing brother phases in order to compute the spectrum. It can be shown that each configuration leads to only one inequivalent model. Instead of looking at the brother phases associated to the Wilson lines, we can focus on those configurations which are fully modular invariant: one of such candidates is

$$W_1 = \frac{1}{4}(17,-17,-15,-19,-15,-15,5,7) \oplus \frac{1}{4}(1,1,-3,1,-1,-3,-1,1), \tag{4.33}$$

$$W_2 = \frac{1}{2}(2,-2,-2,-2,-2,-1,0,1) \oplus \frac{1}{2}(0,1,-1,1,0,-1,1,1). \tag{4.34}$$

This choice for the Wilson lines induces the breaking

$$\mathrm{SO}(10)\times\mathrm{U}(1)^3|\mathrm{SO}(10)\times\mathrm{SU}(4) \xrightarrow{W_1,W_2} \mathrm{SU}(3)\times\mathrm{SU}(2)\times\mathrm{U}(1)^5|\mathrm{SU}(4)\times\mathrm{SU}(2)\times\mathrm{U}(1)^4.$$

The matter spectrum is computed in a similar manner as in section 4.3, with the

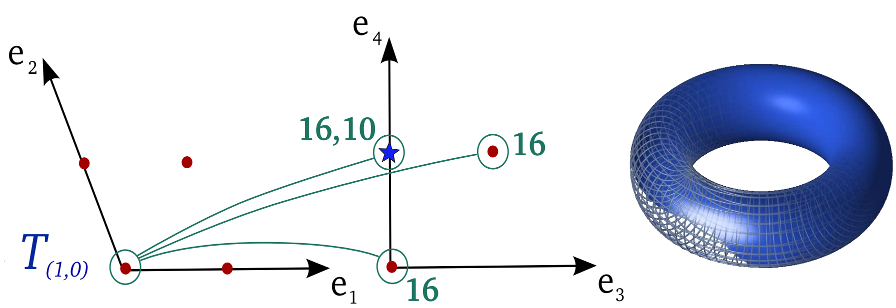

Figure 4.4: Localization of the families in the $T_{(1,0)}$ twisted sector.

only subtlety that now one has to consider the breaking of the degeneracies due to the Wilson lines and the appropriate projection conditions within degenerate fixed points. This permits to compute the actual multplicity of states. The complete spectrum of states is presented in table 4.3. Let us simply remind that in the twisted sector of our interest there are two fixed points where the local gauge group is SO(10) from where one gets two **16**-plets there is also one special fixed point which is protected where the local gauge group is E_6. There one has a **27** which decomposes under SO(10) as $\mathbf{16} \oplus \mathbf{10} \oplus \mathbf{1}$. The **16** gives us the third family whereas the **10** can be probably used to get the Higgses. Note that, as expected from modular invariance, all non-Abelian factors are anomaly free.

Untwisted	(0,1)	(0,2)	(0,3)	(1,0)	(1,0)	(1,1)	(1,2)	(1,3)
$4(\overline{\mathbf{3}},\mathbf{1})$	$16(\mathbf{1},\mathbf{1})$	$16(\mathbf{1},\mathbf{1})$	$8(\mathbf{1},\mathbf{1})$	$4(\mathbf{3},\mathbf{1})$	$3(\overline{\mathbf{3}},\mathbf{2})$		$(\mathbf{3},\mathbf{1})$	$8(\mathbf{1},\mathbf{1})$
$2(\mathbf{3},\mathbf{1})$				$2(\overline{\mathbf{3}},\mathbf{1})$	$6(\mathbf{3},\mathbf{1})$		$(\overline{\mathbf{3}},\mathbf{1})$	
$8(\mathbf{1},\mathbf{1})$				$8(\mathbf{1},\mathbf{2})$	$3(\mathbf{1},\mathbf{2})$		$2(\mathbf{1},\mathbf{2})$	
				$40(\mathbf{1},\mathbf{1})$	$3(\mathbf{1},\mathbf{1})$		$33(\mathbf{1},\mathbf{1})$	
$(\overline{\mathbf{4}},\mathbf{2})$				$4(\mathbf{1},\mathbf{2})$	$4(\mathbf{4},\mathbf{1})$		$7(\overline{\mathbf{4}},\mathbf{1})$	$4(\mathbf{1},\mathbf{2})$
$(\mathbf{6},\mathbf{1})$							$5(\mathbf{4},\mathbf{1})$	
							$4(\mathbf{1},\mathbf{2})$	

Table 4.3: Matter spectrum for the embedding under consideration. The states in blue arise from the three protected **16** and the **10** in the $T_{(1,0)}$ sector. The states in red are exotics arising from other points on the orbifold. For the fields in the observable sector we give their corresponding representations under SU(3) × SU(2) and for the hidden matter the representations under SU(4) × SU(2)

4.5.1 Brief Comments on the Hypercharge Generator

So far we have not discussed the charges of the matter states under the U(1) factors. This will be a matter of further research and is out of the scope of this work. Nevertheless, we want to sketch briefly the procedure to follow in order to obtain these charges. On the one hand it is known that among all those U(1) factors there is at most one which is anomalous [55]. At our stage, the most important U(1) is that which gives the correct hypercharge for the families of the standard model, Our aim is to relate this hypercharge generator to the one obtained in the context of SU(5) GUTs. In order to achieve a satisfactory phenomenology one has to guarantee that the following requirements hold

(i) The $U(1)_Y$ generator has to be non-anomalous.

(ii) Since our model contains exotics, we have to require that for each $(\mathbf{3},\mathbf{1})$ state one has, there has to be a $(\overline{\mathbf{3}},\mathbf{1})$ with the opposite hypercharge and similarly for the $(\mathbf{1},\mathbf{2})$ states. This is so, because one will be interested in generating mass terms for such fields in order to decouple them from the spectrum without breaking $\mathrm{U}(1)_Y$.

It is not guaranteed that this is possible in our model. In such circumstances we have to explore some other alternatives which lead to the right charge assignment for the SM fields and satisfies the previous two conditions. In that case one has to guarantee that such U(1) factor leads to a realistic Weinberg angle.

If there is no way in which a realistic U(1) generator is achieved, one has to look for a different embedding for the Wilson lines. As discussed previously one has 8365 alternatives to play with, so that a systematic search can be performed. In the ultimate case when this search is completely unsuccessful, we are still left with another 8 promising models with SO(10) GUT structures and 10 with E_6

4.6 *R*-symmetries for $\mathbb{Z}_2 \times \mathbb{Z}_4$

Finally we would like to discuss the possible R-symmetries one can find for the orbifold of our interest. Based upon the general results we found in section 3.3, this is a very exciting example because the point group has two generators, where only one contains prime twists.

The automorphism group of the lattice under consideration consists 1024 elements out of which only 128 commute with the orbifold generators, from this subgroup only 64 belong to SO(6).

By looking at the symmetries which also leave the conjugacy classes invariant, we found that they form an abelian group $\mathcal{G}_u$ of order 16 which can be generated by powers and products of the $\mathbb{Z}_4$ orbifold twist and two $\mathbb{Z}_2$ symmetries acting independently in the first and second planes. Invariance of the L point correlator under such transformations imposes the following constraints

$$\sum_{\alpha=1}^{L} R^1_\alpha = 1 \bmod 2, \qquad \sum_{\alpha=1}^{L} R^2_\alpha = 1 \bmod 2, \qquad \sum_{\alpha=1}^{L} (R^2_\alpha - R^3_\alpha) = 0 \bmod 4. \quad (4.35)$$

To study the effects of the remaining 48 elements which are left in SO(6), we first divide it by $\mathcal{G}_u$. Then we find that the only inequivalent symmetries which exchange some conjugacy classes are two $\mathbb{Z}_4$ twists acting in the first and third planes as well as the combined action of them. Since the first two symmetries do not have common unprotected conjugacy classes all the constraints that the third imposes on the couplings can be inferred from the others.

The rotation $\varrho_1 = \text{diag}(1, 1, e^{2\pi i/4})$ leaves invariant all conjugacy classes but those related to the fixed tori

$$\frac{1}{2}(e_4 + e_5) \otimes \mathbb{T}_1^2 \leftrightarrow \frac{1}{2}(e_4 + e_6) \otimes \mathbb{T}_1^2, \tag{4.36}$$

of the $(0, 2)$ twisted sector. Since this twist does not affect the gauge embedding of the model it will not be broken by the Wilson lines. Under the action of ϱ_1, any state ϕ_1 wrapping around one of the unprotected tori will get mapped to its counterpart ϕ_2 in the other. Note also that couplings which involve invariant physical states (up to phases) must be invariant under ϱ_1:

$$\sum_{\alpha=1}^{L} R_\alpha^3 = 1 \mod 4, \tag{4.37}$$

while couplings containing fields such as ϕ_1 or ϕ_2 get interchanged with some others. Consider some fields ζ_α, $\alpha = 2, ..., L$ which sit at invariant fixed points and assume that a term $y_1\phi_1\zeta_2...\zeta_L$ is present in the superpotential, with y_1 being the Yukawa coupling. Since ϱ_1 commutes with the orbifold action it can be shown that the space group selection rule also allows for the presence of a term $y_2\phi_2\zeta_2...\zeta_L$. Under ϱ_1 these terms transform as:

$$y_1\phi_1\zeta_2...\zeta_L \xrightarrow{\varrho_1} e^{i\varphi}y_1\phi_2\zeta_2...\zeta_L, \tag{4.38}$$

$$y_2\phi_2\zeta_2...\zeta_L \xrightarrow{\varrho_1} e^{-i\varphi}y_2\phi_1\zeta_2...\zeta_L, \tag{4.39}$$

where φ is a phase arising from the transformation of the space group part and the oscillators under ϱ_1. The superpotential has to transform trivial under this symmetry. The Yukawa couplings y_1 and y_2 are related by:

$$y_1 = e^{i\varphi}y_2,$$

Although, this is an interesting result which relates some Yukawa couplings and henceforth reduced the number of correlators which need to be calculated, we will be more interested in the cases where such symmetry can be used to forbid certain couplings. As we saw ϱ_1 resembles an R-symmetry for couplings containing only fields sitting at invariant fixed points. Nevertheless the chance of implementing ϱ_1 as a symmetry of the physical states depends on the possibility of having the same spectrum sitting at the fixed points which get mapped onto each other under such transformation. In order to implement a given *local R-symmetry* one has to guarantee that the gauge embedding is not affected by the action of ϱ_1: given two space group elements h and g related by $h = \varrho_1(g)$ it must hold $V_h \sim V_g$ so that $[g]$ and $[h]$ contain states which share the same mass equations and projection conditions so that a one-to-one mapping between them can be stablished. Note that $\varrho_1 \mathbf{e}_5 = \mathbf{e}_6$ but since $\mathbf{e}_5$ and $\mathbf{e}_6$ are identified under the orbifold, they share the same Wilson line (W_4). This means that ϱ_1 can be incorporated as a partial R-symmetry which is independent of the choice of the Wilson lines.

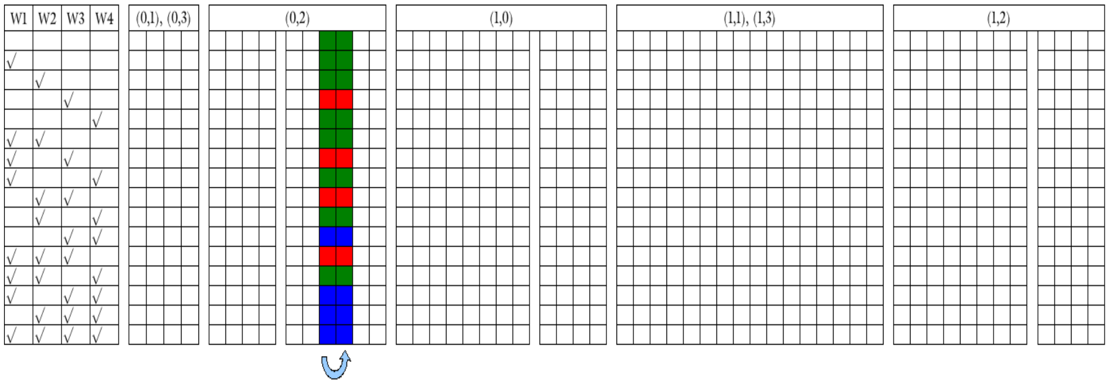

Figure 4.5: The action of ϱ_1 on the fixed points. The arrow illustrates the transformation and the boxes in white represent invariant fixed points.Note that the points which get mapped onto each other share the same set of fields, this is independent of the Wilson line configuration, (see fig. 4.2).

Chapter 5

Conclusions

"And there is no trade or employment but the young man following it may become a hero,
And there is no object so soft but it makes a hub for the wheel´d universe."

Walt Whitman, *Leaves of Grass*

Our exploration of possible R-symmetries, was based on the construction of the automorphism groups for the lattices of $\mathbb{Z}_N$ orbifolds. For those which were factorizable, we could follow a more general approach, by looking at the independent twists which compose the point group generator. If any of those twists leave the fixed points invariant, then there is an R-symmetry associated to it. For the case of non-factorizable orbifolds, we look for the lattice automorphisms which preserve the group structure in order to describe the R-symmetries of the model. For some orbifolds such as the $\mathbb{Z}_7$ and the non-factorizable $\mathbb{Z}_{6\text{-II}}$ there is no Lorentz symmetry which can provide any selection rule. Those models seem to be promising grounds to study the effects of instanton selection rules, such as rule 4 and the recently discovered rule 5 [32].

The $\mathbb{Z}_2 \times \mathbb{Z}_4$ orbifold was considered with the aim of obtaining grand unified schemes in which a three family structure can be realized. We used the properties of the $E_8 \times E_8$ root lattice to obtain all possible inequivalent embeddings for the point group generators, which are permitted by modular invariance of the one loop partition function. 144 models were found, among which 35 lead to gauge groups which contain an SO(10) factor, 26 contain an E_6 and 25 an SU(5).

Since we considered all inequivalent embeddings up to lattice shifts, we are left with two brother phases per embedding. Each such brother phase was used to compute the complete spectrum of each model at the stage when no Wilson lines are switched on. We were interested in those models in which local GUT schemes can be implemented, and the spectra we previously described were of great help: The GUT factors need to be broken to some group structure which looks like the standard model up to some U(1) factors. This can be achieved by introducing Wilson lines. In our model we have the possibility to switch on four independent Wilson lines. We looked at all possible Wilson line configurations in order to accommodate three families at fixed points where the GUT group remains unbroken.

The simplest model we found requires two Wilson lines to give a three family model. The gauge group in the observable sector is broken down to $\mathrm{SU}(3)\times\mathrm{SU}(2)\times\mathrm{U}(1)^5$. Three SO(10) complete families were found to sit at fixed points in the $T_{(1,0)}$ twisted sector.

In addition to the model described previously there are 9 further feasible models for SO(10) local GUTs and 9 for E_6 in which three net families can be obtained. This makes the $\mathbb{Z}_2 \times \mathbb{Z}_4$ orbifold a very promising scenario for phenomenology.

There is still a long way in this heterotic road down to the MSSM. First we have to look for the generators of the U(1) factors. Among them, there is one which is anomalous. This U(1) introduces a Fayet Iliopoulos term [55] which has to be canceled by assigning VEVs to some singlet fields while preserving supersymmetry. Such configurations have to be choosen in such a way that the hypercharge generator remains unbroken, all exotics are decoupled and all Yukawa couplings for the MSSM fields are generated.

Appendix A

Inequivalent Vectors for One E_8 Embedding

$4 \cdot A_4$	$2 \cdot A_2$	Group Decomposition
(0,0,0,0,0,0,0,0,0)	$(0,0,0,0,0,0,0,0)$	E_8
	$(-1,0,0,0,0,0,1,0)$	$E_7 \times SU(2)$
	$(-1,-1,-1,0,0,0,0,1)$	$SO(16)$
(2,2,0,0,0,0,0,0,0)	$(0,0,0,0,0,0,0,0)$	$E_7 \times SU(2)$
	$(-1,1,0,0,0,0,0,0)$	$E_7 \times SU(2)$
	$(-1,0,0,0,0,0,1,0)$	$E_6 \times U(1)^2$
	$(-1,1,0,0,0,0,0,0)$	$SO(12) \times SU(2)^2$
	$(-1,-1,-1,0,0,0,0,1)$	$SO(12) \times SU(2)^2$
	$(-1,0,-1,-1,0,0,0,1)$	$SU(8) \times U(1)$
(1,1,0,0,0,0,0,0,0)	$(0,0,0,0,0,0,0,0)$	$E_7 \times U(1)$
	$(-1,1,0,0,0,0,0,0)$	$E_7 \times U(1)$
	$(-1,0,0,0,0,0,1,0)$	$E_6 \times U(1)^2$
	$(-1,1,0,0,0,0,0,0)$	$SO(12) \times SU(2) \times U(1)$
	$(-1,-1,-1,0,0,0,0,1)$	$SO(12) \times SU(2) \times U(1)$
	$(-1,0,-1,-1,0,0,0,1)$	$SU(8) \times U(1))$
(2,1,1,0,0,0,0,0,0)	$(0,0,0,0,0,0,0,0)$	$E_6 \times SU(2) \times U(1)$
	$(-1,0,1,0,0,0,0,0)$	$E_6 \times U(1)^2$
	$(-1,0,0,0,0,0,1,0)$	$SO(10) \times SU(2) \times U(1)^2$
	$(-1,0,1,0,0,0,0,0)$	$SO(10) \times U(1)^3$
	$(0,0,0,-1,0,0,1,0)$	$SU(6) \times SU(2)^2 \times U(1)$

$4{\cdot}A_4$	$2{\cdot}A_2$	Group Decomposition
(2,1,1,0,0,0,0,0)	$(0,-1,1,0,0,0,0,0)$	$E_6\times SU(2)\times U(1)$
	$(-1,-1,-1,0,0,0,0,1)$	$SO(10)\times SU(2)\times U(1)^2$
	$(-1,-1,0,-1,0,0,0,1)$	$SU(6)\times SU(2)\times U(1)^2$
	$(-1,0,0,-1,-1,0,0,1)$	$SU(6)\times SU(2)^2)\times U(1)$
(4,0,0,0,0,0,0,0)	$(0,0,0,0,0,0,0,0)$	$SO(16)$
	$(-1,0,0,0,0,0,1,0)$	$SO(12)\times SU(2)^2$
	$-\frac{1}{2}(1,1,1,1,1,1,-1,-1)$	$SU(8)\times U(1)$
	$(0,0,0,0,0,-2,0,0)$	$SO(16)$
	$\frac{1}{2}(-3,-1,-1,1,1,1,1,1)$	$SU(8)\times U(1)$
	$(-1,-1,-1,0,0,0,0,1)$	$SO(8)^2$
(2,0,0,0,0,0,0,0)	$(0,0,0,0,0,0,0,0)$	$SO(14)\times U(1)$
	$(-1,0,0,0,0,0,1,0)$	$SO(12)\times U(1)^2$
	$-\frac{1}{2}(1,1,1,1,1,1,-1,-1)$	$SU(7)\times U(1)^2$
	$(0,-1,0,0,0,0,1,0)$	$SO(10)\times SU(2)^2\times U(1)$
	$(0,0,0,0,0,-2,0,0)$	$SO(14)\times U(1)$
	$\frac{1}{2}(-3,-1,-1,1,1,1,1,1)$	$SU(7)\times U(1)^2$
	$(-1,-1,-1,0,0,0,0,1)$	$SU(4)\times SO(8)\times U(1)$
(3,1,0,0,0,0,0,0)	$(0,0,0,0,0,0,0,0)$	$SO(12)\times SU(2)\times U(1)$
	$(-1,1,0,0,0,0,0,0)$	$SO(12)\times SU(2)\times U(1)$
	$(-1,0,0,0,0,0,1,0)$	$SO(10)\times U(1)^3$
	$(-1,1,0,0,0,0,0,0)$	$SO(12)\times SU(2)\times U(1)$
	$-\frac{1}{2}(1,1,1,1,1,1,-1,-1)$	$SU(6)\times U(1)^3$
	$-\frac{1}{2}(1,-1,1,1,1,1,1,-1)$	$SU(6)\times SU(2)\times U(1)^2$
	$(0,0,-1,0,0,0,1,0)$	$SO(8)\times SU(2)^3\times U(1)$
	$(0,0,0,0,0,-2,0,0)$	$SO(12)\times SU(2)\times U(1)$
	$\frac{1}{2}(-3,-1,-1,1,1,1,1,1)$	$SU(6)\times SU(2)\times U(1)^2$
	$\frac{1}{2}(-3,1,-1,-1,1,1,1,1)$	$SU(6)\times U(1)^3$
	$(-1,-1,-1,0,0,0,0,1)$	$SO(8)\times SU(2)^3\times U(1)$
	$(-1,0,-1,-1,0,0,0,1)$	$SU(4)^2\times U(1)^2$

$4{\cdot}V_{\mathbb{Z}_4}$	$2{\cdot}V_{\mathbb{Z}_4}$	Group Decomposition
(2,2,2,0,0,0,0,0)	$(0,0,0,0,0,0,0,0)$	SO(10)×SU(4)
	$(-1,-1,0,0,0,0,0,0)$	SO(10)×SU(2)^2×U(1)
	$(-1,0,0,0,0,0,1,0)$	SO(8)×SU(2)^2×U(1)^2
	$-\frac{1}{2}(1,1,1,1,1,1,-1,-1)$	SU(5)×SU(3)×U(1)^2
	$(0,0,0,-1,0,0,1,0)$	SU(4)^2×SU(2)^2
	$(0,0,0,0,0,-2,0,0)$	SO(10)×SU(4)
	$\frac{1}{2}(-3,-1,-1,1,1,1,1,1)$	SU(5)×SU(3)×U(1)^2
	$(-1,-1,-1,0,0,0,0,1)$	SO(8)×SU(4)×U(1)
	$(-1,-1,0,-1,0,0,0,1)$	SU(4)×SU(2)^4×U(1)
(3,1,1,1,1,1,0,0)	$(0,0,0,0,0,0,0,0)$	SU(8)×SU(2)
	$(-1,-1,0,0,0,0,0,0)$	SU(6)×SU(2)^2×U(1)
	$(-1,0,0,0,0,0,1,0)$	SU(6)×SU(2)×U(1)^2
	$(-1,0,0,0,0,1,0,0)$	SU(4)^2×SU(2)×U(1)
	$(0,0,0,0,0,0,-1,1)$	SU(8)×SU(2)
	$(-1,0,0,1,-1,1,0,0)$	SU(6)×SU(2)^2)×U(1)
	$\frac{1}{2}(-1,-3,1,1,1,1,-1,1)$	SU(8)×U(1)
	$(-1,-1,-1,0,0,0,0,1)$	SU(4)^2×U(1)^2
	$(0,-1,-1,0,0,0,-1,1)$	SU(4)^2×SU(2)×U(1)
(1,1,1,1,1,1,1,-1)	$(0,0,0,0,0,0,0,0)$	SU(8)×U(1)
	$(0,0,0,0,-1,-1,0,0)$	SU(6)×SU(2)×U(1)^2
	$(-1,0,0,0,0,0,1,0)$	SU(6)×SU(2)×U(1)^2
	$-\frac{1}{2}(1,1,1,1,1,1,-1,-1)$	SU(7)×U(1)^2
	$\frac{1}{2}(-1,-1,-1,-1,1,1,1,1)$	SU(5)×SU(3)×U(1)^2
	$(0,0,0,0,0,-2,0,0)$	SU(8)×U(1)
	$\frac{1}{2}(-1,1,1,1,1,-3,1,-1)$	SU(7)×U(1)^2
	$-\frac{1}{2}(1,1,1,1,-1,3,-1,-1)$	SU(5)×SU(3)×U(1)^2
	$(-1,-1,-1,0,0,0,0,1)$	SU(4)^2×U(1)^2

Appendix B

Inequivalent Embeddings for $\mathbb{Z}_2 \times \mathbb{Z}_4$

The brother phases Φ allowed by modular invariance are also presented.

	$2 \cdot V_2$	Φ	Group Decomposition
	$4 \cdot V_4 = (0,0,0,0,0,0,0,0) \oplus (1,1,0,0,0,0,0,0)$		
1	$(0,0,0,0,0,0,0,0)\oplus(-1,0,0,0,0,0,1,0)$	$0, \frac{1}{2}$	$E_8|E_6{\times}U(1)^2$
2	$(-1,0,0,0,0,0,1,0)\oplus(-1,0,-1,-1,0,0,0,1)$	$0, \frac{1}{2}$	$E_7{\times}SU(2)|SU(8){\times}U(1)$
3	$(-1,-1,-1,0,0,0,0,1)\oplus(-1,0,0,0,0,0,1,0)$	$0, \frac{1}{2}$	$SO(16)|E_6{\times}U(1)^2$
	$4 \cdot V_4 = (0,0,0,0,0,0,0,0) \oplus (3,1,0,0,0,0,0,0)$		
4	$(0,0,0,0,0,0,0,0)\oplus(-1,0,0,0,0,0,1,0)$	$-\frac{1}{8}, \frac{3}{8}$	$E_8|SO(10){\times}U(1)^3$
5	$(0,0,0,0,0,0,0,0)\oplus-\frac{1}{2}(1,-1,1,1,1,1,1,-1)$	$0, \frac{1}{2}$	$E_8|SU(6){\times}SU(2){\times}U(1)^2$
6	$(-1,0,0,0,0,0,1,0)\oplus\frac{1}{2}(-3,-1,-1,1,1,1,1,1)$	$-\frac{1}{4}, \frac{1}{4}$	$E_7{\times}SU(2)|SU(6){\times}SU(2){\times}U(1)^2$
7	$(-1,0,0,0,0,0,1,0)\oplus(-1,0,-1,-1,0,0,0,1)$	$-\frac{1}{8}, \frac{3}{8}$	$E_7{\times}SU(2)|SU(4)^2{\times}U(1)^2$
8	$(-1,-1,-1,0,0,0,0,1)\oplus(-1,0,0,0,0,0,1,0)$	$-\frac{1}{8}, \frac{3}{8}$	$SO(16)|SO(10){\times}U(1)^3$
9	$(-1,-1,-1,0,0,0,0,1)\oplus-\frac{1}{2}(1,-1,1,1,1,1,1,-1)$	$0, \frac{1}{2}$	$SO(16)|SU(6){\times}SU(2){\times}U(1)^2$
	$4 \cdot V_4 = (2,2,0,0,0,0,0,0) \oplus (1,1,0,0,0,0,0,0)$		
10	$(0,0,0,0,0,0,0,0)\oplus(-1,0,0,0,0,0,1,0)$	$0, \frac{1}{2}$	$E_7{\times}SU(2)|E_6{\times}U(1)^2$
11	$(-1,-1,0,0,0,0,0,0)\oplus(-1,0,-1,-1,0,0,0,1)$	$-\frac{1}{4}, \frac{1}{4}$	$E_7{\times}SU(2)|SU(8){\times}U(1)$
12	$(-1,0,0,0,0,0,1,0)\oplus(-1,0,-1,-1,0,0,0,1)$	$-\frac{1}{8}, \frac{3}{8}$	$E_6{\times}U(1)^2|SU(8){\times}U(1)$
13	$(-1,1,0,0,0,0,0,0)\oplus(-1,0,-1,-1,0,0,0,1)$	$0, \frac{1}{2}$	$SO(12){\times}SU(2)^2|SU(8){\times}U(1)$
14	$(-1,-1,-1,0,0,0,0,1)\oplus(-1,0,0,0,0,0,1,0)$	$-\frac{1}{4}, \frac{1}{4}$	$SO(12){\times}SU(2)^2|E_6{\times}U(1)^2$
15	$(-1,0,-1,-1,0,0,0,1)\oplus(-1,0,0,0,0,0,1,0)$	$-\frac{1}{8}, \frac{3}{8}$	$SU(8){\times}U(1)|E_6{\times}U(1)^2$
	$4 \cdot V_4 = (2,2,0,0,0,0,0,0) \oplus (3,1,0,0,0,0,0,0)$		
16	$(0,0,0,0,0,0,0,0)\oplus(-1,0,0,0,0,0,1,0)$	$-\frac{1}{8}, \frac{3}{8}$	$E_7{\times}SU(2)|SO(10){\times}U(1)^3$
17	$(0,0,0,0,0,0,0,0)\oplus-\frac{1}{2}(1,-1,1,1,1,1,1,-1)$	$0, \frac{1}{2}$	$E_7{\times}SU(2)|SU(6){\times}SU(2){\times}U(1)^2$
18	$(-1,-1,0,0,0,0,0,0)\oplus\frac{1}{2}(-3,-1,-1,1,1,1,1,1)$	$0, \frac{1}{2}$	$E_7{\times}SU(2)|SU(6){\times}SU(2){\times}U(1)^2$
19	$(-1,-1,0,0,0,0,0,0)\oplus(-1,0,-1,-1,0,0,0,1)$	$-\frac{3}{8}, \frac{1}{8}$	$E_7{\times}SU(2)|SU(4)^2{\times}U(1)^2$

	$2 \cdot V_2$	Φ	Group Decomposition
	$4 \cdot V_4 = (2,2,0,0,0,0,0,0) \oplus (3,1,0,0,0,0,0,0)$		
20	$(-1,0,0,0,0,0,1,0)\oplus\frac{1}{2}(-3,-1,-1,1,1,1,1,1)$	$-\frac{3}{8},\frac{1}{8}$	$E_6{\times}U(1)^2\vert SU(6){\times}SU(2){\times}U(1)^2$
21	$(-1,0,0,0,0,0,1,0)\oplus(-1,0,-1,-1,0,0,0,1)$	$-\frac{1}{4},\frac{1}{4}$	$E_6{\times}U(1)^2\vert SU(4)^2{\times}U(1)^2$
22	$(-1,1,0,0,0,0,0,0)\oplus\frac{1}{2}(-3,-1,-1,1,1,1,1,1)$	$-\frac{1}{4},\frac{1}{4}$	$SO(12){\times}SU(2)^2\vert SU(6){\times}SU(2){\times}U(1)^2$
	$4 \cdot V_4 = (2,2,0,0,0,0,0,0) \oplus (3,1,0,0,0,0,0,0)$		
23	$(-1,1,0,0,0,0,0,0)\oplus(-1,0,-1,-1,0,0,0,1)$	$-\frac{1}{8},\frac{3}{8}$	$SO(12){\times}SU(2)^2\vert SU(4)^2{\times}U(1)^2$
24	$(-1,-1,-1,0,0,0,0,1)\oplus(-1,0,0,0,0,0,1,0)$	$-\frac{3}{8},\frac{1}{8}$	$SO(12){\times}SU(2)^2\vert SO(10){\times}U(1)^3$
25	$(-1,-1,-1,0,0,0,0,1)\oplus-\frac{1}{2}(1,-1,1,1,1,1,1,-1)$	$-\frac{1}{4},\frac{1}{4}$	$SO(12){\times}SU(2)^2\vert SU(6){\times}SU(2){\times}U(1)^2$
26	$(-1,0,-1,-1,0,0,0,1)\oplus(-1,0,0,0,0,0,1,0)$	$-\frac{1}{4},\frac{1}{4}$	$SU(8){\times}U(1)\vert SO(10){\times}U(1)^3$
27	$(-1,0,-1,-1,0,0,0,1)\oplus-\frac{1}{2}(1,-1,1,1,1,1,1,-1)$	$-\frac{1}{8},\frac{3}{8}$	$SU(8){\times}U(1)\vert SU(6){\times}SU(2){\times}U(1)^2$
	$4 \cdot V_4 = (1,1,0,0,0,0,0,0) \oplus (4,0,0,0,0,0,0,0)$		
28	$(-1,0,0,0,0,0,1,0)\oplus(0,0,0,0,0,0,0,0)$	$0,\frac{1}{2}$	$E_6{\times}U(1)^2\vert SO(16)$
29	$(-1,0,0,0,0,0,1,0)\oplus(0,0,0,0,0,-2,0,0)$	$0,\frac{1}{2}$	$E_6{\times}U(1)^2\vert SO(16)$
30	$(-1,0,0,0,0,0,1,0)\oplus\frac{1}{2}(-3,-1,-1,1,1,1,1,1)$	$-\frac{3}{8},\frac{1}{8}$	$E_6{\times}U(1)^2\vert SU(8){\times}U(1)$
31	$(-1,0,0,0,0,0,1,0)\oplus(-1,-1,-1,0,0,0,0,1)$	$-\frac{1}{4},\frac{1}{4}$	$E_6{\times}U(1)^2\vert SO(8){\times}SO(8)$
32	$(-1,0,-1,-1,0,0,0,1)\oplus(-1,0,0,0,0,0,1,0)$	$-\frac{1}{4},\frac{1}{4}$	$SU(8){\times}U(1)\vert SO(12){\times}SU(2)2$
33	$(-1,0,-1,-1,0,0,0,1)\oplus-\frac{1}{2}(1,1,1,1,1,1,-1,-1)$	$-\frac{1}{8},\frac{3}{8}$	$SU(8){\times}U(1)\vert SU(8){\times}U(1)$
	$4 \cdot V_4 = (1,1,0,0,0,0,0,0) \oplus (1,1,1,1,1,1,1,-1)$		
34	$(0,0,0,0,0,0,0,0)\oplus-\frac{1}{2}(1,1,1,1,1,1,-1,-1)$	$-\frac{1}{8},\frac{3}{8}$	$E_7{\times}U(1)\vert SU(7){\times}U(1)^2$
35	$(0,0,0,0,0,0,0,0)\oplus\frac{1}{2}(-1,-1,-1,-1,1,1,1,1)$	$0,\frac{1}{2}$	$E_7{\times}U(1)\vert SU(5){\times}SU(3){\times}U(1)^2$
36	$(-1,-1,0,0,0,0,0,0)\oplus\frac{1}{2}(-1,1,1,1,1,-3,1,-1)$	$0,\frac{1}{2}$	$E_7{\times}U(1)\vert SU(7){\times}U(1)^2$
37	$(-1,-1,0,0,0,0,0,0)\oplus-\frac{1}{2}(1,1,1,1,-1,3,-1,-1)$	$-\frac{1}{4},\frac{1}{4}$	$E_7{\times}U(1)\vert SU(5){\times}SU(3){\times}U(1)^2$
38	$(-1,0,0,0,0,0,1,0)\oplus(0,0,0,0,0,0,0,0)$	$0,\frac{1}{2}$	$E_6{\times}U(1)^2\vert SU(8){\times}U(1)$
39	$(-1,0,0,0,0,0,1,0)\oplus(0,0,0,0,0,-2,0,0)$	$-\frac{1}{8},\frac{3}{8}$	$E_6{\times}U(1)^2\vert SU(8){\times}U(1)$
40	$(-1,0,0,0,0,0,1,0)\oplus(-1,-1,-1,0,0,0,0,1)$	$-\frac{1}{4},\frac{1}{4}$	$E_6{\times}U(1)\vert SU(4)^2{\times}U(1)^2$
41	$(-1,1,0,0,0,0,0,0)\oplus\frac{1}{2}(-1,1,1,1,1,-3,1,-1)$	$-\frac{3}{8},\frac{1}{8}$	$SO(12){\times}SU(2){\times}U(1)\vert SU(7){\times}U(1)^2$
42	$(-1,1,0,0,0,0,0,0)\oplus-\frac{1}{2}(1,1,1,1,-1,3,-1,-1)$	$-\frac{1}{8},\frac{3}{8}$	$SO(12){\times}SU(2){\times}U(1)\vert SU(5){\times}SU(3){\times}U(1)^2$
43	$(-1,-1,-1,0,0,0,0,1)\oplus-\frac{1}{2}(1,1,1,1,1,1,-1,-1)$	$-\frac{1}{4},\frac{1}{4}$	$SO(12){\times}SU(2){\times}U(1)\vert SU(7){\times}U(1)^2$
44	$(-1,-1,-1,0,0,0,0,1)\oplus\frac{1}{2}(-1,-1,-1,-1,1,1,1,1)$	$-\frac{1}{8},\frac{3}{8}$	$SO(12){\times}SU(2){\times}U(1)\vert SU(5){\times}SU(3){\times}U(1)^2$
45	$(-1,0,-1,-1,0,0,0,1)\oplus(0,0,0,0,-1,-1,0,0)$	$-\frac{1}{8},\frac{3}{8}$	$SU(8){\times}U(1)\vert SU(6){\times}SU(2){\times}U(1)^2$
46	$(-1,0,-1,-1,0,0,0,1)\oplus(-1,0,0,0,0,0,1,0)$	$0,\frac{1}{2}$	$SU(8){\times}U(1)\vert SU(6){\times}SU(2){\times}U(1)^2$
	$4 \cdot V_4 = (2,1,1,0,0,0,0,0) \oplus (2,0,0,0,0,0,0,0)$		
47	$(0,0,0,0,0,0,0,0)\oplus-\frac{1}{2}(1,1,1,1,1,1,-1,-1)$	$0,\frac{1}{2}$	$E_6{\times}SU(2){\times}U(1)\vert SU(7){\times}U(1)^2$
48	$(-1,-1,0,0,0,0,0,0)\oplus(0,0,0,0,0,0,0,0)$	$-\frac{1}{8},\frac{3}{8}$	$E_6{\times}U(1)^2\vert SO(14){\times}U(1)$
49	$(-1,-1,0,0,0,0,0,0)\oplus(0,0,0,0,0,-2,0,0)$	$-\frac{1}{8},\frac{3}{8}$	$E_6{\times}U(1)^2\vert SO(14){\times}U(1)$
50	$(-1,-1,0,0,0,0,0,0)\oplus(-1,-1,-1,0,0,0,0,1)$	$-\frac{1}{4},\frac{1}{4}$	$E_6{\times}U(1)^2\vert SO(8){\times}SU(4){\times}U(1)$
51	$(-1,0,0,0,0,0,1,0)\oplus\frac{1}{2}(-3,-1,-1,1,1,1,1,1)$	$-\frac{1}{4},\frac{1}{4}$	$SO(10){\times}SU(2){\times}U(1)^2\vert SU(7){\times}U(1)^2$
52	$(-1,0,1,0,0,0,0,0)\oplus(0,0,0,0,0,0,0,0)$	$0,\frac{1}{2}$	$SO(10){\times}U(1)^3\vert SO(14){\times}U(1)$
53	$(-1,0,1,0,0,0,0,0)\oplus(0,0,0,0,0,-2,0,0)$	$0,\frac{1}{2}$	$SO(10){\times}U(1)^3\vert SO(14){\times}U(1)$
54	$(-1,0,1,0,0,0,0,0)\oplus(-1,-1,-1,0,0,0,0,1)$	$-\frac{1}{8},\frac{3}{8}$	$SO(10){\times}U(1)^3\vert SO(8){\times}SU(4){\times}U(1)$
55	$(0,0,0,-1,0,0,1,0)\oplus\frac{1}{2}(-3,-1,-1,1,1,1,1,1)$	$-\frac{1}{8},\frac{3}{8}$	$SU(6){\times}SU(2)^2{\times}U(1)\vert SU(7){\times}U(1)^2$
56	$(0,-1,1,0,0,0,0,0)\oplus\frac{1}{2}(-3,-1,-1,1,1,1,1,1)$	$-\frac{1}{8},\frac{3}{8}$	$E_6{\times}SU(2){\times}U(1)\vert SU(7){\times}U(1)^2$
57	$(-1,-1,-1,0,0,0,0,1)\oplus-\frac{1}{2}(1,1,1,1,1,1,-1,-1)$	$-\frac{1}{4},\frac{1}{4}$	$SO(10){\times}SU(2){\times}U(1)^2\vert SU(7){\times}U(1)^2$
58	$(-1,-1,0,-1,0,0,0,1)\oplus(-1,0,0,0,0,0,1,0)$	$-\frac{1}{4},\frac{1}{4}$	$SU(6){\times}SU(2){\times}U(1)^2\vert SO(12){\times}U(1)^2$
59	$(-1,-1,0,-1,0,0,0,1)\oplus(0,-1,0,0,0,0,1,0)$	$-\frac{1}{8},\frac{3}{8}$	$SU(6){\times}SU(2)^2{\times}U(1)\vert SO(10){\times}SU(2)^2{\times}U(1)$
60	$(-1,0,0,-1,-1,0,0,1)\oplus-\frac{1}{2}(1,1,1,1,1,1,-1,-1)$	$-\frac{1}{8},\frac{3}{8}$	$SU(6){\times}SU(2)^2{\times}U(1)\vert SU(7){\times}U(1)^2$

	$2 \cdot V_2$	Φ	Group Decomposition
	$4 \cdot V_4 = (2,1,1,0,0,0,0,0) \oplus (2,2,2,0,0,0,0,0)$		
61	$(0,0,0,0,0,0,0,0)\oplus-\frac{1}{2}(1,1,1,1,1,1,-1,-1)$	$-\frac{1}{8},\frac{3}{8}$	$E_6\times SU(2)\times U(1)\vert SU(5)\times SU(3)\times U(1)^2$
62	$(-1,-1,0,0,0,0,0,0)\oplus(0,0,0,0,0,0,0,0)$	$-\frac{1}{8},\frac{3}{8}$	$E_6\times U(1)^2\vert SO(10)\times SU(4)$
63	$(-1,-1,0,0,0,0,0,0)\oplus(0,0,0,0,0,-2,0,0)$	$-\frac{1}{8},\frac{3}{8}$	$E_6\times U(1)^2\vert SO(10)\times SU(4)$
64	$(-1,-1,0,0,0,0,0,0)\oplus(-1,-1,-1,0,0,0,0,1)$	$0,\frac{1}{2}$	$E_6\times U(1)^2\vert SO(8)\times SU(4)\times U(1)$
65	$(-1,-1,0,0,0,0,0,0)\oplus(-1,-1,0,-1,0,0,0,1)$	$-\frac{3}{8},\frac{1}{8}$	$E_6\times U(1)^2\vert SU(4)\times SU(2)^4\times U(1)$
66	$(-1,0,0,0,0,0,1,0)\oplus\frac{1}{2}(-3,-1,-1,1,1,1,1,1)$	$-\frac{3}{8},\frac{1}{8}$	$SO(10)\times SU(2)\times U(1)^2\vert SU(5)\times SU(3)\times U(1)^2$
67	$(-1,0,1,0,0,0,0,0)\oplus(0,0,0,0,0,0,0,0)$	$0,\frac{1}{2}$	$SO(10)\times U(1)^3\vert SO(10)\times SU(4)$
68	$(-1,0,1,0,0,0,0,0)\oplus(0,0,0,0,0,-2,0,0)$	$0,\frac{1}{2}$	$SO(10)\times U(1)^3\vert SO(10)\times SU(4)$
69	$(-1,0,1,0,0,0,0,0)\oplus(-1,-1,-1,0,0,0,0,1)$	$-\frac{1}{8},\frac{3}{8}$	$SO(10)\times U(1)^3\vert SO(8)\times SU(4)\times U(1)$
70	$(-1,0,1,0,0,0,0,0)\oplus(-1,-1,0,-1,0,0,0,1)$	$-\frac{3}{8},\frac{1}{8}$	$SO(10)\times U(1)^3\vert SU(4)\times SU(2)^4\times U(1)$
71	$(0,0,0,-1,0,0,1,0)\oplus\frac{1}{2}(-3,-1,-1,1,1,1,1,1)$	$-\frac{1}{4},\frac{1}{4}$	$SU(6)\times SU(2)^2\times U(1)\vert SU(5)\times SU(3)\times U(1)^2$
72	$(0,-1,1,0,0,0,0,0)\oplus\frac{1}{2}(-3,-1,-1,1,1,1,1,1)$	$-\frac{1}{4},\frac{1}{4}$	$E_6\times SU(2)\times U(1)\vert SU(5)\times SU(3)\times U(1)^2$
73	$(-1,-1,-1,0,0,0,0,1)\oplus-\frac{1}{2}(1,1,1,1,1,1,-1,-1)$	$-\frac{3}{8},\frac{1}{8}$	$SO(10)\times SU(2)\times U(1)^2\vert SU(5)\times SU(3)\times U(1)^2$
74	$(-1,-1,0,-1,0,0,0,1)\oplus(-1,-1,0,0,0,0,0,0)$	$-\frac{3}{8},\frac{1}{8}$	$SU(6)\times SU(2)\times U(1)^2\vert SO(10)\times SU(2)^2\times U(1)$
75	$(-1,-1,0,-1,0,0,0,1)\oplus(-1,0,0,0,0,0,1,0)$	$-\frac{1}{4},\frac{1}{4}$	$SU(6)\times SU(2)\times U(1)^2\vert SO(8)\times SU(2)^2\times U(1)^2$
76	$(-1,-1,0,-1,0,0,0,1)\oplus(0,0,0,-1,0,0,1,0)$	$-\frac{1}{8},\frac{3}{8}$	$SU(6)\times SU(2)\times U(1)^2\vert SU(4)^2\times SU(2)^2$
77	$(-1,0,0,-1,-1,0,0,1)\oplus-\frac{1}{2}(1,1,1,1,1,1,-1,-1)$	$-\frac{1}{4},\frac{1}{4}$	$SU(6)\times SU(2)^2\times U(1)\vert SU(5)\times SU(3)\times U(1)^2$
	$4 \cdot V_4 = (4,0,0,0,0,0,0,0) \oplus (3,1,0,0,0,0,0,0)$		
78	$(0,0,0,0,0,0,0,0)\oplus(-1,0,0,0,0,0,1,0)$	$-\frac{1}{8},\frac{3}{8}$	$SO(16)\vert SO(10)\times U(1)^3$
79	$(0,0,0,0,0,0,0,0)\oplus-\frac{1}{2}(1,-1,1,1,1,1,1,-1)$	$0,\frac{1}{2}$	$SO(16)\vert SU(6)\times SU(2)\times U(1)^2$
80	$(-1,0,0,0,0,0,1,0)\oplus\frac{1}{2}(-3,-1,-1,1,1,1,1,1)$	$0,\frac{1}{2}$	$SO(12)\times SU(2)^2\vert SU(6)\times SU(2)\times U(1)^2$
81	$(-1,0,0,0,0,0,1,0)\oplus(-1,0,-1,-1,0,0,0,1)$	$-\frac{3}{8},\frac{1}{8}$	$SO(12)\times SU(2)^2\vert SU(4)^2\times U(1)^2$
82	$-\frac{1}{2}(1,1,1,1,1,1,-1,-1)\oplus\frac{1}{2}(-3,-1,-1,1,1,1,1,1)$	$-\frac{3}{8},\frac{1}{8}$	$SU(8)\times U(1)\vert SU(6)\times SU(2)\times U(1)^2$
83	$-\frac{1}{2}(1,1,1,1,1,1,-1,-1)\oplus(-1,0,-1,-1,0,0,0,1)$	$-\frac{1}{4},\frac{1}{4}$	$SU(8)\times U(1)\vert SU(4)^2\times U(1)^2$
84	$(0,0,0,0,0,-2,0,0)\oplus(-1,0,0,0,0,0,1,0)$	$-\frac{1}{8},\frac{3}{8}$	$SO(14)\times U(1)\vert SO(10)\times U(1)^3$
85	$(0,0,0,0,0,-2,0,0)\oplus-\frac{1}{2}(1,-1,1,1,1,1,1,-1)$	$0,\frac{1}{2}$	$SO(14)\times U(1)\vert SU(6)\times SU(2)\times U(1)^2$
86	$\frac{1}{2}(-3,-1,-1,1,1,1,1,1)\oplus(-1,0,0,0,0,0,1,0)$	$0,\frac{1}{2}$	$SU(7)\times U(1)^2\vert SO(10)\times U(1)^3$
87	$\frac{1}{2}(-3,-1,-1,1,1,1,1,1)\oplus-\frac{1}{2}(1,-1,1,1,1,1,1,-1)$	$-\frac{3}{8},\frac{1}{8}$	$SU(7)\times U(1)^2\vert SU(6)\times SU(2)\times U(1)^2$
88	$(-1,-1,-1,0,0,0,0,1)\oplus(-1,0,0,0,0,0,1,0)$	$-\frac{3}{8},\frac{1}{8}$	$SO(8)\times SU(4)\times U(1)\vert SO(10)\times U(1)^3$
89	$(-1,-1,-1,0,0,0,0,1)\oplus-\frac{1}{2}(1,-1,1,1,1,1,1,-1)$	$-\frac{1}{4},\frac{1}{4}$	$SO(8)\times SU(4)\times U(1)\vert SU(6)\times SU(2)\times U(1)^2$
	$4 \cdot V_4 = (2,0,0,0,0,0,0,0) \oplus (3,1,1,1,1,1,0,0)$		
90	$(0,0,0,0,0,0,0,0)\oplus(-1,0,0,0,0,0,1,0)$	$-\frac{1}{8},\frac{3}{8}$	$SO(14)\times U(1)\vert SU(6)\times SU(2)\times U(1)^2$
91	$(-1,0,0,0,0,0,1,0)\oplus\frac{1}{2}(-1,-3,1,1,1,1,-1,1)$	$-\frac{1}{8},\frac{3}{8}$	$SO(12)\times U(1)^2\vert SU(8)\times U(1)$
92	$(-1,0,0,0,0,0,1,0)\oplus(-1,-1,-1,0,0,0,0,1)$	$-\frac{3}{8},\frac{1}{8}$	$SO(12)\times U(1)^2\vert SU(4)^2\times U(1)^2$
93	$-\frac{1}{2}(1,1,1,1,1,1,-1,-1)\oplus(0,0,0,0,0,0,0,0)$	$0,\frac{1}{2}$	$SU(7)\times U(1)^2\vert SU(8)\times SU(2)$
94	$-\frac{1}{2}(1,1,1,1,1,1,-1,-1)\oplus(-1,0,0,1,-1,1,0,0)$	$-\frac{1}{8},\frac{3}{8}$	$SU(7)\times U(1)^2\vert SU(6)\times SU(2)^2\times U(1)$
95	$-\frac{1}{2}(1,1,1,1,1,1,-1,-1)\oplus(0,-1,-1,0,0,0,-1,1)$	$-\frac{1}{8},\frac{3}{8}$	$SU(7)\times U(1)^2\vert SU(4)^2\times SU(2)\times U(1)$
96	$(0,-1,0,0,0,0,1,0)\oplus\frac{1}{2}(-1,-3,1,1,1,1,-1,1)$	$0,\frac{1}{2}$	$SO(10)\times SU(2)^2\times U(1)\vert SU(8)\times U(1)$
97	$(0,-1,0,0,0,0,1,0)\oplus(-1,-1,-1,0,0,0,0,1)$	$-\frac{1}{4},\frac{1}{4}$	$SO(10)\times SU(2)^2\times U(1)\vert SU(4)^2\times U(1)^2$
98	$(0,0,0,0,0,-2,0,0)\oplus(-1,0,0,0,0,0,1,0)$	$-\frac{1}{8},\frac{3}{8}$	$SO(14)\times U(1)\vert SU(6)\times SU(2)\times U(1)^2$
99	$\frac{1}{2}(-3,-1,-1,1,1,1,1,1)\oplus(-1,-1,0,0,0,0,0,0)$	$-\frac{3}{8},\frac{1}{8}$	$SU(7)\times U(1)^2\vert SU(6)\times SU(2)^2\times U(1)$
100	$\frac{1}{2}(-3,-1,-1,1,1,1,1,1)\oplus(-1,0,0,0,0,1,0,0)$	$-\frac{1}{4},\frac{1}{4}$	$SU(7)\times U(1)^2\vert SU(4)^2\times SU(2)\times U(1)$
101	$\frac{1}{2}(-3,-1,-1,1,1,1,1,1)\oplus(0,0,0,0,0,0,-1,1)$	$-\frac{1}{8},\frac{3}{8}$	$SU(7)\times U(1)^2\vert SU(8)\times SU(2)$
102	$(-1,-1,-1,0,0,0,0,1)\oplus(-1,0,0,0,0,0,1,0)$	$-\frac{1}{4},\frac{1}{4}$	$SO(8)\times SU(4)\times U(1)\vert SU(6)\times SU(2)\times U(1)^2$

	$2 \cdot V_2$	Φ	Group Decomposition
	$4 \cdot V_4 = (3,1,0,0,0,0,0,0) \oplus (1,1,1,1,1,1,1,-1)$		
103	$(0,0,0,0,0,0,0,0) \oplus -\frac{1}{2}(1,1,1,1,1,1,-1,-1)$	$-\frac{1}{8}, \frac{3}{8}$	$SO(12) \times SU(2) \times U(1) \vert SU(7) \times U(1)^2$
104	$(0,0,0,0,0,0,0,0) \oplus \frac{1}{2}(-1,-1,-1,-1,1,1,1,1)$	$0, \frac{1}{2}$	$SO(12) \times SU(2) \times U(1) \vert SU(5) \times SU(3) \times U(1)^2$
105	$(-1,-1,0,0,0,0,0,0) \oplus \frac{1}{2}(-1,1,1,1,1,-3,1,-1)$	$-\frac{1}{8}, \frac{3}{8}$	$SO(12) \times SU(2) \times U(1) \vert SU(7) \times U(1)^2$
106	$(-1,-1,0,0,0,0,0,0) \oplus -\frac{1}{2}(1,1,1,1,-1,3,-1,-1)$	$-\frac{3}{8}, \frac{1}{8}$	$SO(12) \times SU(2) \times U(1) \vert SU(5) \times SU(3) \times U(1)^2$
107	$(-1,0,0,0,0,0,1,0) \oplus (0,0,0,0,0,0,0,0)$	$-\frac{1}{8}, \frac{3}{8}$	$SO(10) \times U(1)^3 \vert SU(8) \times U(1)$
108	$(-1,0,0,0,0,0,1,0) \oplus (0,0,0,0,0,-2,0,0)$	$-\frac{1}{4}, \frac{1}{4}$	$SO(10) \times U(1)^3 \vert SU(8) \times U(1)$
109	$(-1,0,0,0,0,0,1,0) \oplus (-1,-1,-1,0,0,0,0,1)$	$-\frac{3}{8}, \frac{1}{8}$	$SO(10) \times U(1)^3 \vert SU(4)^2 \times U(1)^2$
110	$(-1,1,0,0,0,0,0,0) \oplus \frac{1}{2}(-1,1,1,1,1,-3,1,-1)$	$0, \frac{1}{2}$	$SO(12) \times SU(2) \times U(1) \vert SU(7) \times U(1)^2$
111	$(-1,1,0,0,0,0,0,0) \oplus -\frac{1}{2}(1,1,1,1,-1,3,-1,-1)$	$-\frac{1}{4}, \frac{1}{4}$	$SO(12) \times SU(2) \times U(1) \vert SU(5) \times SU(3) \times U(1)^2$
112	$-\frac{1}{2}(1,1,1,1,1,1,-1,-1) \oplus \frac{1}{2}(-1,1,1,1,1,-3,1,-1)$	$0, \frac{1}{2}$	$SU(6) \times U(1)^3 \vert SU(7) \times U(1)^2$
113	$-\frac{1}{2}(1,1,1,1,1,1,-1,-1) \oplus -\frac{1}{2}(1,1,1,1,-1,3,-1,-1)$	$-\frac{1}{4}, \frac{1}{4}$	$SU(6) \times U(1)^3 \vert SU(5) \times SU(3) \times U(1)^2$
114	$-\frac{1}{2}(1,-1,1,1,1,1,1,-1) \oplus (0,0,0,0,0,0,0,0)$	$0, \frac{1}{2}$	$SU(6) \times SU(2) \times U(1)^2 \vert SU(8) \times U(1)$
115	$-\frac{1}{2}(1,-1,1,1,1,1,1,-1) \oplus (0,0,0,0,0,-2,0,0)$	$-\frac{1}{8}, \frac{3}{8}$	$SU(6) \times SU(2) \times U(1)^2 \vert SU(8) \times U(1)$
116	$-\frac{1}{2}(1,-1,1,1,1,1,1,-1) \oplus (-1,-1,-1,0,0,0,0,1)$	$-\frac{1}{4}, \frac{1}{4}$	$SU(6) \times SU(2) \times U(1)^2 \vert SU(4)^2 \times U(1)^2$
117	$(0,0,-1,0,0,0,1,0) \oplus \frac{1}{2}(-1,1,1,1,1,-3,1,-1)$	$-\frac{3}{8}, \frac{1}{8}$	$SO(8) \times SU(2)^3 \times U(1) \vert SU(7) \times U(1)^2$
118	$(0,0,-1,0,0,0,1,0) \oplus -\frac{1}{2}(1,1,1,1,-1,3,-1,-1)$	$-\frac{1}{8}, \frac{3}{8}$	$SO(8) \times SU(2)^3 \times U(1) \vert SU(5) \times SU(3) \times U(1)^2$
119	$(0,0,0,0,0,-2,0,0) \oplus -\frac{1}{2}(1,1,1,1,1,1,-1,-1)$	$-\frac{1}{8}, \frac{3}{8}$	$SO(12) \times SU(2) \times U(1) \vert SU(7) \times U(1)^2$
120	$(0,0,0,0,0,-2,0,0) \oplus \frac{1}{2}(-1,-1,-1,-1,1,1,1,1)$	$0, \frac{1}{2}$	$SO(12) \times SU(2) \times U(1) \vert SU(5) \times SU(3) \times U(1)^2$
121	$\frac{1}{2}(-3,-1,-1,1,1,1,1,1) \oplus (0,0,0,0,-1,-1,0,0)$	$-\frac{3}{8}, \frac{1}{8}$	$SU(6) \times SU(2) \times U(1)^2 \vert SU(6) \times SU(2) \times U(1)^2$
122	$\frac{1}{2}(-3,-1,-1,1,1,1,1,1) \oplus (-1,0,0,0,0,0,1,0)$	$-\frac{1}{4}, \frac{1}{4}$	$SU(6) \times SU(2) \times U(1)^2 \vert SU(6) \times SU(2) \times U(1)^2$
123	$\frac{1}{2}(-3,1,-1,-1,1,1,1,1) \oplus -\frac{1}{2}(1,1,1,1,1,1,-1,-1)$	$-\frac{3}{8}, \frac{1}{8}$	$SU(6) \times U(1)^3 \vert SU(7) \times U(1)^2$
124	$\frac{1}{2}(-3,1,-1,-1,1,1,1,1) \oplus \frac{1}{2}(-1,-1,-1,-1,1,1,1,1)$	$-\frac{1}{4}, \frac{1}{4}$	$SU(6) \times U(1)^3 \vert SU(5) \times SU(3) \times U(1)^2$
125	$(-1,-1,-1,0,0,0,0,1) \oplus -\frac{1}{2}(1,1,1,1,1,1,-1,-1)$	$-\frac{3}{8}, \frac{1}{8}$	$SO(8) \times SU(2)^3 \times U(1) \vert SU(7) \times U(1)^2$
126	$(-1,-1,-1,0,0,0,0,1) \oplus \frac{1}{2}(-1,-1,-1,-1,1,1,1,1)$	$-\frac{1}{4}, \frac{1}{4}$	$SO(8) \times SU(2)^3 \times U(1) \vert SU(5) \times SU(3) \times U(1)^2$
127	$(-1,0,-1,-1,0,0,0,1) \oplus (0,0,0,0,-1,-1,0,0)$	$-\frac{1}{4}, \frac{1}{4}$	$SU(4)^2 \times U(1)^2 \vert SU(6) \times SU(2) \times U(1)^2$
128	$(-1,0,-1,-1,0,0,0,1) \oplus (-1,0,0,0,0,0,1,0)$	$-\frac{1}{8}, \frac{3}{8}$	$SU(4)^2 \times U(1)^2 \vert SU(6) \times SU(2) \times U(1)^2$
	$4 \cdot V_4 = (2,2,2,0,0,0,0,0) \oplus (3,1,1,1,1,1,0,0)$		
129	$(0,0,0,0,0,0,0,0) \oplus (-1,0,0,0,0,0,1,0)$	$-\frac{1}{8}, \frac{3}{8}$	$SO(10) \times SU(4) \vert SU(6) \times SU(2) \times U(1)^2$
130	$(-1,-1,0,0,0,0,0,0) \oplus \frac{1}{2}(-1,-3,1,1,1,1,-1,1)$	$-\frac{1}{4}, \frac{1}{4}$	$SO(10) \times SU(2)^2 \times U(1) \vert SU(8) \times U(1)$
131	$(-1,-1,0,0,0,0,0,0) \oplus (-1,-1,-1,0,0,0,0,1)$	$0, \frac{1}{2}$	$SO(10) \times SU(2)^2 \times U(1) \vert SU(4)^2 \times U(1)^2$
132	$(-1,0,0,0,0,0,1,0) \oplus \frac{1}{2}(-1,-3,1,1,1,1,-1,1)$	$-\frac{1}{8}, \frac{3}{8}$	$SO(8) \times SU(2)^2 \times U(1)^2 \vert SU(8) \times U(1)$
133	$(-1,0,0,0,0,0,1,0) \oplus (-1,-1,-1,0,0,0,0,1)$	$-\frac{3}{8}, \frac{1}{8}$	$SO(8) \times SU(2)^2 \times U(1)^2 \vert SU(4)^2 \times U(1)^2$
134	$-\frac{1}{2}(1,1,1,1,1,1,-1,-1) \oplus (0,0,0,0,0,0,0,0)$	$-\frac{1}{8}, \frac{3}{8}$	$SU(5) \times SU(3) \times U(1)^2 \vert SU(8) \times SU(2)$
135	$-\frac{1}{2}(1,1,1,1,1,1,-1,-1) \oplus (-1,0,0,1,-1,1,0,0)$	$-\frac{1}{4}, \frac{1}{4}$	$SU(5) \times SU(3) \times U(1)^2 \vert SU(8) \times SU(2)$
136	$-\frac{1}{2}(1,1,1,1,1,1,-1,-1) \oplus (0,-1,-1,0,0,0,-1,1)$	$-\frac{1}{4}, \frac{1}{4}$	$SU(5) \times SU(3) \times U(1)^2 \vert SU(4)^2 \times SU(2) \times U(1)$
137	$(0,0,0,-1,0,0,1,0) \oplus \frac{1}{2}(-1,-3,1,1,1,1,-1,1)$	$0, \frac{1}{2}$	$SU(4)^2 \times SU(2)^2 \vert SU(8) \times U(1)$
138	$(0,0,0,-1,0,0,1,0) \oplus (-1,-1,-1,0,0,0,0,1)$	$-\frac{1}{4}, \frac{1}{4}$	$SU(4)^2 \times SU(2)^2 \vert SU(4)^2 \times U(1)^2$
139	$(0,0,0,0,0,-2,0,0) \oplus (-1,0,0,0,0,0,1,0)$	$-\frac{1}{8}, \frac{3}{8}$	$SO(10) \times SU(4) \vert SU(6) \times SU(2) \times U(1)^2$
140	$\frac{1}{2}(-3,-1,-1,1,1,1,1,1) \oplus (-1,-1,0,0,0,0,0,0)$	$0, \frac{1}{2}$	$SU(5) \times SU(3) \times U(1)^2 \vert SU(6) \times SU(2)^2 \times U(1)$
141	$\frac{1}{2}(-3,-1,-1,1,1,1,1,1) \oplus (-1,0,0,0,0,1,0,0)$	$-\frac{3}{8}, \frac{1}{8}$	$SU(5) \times SU(3) \times U(1)^2 \vert SU(4)^2 \times SU(2) \times U(1)$
142	$\frac{1}{2}(-3,-1,-1,1,1,1,1,1) \oplus (0,0,0,0,0,0,-1,1)$	$-\frac{1}{4}, \frac{1}{4}$	$SU(5) \times SU(3) \times U(1)^2 \vert SU(8) \times SU(2)$
143	$(-1,-1,-1,0,0,0,0,1) \oplus (-1,0,0,0,0,0,1,0)$	$0, \frac{1}{2}$	$SO(8) \times SU(4) \times U(1) \vert SU(6) \times SU(2) \times U(1)^2$
144	$(-1,-1,0,-1,0,0,0,1) \oplus (-1,0,0,0,0,0,1,0)$	$-\frac{3}{8}, \frac{1}{8}$	$SU(4) \times SU(2)^4 \times U(1) \vert SU(6) \times SU(2) \times U(1)^2$

Appendix C

Matter Content for Models with an SO(10) Factor

In the above table only the chiral fields which are charged under SO(10) are shown.

Model	Untwisted	(0,1)	(0,2)		(0,3)	(1,0)		(1,1)	(1,2)		(1,3)
4	4(10), 2($\overline{16}$)	10,16	10	10,16	10,16	16	10,16		16	10,16	10
		10,$\overline{16}$	10	10,16	10,$\overline{16}$	10	10,16		10	10,16	$\overline{16}$
8	4(10), 2($\overline{16}$)	10,16	10	10,16	10,16						
		10,$\overline{16}$	10	10,16	10,$\overline{16}$						
16	4(10), 2($\overline{16}$)		16	10,16	10	16	10,16		10	10,16	
		10	16	10,16		10	10,16		16	10,16	
24	4(10), 2($\overline{16}$)		16	10,16	10						
		10	16	10,16							
26	4(10), 2($\overline{16}$)	10	$\overline{16}$	10,$\overline{16}$	10						
51	(1, 16), (1, $\overline{16}$), (2, 10), (2, $\overline{16}$)	(1, 10)			(1, $\overline{16}$)					(1, 10)	
		(1, 16)			(1, 10)				(1, 10)	(1, 10)	
52	2(10), 2(16), 2($\overline{16}$)	16			10	16	10,16				10
		10			$\overline{16}$	10	10,16				
53	2(10), 2(16), 2($\overline{16}$)	16			10				10	10,$\overline{16}$	
		10			$\overline{16}$				$\overline{16}$	10,$\overline{16}$	10
54	2(10), 2(16), 2($\overline{16}$)	16			10						
		10			$\overline{16}$						
57	(1, 16), (1, $\overline{16}$), (2, 10), (2, $\overline{16}$)	(1, 10)			(1, 16)	(1, 10)	(1, 10)				
		(1, $\overline{16}$)			(1, 10)		(1, 10)				
59	(2, 2, 10),(1, 2, 16), (2, 1, 16)	(1, 1, 10)		(1, 1, 10)							
				(1, 1, 10)	(1, 1, 10)						
62	(4, $\overline{16}$)	(1, 16)	(1, 10)	(1, 10)						(1, 10)	
			(1, 10)	(1, 10)	(1, $\overline{16}$)				(1, 10)	(1, 10)	(1, $\overline{16}$)
63	(4, 16)		(1, 10)	(1, 10)	(1, $\overline{16}$)		(1, 10)				
		(1, 16)	(1, 10)	(1, 10)		(1, 10)	(1, 10)				(1, 16)
66	(1, 16), (1, $\overline{16}$), (2, 10), (2, $\overline{16}$)								(1, 10)	(1, 10)	
										(1, 10)	

Model	Untwisted	(0,1)	(0,2)		(0,3)	(1,0)		(1,1)	(1,2)		(1,3)
67_1	2(10), 2(16),					16	10,16				
	2($\overline{16}$)					10	10,16				
67_2	(4, $\overline{16}$)	(1, 16)	(1, 10)	(1, 10)					(1, 10)	(1, 10)	
			(1, 10)	(1, 10)	(1, $\overline{16}$)					(1, 10)	
68_1	2(10), 2(16),								$\overline{16}$	10,$\overline{16}$	
	2($\overline{16}$)								10	10,$\overline{16}$	
68_2	(4, 16)		(1, 10)	(1, 10)	(1, $\overline{16}$)	(1, 10)	(1, 10)				
		(1, 16)	(1, 10)	(1, 10)			(1, 10)				
69	2(10), 2(16),										
	2($\overline{16}$)										10
70	2(10), 2(16),										
	2($\overline{16}$)										
73	(1, 16), (1, $\overline{16}$),						(1, 10)				
	(2, 10), (2, $\overline{16}$)					(1, 10)	(1, 10)				
74	(2, 1, 16)		(1, 1, 10)	(1, 1, 10)	(1, 1, $\overline{16}$)						
	(1, 2, $\overline{16}$)	(1, 1, 16)	(1, 1, 10)	(1, 1, 10)							
78	4(10), 2($\overline{16}$)		10	10,16		16	10,16		16	10,16	
			10	10,16		10	10,16		10	10,16	
84	4(10), 2($\overline{16}$)		10	10,16							$\overline{16}$
			10	10,16							10
86	4(10), 2($\overline{16}$)		10	10,$\overline{16}$							
			10	10,$\overline{16}$							
88	4(10), 2($\overline{16}$)		10	10,16							
			10	10,16							
96	(2, 2, 10),(1, 2, 16),		(1, 1, 10)	(1, 1, 10)							
	(2, 1, 16)		(1, 1, 10)	(1, 1, 10)							(1, 1, 16)
97	(2, 2, 10),(1, 2, 16),		(1, 1, 10)	(1, 1, 10)							
	(2, 1, 16)		(1, 1, 10)	(1, 1, 10)							
107	4(10), 2($\overline{16}$)				10	16	10,16				
		10				10	10,16				
108	4(10), 2($\overline{16}$)				10				10	10,16	
		10							16	10,16	
109	4(10), 2($\overline{16}$)				10						
		10									
129	(4, $\overline{16}$)			(1, 10)					(1, 10)	(1, 10)	
				(1, 10)						(1, 10)	
130	(2, 1, 16)			(1, 1, 10)							(1, 1, 10)
	(1, 2, $\overline{16}$)			(1, 1, 10)							
131	(2, 1, 16)			(1, 1, 10)							
	(1, 2, $\overline{16}$)			(1, 1, 10)							
139	(4, 16)			(1, 10)			(1, 10)				
				(1, 10)		(1, 10)	(1, 10)				

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