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Phenomenological Aspects of Local F-Theory Models

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Contents

1. Introduction

The world, as we observe it, is made up of particles belonging to the Standard Model (SM) of particle physics, which is a quantum field theory based on the gauge symmetry $SU(3)_C \times SU(2)_L \times U(1)_Y$ governing the strong, weak and electromagnetic forces. Also, there is Einstein's theory of general relativity, which is a classical theory. The history of theoretical physics shows that it has always been an important goal to find a unified description for phenomena that at first sight don't seem to have much in common. Electromagnetism, at the heart of which lie Maxwell's equations, as well as the electroweak unification within the SM at a scale of about 100 GeV are examples for unified frameworks, and there is evidence that the three gauge couplings of the SM also meet at a certain energy scale.

This leads to the idea of grand unified theories (GUTs), which aim at embedding the SM gauge group into a larger group with a single gauge coupling, such as $SU(5)$ [1], $SO(10)$ or E_6. A precise gauge coupling unification at a scale of 10^{16} GeV can only be achieved within the Minimal Supersymmetric Standard Model (MSSM), that is, when supersymmetry (SUSY) is taken into account [2]. This symmetry relates bosons and fermions and, apart from achieving a gauge coupling unification within the MSSM, has the important property of solving the naturalness problem of explaining the small Higgs mass by removing quadratically divergent contributions from the scalar mass terms. SUSY was invented in the '70s, and although there has not been a single experimental proof for its actual existence, theoretical arguments are so strong that it has become a vital ingredient in today's search for a fundamental theory of nature and is hoped to be discovered at the Large Hadron Collider in Geneva within the next few years.

Following the path physicists have successfully been taking for the past centuries, the next step is to arrive at a theory that brings together the forces of the SM and gravity. The most promising candidate is string theory, the fundamental objects of which are not point-like particles but strings. Depending on the way that the strings vibrate and move through space, they are identified with particles of specific quantum numbers, and even the graviton, a spin-two particle mediating gravity, has a natural appearance in form of a string excitation. Working with strings as the smallest ingredients has the advantage that there exists a minimal length scale, the string scale. Therefore, the theory is free of ultraviolet divergences and can serve as a fundamental description of gravity.

A string theory that contains fermions has a supersymmetric spectrum and is therefore called superstring theory. Since this theory is ten-dimensional, it must be compactified on an internal six-dimensional space, which can be done in many ways, leading to a very large number of vacua. It is assumed that some of them can give rise to the MSSM as a low-energy effective theory but so far not a single vacuum has been found that satisfies all requirements. A reasonable approach for model building towards the MSSM is to impose suitable restrictions to narrow the search. String phenomenology has the task to construct certain vacua and deduce their consequences for particle physics.

There are five superstring theories, which are related by dualities and conjectured to be limits of a single theory, called M-theory. For the incorporation of GUTs in the framework of string theory the availability of exceptional symmetries is essential, and these are naturally present in the heterotic $E_8 \times E_8$ string theory [3] compactified on smooth Calabi–Yau (CY) manifolds with vector bundles or on orbifolds. In both cases all aspects of four-dimensional physics are sensitive to the global structure, which means that one must specify a global model. In particular when dealing with CY manifolds, this can be difficult. Another attractive corner for string phenomenology in the perturbative region of the string landscape is based on Type IIB orientifolds with intersecting D-branes, which lead to classical groups. Their advantage is that the particle physics degrees of freedom are confined to branes and their intersections, whereas gravity lives in the bulk and is therefore of different geometrical origin. This allows for so-called *local* model building, where only small parts of the compact space are considered and many questions concerning particle physics can be answered easily, regardless of the details of the full compactification. This bottom-up approach has the benefit that, as long as one works on a local patch, global consistency requirements need not to be taken into account. The obvious shortcoming of local models is that the actual existence of a global model is never guaranteed. Also, certain questions, e.g. moduli stabilization, cannot be answered.

Besides D-branes and O-planes there are more general branes which give rise to exceptional groups. These play an essential role in F-theory, which, unlike M-theory, is not a fundamental theory, but should be thought of as a genuinely non-perturbative description of a class of string vacua accessible from different sides via string dualities [4]. It can be thought of as the correct way to describe Type IIB theory with seven-branes in generic situations where the string coupling cannot be assumed to be small. Two additional auxiliary dimensions are introduced to encode the backreaction of the seven-branes on the ambient, which causes the string coupling to vary along the compact coordinates. Thus, F-theory is formulated in twelve dimensions, the two additional dimensions of which make up a torus which is the fiber of a four-complex-dimensional CY manifold Y. Seven-branes are indicated by the degeneration locus of the elliptic fibration.

So far, F-theory model building has relied on a bottom-up approach similar to Type IIB orientifolds, but it has the great advantage that exceptional gauge groups are available [5, 6]. The general idea of local F-theory GUT models is to decouple the bulk of Y and focus instead on a seven-brane wrapping the submanifold S which carries the gauge group G_{GUT}. The intersections with other branes form curves Σ of complex codimension one, along which matter localized. These are visible as symmetry enhancements of G_{GUT} to $G_\Sigma \supset G_{\mathrm{GUT}}$. Furthermore, these matter curves can intersect in points, where the gauge group enhances even further to G_{P}, leading to localized Yukawa couplings. For an $SU(5)$ GUT, the up- and down-type Yukawa couplings require enhancements to E_6 and $SO(12)$, respectively. The natural presence of the crucial up-type Yukawa coupling is the prime motivation to pursue F-theory as the framework for GUT models.

As usually when dealing with GUTs, a major challenge is to achieve proton longevity. Many processes that lead to proton decay can be evaded by imposing a symmetry, called matter parity. Requiring the existence of this symmetry and furthermore the total absence of proton decay at the local point is a central aspect of this work.

The thesis is structured as follows: Chapter 2 summarizes aspects of the SM, SUSY, the MSSM and GUTs that are needed for the remainder of the work. In chapter 3 I will give the proper definition of F-theory and introduce its mathematical description as well as the setup for the local model building. For a detailed review of F-theory, see for example [7–9]. Chapters 4 and 5 form the main part of this thesis. They are based on [10] and contain an analysis of local F-theory $SU(5)$ GUTs at a point of E_8 symmetry enhancement. The approach differs from previous works that use the same setup, e.g. [11], in that the first priority is to guarantee the absence of proton decay. It turns out that there are exactly two possible definitions of matter parity P_M within the local framework. For each case I will show the possible assignment of matter and Higgs fields to the curves that is consistent with matter parity, proton stability and masses for all SM families. In order to obtain a heavy top quark, its mass is required to be generated at the trilinear level, whereas the other masses, if absent at the trilinear level, are induced at higher order via a mechanism similar to the Froggatt-Nielsen mechanism [12]. For each case essentially only one model remains. Chapter 5 describes the attempt to embed these models in a semilocal scheme using the spectral cover approach [13, 14] as the so far only available tool for the discussion. This framework allows for a description of the eight-dimensional GUT surface and thus constitutes the first step towards a global completion. This effort fails because some assumptions about fluxes that determine the chirality of matter fields and split the Higgs multiplets are not consistent with the semilocal constraints. The remaining part of the thesis deals with a new concept developed in 2010 which goes under the name *T-branes* [15]. In chapter 6 I will explain the idea, introduce the calculus and motivate why T-branes might have the potential to validate or improve the local models that were found. Since this scheme is still very poorly developed, I started to search for rules how to make use of it for practical intentions. The results are shown at the end of chapter 6. Chapter 7 contains a summary and conclusion as well as a detailed outlook offering suggestions how to proceed with my work on T-branes.

2. Preliminaries

2.1. The Standard Model of particle physics

The SM of particle physics describes strong and electroweak interactions based on the gauge group $SU(3)_C \times SU(2)_L \times U(1)_Y$. The twelve spin-one gauge bosons are made up of the eight gluons of $SU(3)_C$, the three W bosons of $SU(2)_L$ and the hypercharge boson B of $U(1)_Y$. The photon and the Z^0 boson are linear combinations of the electrically neutral W^3 boson and the hypercharge boson B. The non-Abelian and Abelian field strength tensors are

$$\begin{aligned} W^a_{\mu\nu} &= \partial_\mu W^a_\nu - \partial_\nu W^a_\mu + g_2 \epsilon^{abc} W^b_\mu W^c_\nu \,, \\ G^m_{\mu\nu} &= \partial_\mu G^m_\nu - \partial_\nu G^m_\mu + g_3 f^{mnp} G^n_\mu G^p_\nu \,, \\ B_{\mu\nu} &= \partial_\mu B_\nu - \partial_\nu B_\mu \,, \end{aligned} \tag{2.1}$$

where f^{abc} denote the structure constants of $SU(3)$.

The fermionic content is given by the following list of three families of left-handed Weyl spinors ($i = 1, 2, 3$ is the family index):

$$\begin{aligned} Q_i &= \begin{pmatrix} u \\ d \end{pmatrix} , \begin{pmatrix} c \\ s \end{pmatrix} , \begin{pmatrix} t \\ b \end{pmatrix} , \\ \bar{u}_i &= \bar{u} \,, \bar{c} \,, \bar{t} \,, \\ \bar{d}_i &= \bar{d} \,, \bar{s} \,, \bar{b} \,, \\ L_i &= \begin{pmatrix} \nu_e \\ e \end{pmatrix} , \begin{pmatrix} \nu_\mu \\ \mu \end{pmatrix} , \begin{pmatrix} \nu_\tau \\ \tau \end{pmatrix} , \\ \bar{e}_i &= \bar{e} \,, \bar{\mu} \,, \bar{\tau} \,. \end{aligned} \tag{2.2}$$

They transform under the SM gauge group as displayed in table 2.1. Their electric charge is given in terms of the third component of the weak isospin T_3 and the hypercharge Y by $Q = T_3 + Y$.

The part of the SM Lagrangean containing the kinetic terms for the fermions and the gauge bosons as well as the interactions between them is

$$\begin{aligned} \mathscr{L} = \;& \mathrm{i} Q^{\dagger i} \bar{\sigma}^\mu D_\mu Q_i + \mathrm{i} \bar{u}^\dagger_i \bar{\sigma}^\mu D_\mu \bar{u}^i + \mathrm{i} \bar{d}^\dagger_i \bar{\sigma}^\mu D_\mu \bar{d}^i + \mathrm{i} L^{\dagger i} \bar{\sigma}^\mu D_\mu L_i + \mathrm{i} \bar{e}^\dagger_i \bar{\sigma}^\mu D_\mu \bar{e} \\ & - \frac{1}{4} W^a_{\mu\nu} W^{\mu\nu a} - \frac{1}{4} G^a_{\mu\nu} G^{\mu\nu a} - \frac{1}{4} B_{\mu\nu} B^{\mu\nu} \,, \end{aligned} \tag{2.3}$$

where D_μ denotes the appropriate covariant derivative. The $\bar{\sigma}^\mu$ are related to the Pauli matrices σ^i through:

$$\sigma^\mu = (\mathbb{1}, \sigma^i) \,, \quad \bar{\sigma}^\mu = (\mathbb{1}, -\sigma^i) \,. \tag{2.4}$$

Since all fermions of the SM are chiral, the gauge symmetries forbid mass terms such that they can only occur after the $SU(2)_L \times U(1)_Y$ symmetry has been spontaneously broken to $U(1)_{\text{EM}}$ by the Higgs mechanism. The Higgs sector is made up of a scalar $SU(2)$ doublet $h = \begin{pmatrix} h^+ \\ h^0 \end{pmatrix}$ with hypercharge $Y = +\frac{1}{2}$ and potential

$$V = \mu^2 h^\dagger h + \lambda (h^\dagger h)^2 \,. \tag{2.5}$$

If μ^2 is negative, h develops a vacuum expectation value (VEV), which can be rotated to

$$\langle h \rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ v \end{pmatrix} , \quad v = \left(\frac{-\mu^2}{\lambda} \right)^{\frac{1}{2}} \,. \tag{2.6}$$

The kinetic term for the Higgs scalars,

$$(D_\mu h)(D^\mu h^\dagger) \,, \tag{2.7}$$

leads to masses for the $W^\pm$ and Z^0 bosons but leaves the photon massless. Furthermore, the Yukawa couplings of the quarks and leptons to the Higgs,

$$\mathbf{y}_u Q h \bar{u} + \mathbf{y}_d Q \tilde{h}^\dagger \bar{d} + \mathbf{y}_e L \tilde{h}^\dagger \bar{e} \,, \tag{2.8}$$

where $\tilde{h} = i\sigma^2 h^*$, give rise to the fermion masses. The terms (2.3), (2.5), (2.7) and (2.8) form the SM Lagrangean.

2.2. Supersymmetry

SUSY is a symmetry between fermions and bosons and it turns out that one new particle needs to be introduced for each known particle. The single-particle states of a supersymmetric theory are contained in irreducible representations of the SUSY algebra, known as supermultiplets, each of which is inhabited by a fermion and a boson. They are called superpartners of each other. One of the most important properties of SUSY is that it removes the quadratic divergences from the mass term of scalar particles, as the quantum corrections involving SM particles are precisely canceled by the contributions of the corresponding superpartners. This protects the Higgs mass from ultraviolet physics. For an introduction, see e.g. [16]. In $\mathcal{N} = 1$ (one supersymmetry generator) SUSY there are two relevant irreducible supermultiplets: The chiral multiplet, containing a Weyl fermion and a complex scalar field, and the vector multiplet, which combines a spin-one vector boson with a Majorana fermion.

Promoting SUSY to a local symmetry leads to supergravity (SUGRA). It is known that the invariance under a local gauge transformation parameterized by a scalar requires the introduction of gauge bosons. Since the transformation parameter of SUSY is a spinorial quantity, the same argument leads to the necessity of a gauge *fermion*. The place of the gauge boson is taken by the spin-3/2 gravitino, whose partner is a spin-two particle with the properties of the graviton, explaining the name supergravity. The full information of a SUGRA theory is contained in three quantities: The superpotential, which is a holomorphic function of the chiral superfields and has the important property that it is not renormalized, the gauge kinetic function, holomorphic as well, and the Kähler potential, a real function that can be written as $K = \Phi^{\dagger i}\Phi_i$ to lowest order, where a sum over all chiral superfields is implied. Particles assigned to the same supermultiplet have the same quantum numbers under the gauge symmetry and, as long as supersymmetry is unbroken, also have the same mass. Since the superpartners of the SM elementary particles have not yet been discovered, SUSY must be broken.

The particles residing in chiral multiplets are all the SM fermions as well as the spin-zero Higgs. Although the left-handed lepton doublet has the same quantum numbers as the Higgs doublet, combining them into a single supermultiplet turns out to have disastrous consequences: It leads to proton decay. Therefore, one new particle has to be introduced for each particle present in the SM. Because a single Higgs chiral supermultiplet would give rise to *one* new fermion, a weak isospin doublet with hypercharge $Y = +1/2$, this would lead to an anomaly of the electroweak gauge symmetry, since the SM fermions already satisfy the anomaly freedom conditions $\mathrm{Tr}[T_3^2 Y] = \mathrm{Tr}[Y^3] = 0$. The problem is evaded by introducing two Higgs supermultiplets: H_u with $Y = +1/2$ and H_d with $Y = -1/2$. They are called up-type Higgs and down-type Higgs, because the former gives mass to up-type-quarks and the latter to down-type-quarks and charged leptons. The second reason why two Higgs supermultiplets are needed is the generation of masses for all particles: Invariance of the Lagrangean under SUSY transformations requires the superpotential to be a holomorphic function of the superfields from which follows that the conjugate fields H_u^* and H_d^* must not appear. Table 2.1 lists all chiral and gauge multiplets that make up the content of the MSSM.

The superpotential for the MSSM is

$$W_{\mathrm{MSSM}} = \bar{u}\mathbf{y}_u Q H_u - \bar{d}\mathbf{y}_d Q H_d - \bar{e}\mathbf{y}_e L H_d + \mu H_u H_d \,, \tag{2.9}$$

where the 3×3 matrices $\mathbf{y}_u$, $\mathbf{y}_d$ and $\mathbf{y}_e$ are the Yukawa coupling parameters. This superpotential defines the MSSM, however, there are more gauge invariant holomorphic terms:

$$\begin{aligned} W_{\Delta L=1} &= \frac{1}{2}\lambda^{ijk} L_i L_j \bar{e}_k + \lambda^{'ijk} L_i Q_j \bar{d}_k + \mu^{'i} L_i H_u \,, \\ W_{\Delta B=1} &= \frac{1}{2}\lambda^{''ijk} \bar{u}_i \bar{d}_j \bar{d}_k \,. \end{aligned} \tag{2.10}$$

Here, $i = 1, 2, 3$ again denotes the family index. These terms violate either baryon number (B) or lepton number (L) conservation. The corresponding processes are not observed, and in particular the absence of proton decay leads to tight constraints. To simply postulate B and L conservation is not a good solution, because on the one hand B and L are anomalous in the SM, and moreover this would lead to problems when trying to build grand unified models, because these do not respect B and L conservation separately. Though $SU(5)$ respects the $U(1)$ symmetry $B - L$, one should rather impose a discrete symmetry, since global $U(1)$ symmetries are in general broken by gravity. A symmetry that accomplishes to forbid all terms in (2.10) but allows for the terms in (2.9) is called matter parity [17, 18]. The basic principle is to assign

Chiral multiplets	Particles	spin 0	spin 1/2	G_{SM}
Q	squarks, quarks	$(\tilde{u}_L, \tilde{d}_L)$	(u_L, d_L)	$(\mathbf{3}, \mathbf{2})_{\frac{1}{6}}$
$\bar{u}$	$\times 3$ families	$\tilde{u}_R^*$	$u_R^\dagger$	$(\bar{\mathbf{3}}, 1)_{-\frac{2}{3}}$
$\bar{d}$		$\tilde{d}_R^*$	$d_R^\dagger$	$(\bar{\mathbf{3}}, 1)_{\frac{1}{3}}$
L	sleptons, leptons	$(\tilde{\nu}, \tilde{e}_L)$	(ν, e_L)	$(\mathbf{1}, \mathbf{2})_{-\frac{1}{2}}$
$\bar{e}$	$\times 3$ families	$\tilde{e}_R^*$	$e_R^\dagger$	$(\mathbf{1}, \mathbf{1})_1$
H_u	Higgs, higgsinos	(H_u^+, H_u^0)	$(\tilde{H}_u^+, \tilde{H}_u^0)$	$(\mathbf{1}, \mathbf{2})_{+\frac{1}{2}}$
H_d		(H_d^0, H_d^-)	$(\tilde{H}_d^0, \tilde{H}_d^-)$	$(\mathbf{1}, \mathbf{2})_{-\frac{1}{2}}$
Vector multiplets	Particles	spin 1/2	spin 1	G_{SM}
g	gluino, gluon	$\tilde{g}$	g	$(\mathbf{8}, \mathbf{1})_0$
$W^{\pm,0}$	winos, W bosons	$\widetilde{W}^\pm,\ \widetilde{W}^0$	$W^\pm,\ W^0$	$(\mathbf{1}, \mathbf{3})_0$
B^0	bino, B boson	$\widetilde{B}^0$	B^0	$(\mathbf{1}, \mathbf{1})_0$

Table 2.1.: *The chiral and vector multiplets of the MSSM. The spin-1/2 fields are left-handed Weyl fermions. G_{SM} denotes the SM gauge group $SU(3)_C \times SU(2)_L \times U(1)_Y$.*

a charge P_M to each supermultiplet and only allow invariant terms to appear in the superpotential. Matter parity is a discrete subgroup of $B - L$ and can be defined as

$$P_M = (-1)^{3(B-L)}. \tag{2.11}$$

The quark and lepton supermultiplets all have $P_M = -1$, whereas the Higgs supermultiplets have $P_M = +1$. Even if matter parity is an exact symmetry, B and L violation can still occur due to higher-dimensional terms in the Lagrangean.

Since we deal with ten-dimensional string theory, a natural starting point is to consider $\mathcal{N} = 4$ SUSY, i.e. the minimal supersymmetry in ten dimensions. It possesses a vector multiplet consisting of a ten-dimensional vector and a Majorana-Weyl spinor. In the eight-dimensional gauge theory the vector is decomposed into an eight-dimensional vector and two scalars, which will play an important role in section 3.5, and the spinor becomes a Weyl spinor. Along the six-dimensional matter curves the latter decomposes into two spinors, one of which forms an $\mathcal{N} = 2$ vector multiplet with the six-dimensional vector and the other one of which is combined with the four scalars into a $\mathcal{N} = 2$ hypermultiplet. In the model which I will examine the matter therefore comes in

$\mathcal{N} = 2$ hypermultiplets, each of which results under a further decomposition to four-dimensional $\mathcal{N} = 1$ SUSY in two chiral multiplets in conjugate representations. The number of zero modes for the fields of different chirality will be determined by gauge fluxes, as will be further explained in section 3.6.

An additional motivation to consider SUSY is that within the MSSM, unlike the SM, the gauge couplings unify at a scale of $M_{\text{GUT}} \approx 2 \times 10^{16}$ GeV. This can be taken as a hint for an underlying grand unified theory.

2.3. Grand unified theories

The easiest way to accomplish a unification of the SM gauge group into a single larger group leads to the Georgi-Glashow model [1], where $SU(3)_C \times SU(2)_L \times U(1)_Y$ is embedded in $SU(5)$. The adjoint representation of $SU(5)$, A^i_j, has dimension $5^2-1 = 24$ and decomposes under $SU(3) \times SU(2) \times U(1)$ as

$$\mathbf{24} = (\mathbf{8}, \mathbf{1})_0 \oplus (\mathbf{1}, \mathbf{3})_0 \oplus (\mathbf{1}, \mathbf{1})_0 \oplus (\bar{\mathbf{3}}, \mathbf{2})_{-\frac{5}{6}} \oplus (\mathbf{3}, \mathbf{2})_{\frac{5}{6}} \,. \tag{2.12}$$

Identifying the first three of the $SU(5)$ indices with the $SU(3)_C$ indices $\alpha = 1, 2, 3$ and the last two with the $SU(2)_L$ indices $a = 4, 5$, A^α_β and A^a_b give rise to the eight gluons G^α_β of $SU(3)_C$ and the vector fields $W^{\pm,0}$ of $SU(2)$, respectively. Since $SU(3)$ and $SU(2)$ are already traceless themselves, there is a linear combination of $A^\alpha_\beta \sim \delta^\alpha_\beta$ and $A^a_b \sim \delta^a_b$ such that $A^i_i = 0$, which is identified with the hypercharge boson B. The entries of A with mixed indices are new gauge bosons and called X and Y bosons. Schematically, the matrix A looks like

$$A = \begin{pmatrix} G - 2B & X, Y \\ X^\dagger, Y^\dagger & W + 3B \end{pmatrix} \,. \tag{2.13}$$

The SM fermions fit exactly into the antifundamental and the two-index antisymmetric representations of $SU(5)$, the $\bar{\mathbf{5}}$ and the $\mathbf{10}$, as can be seen by comparing the representations of the SM fields under $SU(3)_C \times SU(2)_L \times U(1)_Y$, displayed in table 2.1, to the decompositions

$$\bar{\mathbf{5}} = (\bar{\mathbf{3}}, \mathbf{1})_{\frac{1}{3}} \oplus (\mathbf{1}, \mathbf{2})_{-\frac{1}{2}} \,, \tag{2.14a}$$

$$\mathbf{10} = (\bar{\mathbf{3}}, \mathbf{1})_{-\frac{2}{3}} \oplus (\mathbf{3}, \mathbf{2})_{\frac{1}{6}} \oplus (\mathbf{1}, \mathbf{1})_1 \,. \tag{2.14b}$$

So we have:

$$\bar{\mathbf{5}} : \psi^i = \left(\bar{d}^1, \bar{d}^2, \bar{d}^3, e, \nu_e\right)_L \,, \tag{2.15a}$$

$$\mathbf{10} : \psi_{ij} = \begin{pmatrix} 0 & \bar{u}^3 & -\bar{u}^2 & u_1 & d_1 \\ & 0 & \bar{u}^1 & u_2 & d_2 \\ & & 0 & u_3 & d_3 \\ & & & 0 & \bar{e} \\ & & & & 0 \end{pmatrix}_L \,. \tag{2.15b}$$

The $SU(5)$ GUT can be promoted to a supersymmetric model by adding the fermionic partners of the gauge bosons as well as the scalar partners of the quarks and leptons in the respective representations. It is apparent that B and L cannot separately be preserved. The decay of two fermions into another two fermions via the exchange of X and Y bosons is a dimension-six operator and suppressed by $\frac{1}{M_X{}^2}$. Since the unification scale of the supersymmetric model is 2×10^{16} GeV, the predicted proton lifetime is $\tau \geq 10^{33} - 10^{34}$ years and therefore still compatible with observations.

The breaking of $SU(5)$ to $SU(3)_C \times U(1)_{\mathrm{EM}}$ occurs in two steps. First, one can arrive at $SU(3)_C \times SU(2)_L \times U(1)_Y$ by the standard procedure of giving a VEV to a Higgs field H_{24} in the adjoint representation $\mathbf{24}$, because this preserves the rank and contains the total SM singlet $(\mathbf{1},\mathbf{1})_0$. In F-theory there exists an alternative way [6,14]: This is switching on a hypercharge flux

$$F_Y \propto \begin{pmatrix} 2 & & & & \\ & 2 & & & \\ & & 2 & & \\ & & & -3 & \\ & & & & -3 \end{pmatrix} . \tag{2.16}$$

Group-theoretically both options are very similar because both F_Y and $\langle H_{24} \rangle$ are proportional to the hypercharge generator in the adjoint representation. In section 3.6 I will say more on GUT breaking by hypercharge flux. For the second step, i.e. the SM Higgs mechanism breaking $SU(3)_C \times SU(2)_L \times U(1)_Y$ further to $SU(3)_C \times U(1)_{\mathrm{EM}}$, a VEV for a Higgs field in the fundamental representation $\mathbf{5}$ is needed, because this contains the familiar $SU(2)$ Higgs doublet. Of course, in the supersymmetric model one fundamental representation for the up-type Higgs H_u and one antifundamental for the down-type Higgs H_d are needed to get the Yukawa couplings

$$(\mathbf{y}_u)_{ij} \mathbf{5}_{H_u} \mathbf{10}_{Mi} \mathbf{10}_{Mj} \,, \quad (\mathbf{y}_d)_{ij} \bar{\mathbf{5}}_{H_d} \bar{\mathbf{5}}_{Mi} \mathbf{10}_{Mj} \,, \tag{2.17}$$

where i and j are family indices. The first term gives masses to up-type quarks and the second term to down-type quarks and charged leptons. Additionally, the two Higgs fields contain triplets that lead to proton decay through the terms in (2.17). The challenge to give a large mass to the triplets while leaving the doublets light, which are important ingredients in the low-energy theory, runs under the name doublet-triplet-splitting problem. In the non-supersymmetric model the decay via Higgs triplets is a dimension-six operator and therefore suppressed by $\frac{1}{M_X{}^2}$. This process can still occur in the supersymmetric theory, but in addition two fermions can now decay into two scalars, which corresponds to a dimension-five operator, suppressed by only $\frac{1}{M_X}$. All possible dimension-five B or L violating operators from the superpotential are given by

$$\begin{aligned} W^{(5\mathrm{d})}_{\not{B},\not{L}} &= W^1_{ijkl} \mathbf{10}^i_M \mathbf{10}^j_M \mathbf{10}^k_M \bar{\mathbf{5}}^l_M + W^2_{ijk} \mathbf{10}^i_M \mathbf{10}^j_M \mathbf{10}^k_M \bar{\mathbf{5}}_{H_d} \\ &\quad + W^3_{ij} \bar{\mathbf{5}}^i_M \bar{\mathbf{5}}^j_M \mathbf{5}_{H_u} \mathbf{5}_{H_u} + W^4_i \bar{\mathbf{5}}^i_M \bar{\mathbf{5}}_{H_d} \mathbf{5}_{H_u} \mathbf{5}_{H_u} \end{aligned} \tag{2.18}$$

and from the Kähler potential

$$K^{(5\mathrm{d})}_{\not{B},\not{L}} = K^1_{ijk} \mathbf{10}^i_M \mathbf{10}^j_M \mathbf{5}^k_M + K^2_i \bar{\mathbf{5}}_{H_u} \bar{\mathbf{5}}_{H_d} \mathbf{10}^i_M \,. \tag{2.19}$$

In addition there is the dimension-four operator that gives rise to the terms in (2.10),

$$W^{(4d)}_{\not B,\not L} = \lambda_{ijk}\mathbf{\overline{5}}^i_M\mathbf{\overline{5}}^j_M\mathbf{10}^k_M \,, \tag{2.20}$$

and the dimension-three operator

$$W^{(3d)}_{\not B,\not L} = \mu'_i\mathbf{\overline{5}}^i_M\mathbf{5}_{H_u} \,. \tag{2.21}$$

Matter parity forbids all operators in (2.18)–(2.21) except for W^1 and W^3. The W^3 operator leads to neutrino masses via the Weinberg operator $LLHH$. This might be allowed if the operator is sufficiently suppressed. W^1 on the other hand, is very tightly constrained by proton decay and must either be absent or very tiny.

The following analysis is performed within the framework of an $SU(5)$ GUT, which uses the smallest group that can unify the SM. Of course, there are also groups of higher rank, like $SO(10)$ or E_6. The very nice feature of $SO(10)$ is that one generation of fermions can be accommodated in a single representation, the spinor representation **16**. From its decomposition under the SM gauge group,

$$\mathbf{16} = (\mathbf{3},\mathbf{2})_{\frac{1}{6}} \oplus (\mathbf{\bar{3}},1)_{-\frac{2}{3}} \oplus (\mathbf{\bar{3}},1)_{\frac{1}{3}} \oplus (\mathbf{1},\mathbf{2})_{-\frac{1}{2}} \oplus (\mathbf{1},\mathbf{1})_1 \oplus (\mathbf{1},\mathbf{1})_0 \,, \tag{2.22}$$

one can see that it predicts the right-handed neutrino corresponding to the total singlet $(\mathbf{1},\mathbf{1})_0$. The two Higgs fields belong to the **10** representation of $SO(10)$. In E_6 one finds the fermionic matter as well as the Higgs fields in the representation **27**, which decomposes under $SO(10)$ as

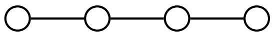
$$\begin{aligned}\mathbf{27}_{\text{Fermions}} &= \mathbf{16}_{\text{Fermions}} \oplus \mathbf{10}_X \oplus \mathbf{1}_X \,,\\ \mathbf{27}_{\text{Higgs}} &= \mathbf{16}_X \oplus \mathbf{10}_{\text{Higgs}} \oplus \mathbf{1}_X \,.\end{aligned} \tag{2.23}$$

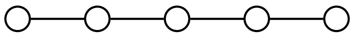
The fields in the representations that carry an index X have to receive a high mass.

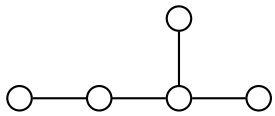
$SU(5)$ is chosen as the GUT group because larger groups would lead to more restrictive model building. Moreover, it has been shown in [19] that F-theory models with $SO(10)$ as the GUT group always contain exotic fields in their spectra. Yet, the groups $SU(6)$, $SU(7)$, $SO(12)$, E_7 and E_8 are needed in addition to $SU(5)$, $SO(10)$ and E_6. The reason is that in the higher-dimensional F-theory picture, as mentioned in the introduction, the matter is localized on curves along which the gauge group is enhanced by one rank, and Yukawa couplings are located at intersections of these curves with at least a rank-two enhancement. The type of the matter in question can be inferred from the decomposition of the adjoint of the enhanced group.

A **10** of $SU(5)$ is present along a curve with the enhanced symmetry group $SO(10)$, and for a **5** of $SU(5)$ one needs an $SU(6)$ symmetry enhancement.

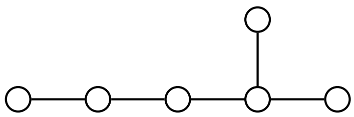
The relations between the groups can be examined best when displaying them in terms of Dynkin diagrams:

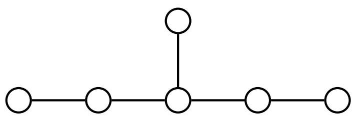
- $SU(5)$: 

- $SU(6)$: 

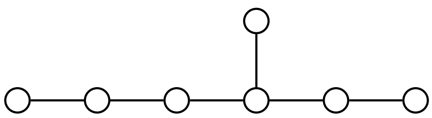
- $SO(10)$: 

Both groups can be obtained from $SU(5)$ by adding one node at the appropriate position. Yukawa couplings descend from the cubic interaction of the adjoint belonging to groups with at least a rank-two enhancement compared to $SU(5)$. An $SO(12)$ enhancement yields the down-type Yukawa coupling $\bar{\mathbf{5}}_{H_d}\bar{\mathbf{5}}_M\mathbf{10}_M$ and an E_6 enhancement the up-type coupling $\mathbf{5}_{H_u}\mathbf{10}_M\mathbf{10}_M$ (see section 3.4). Both groups can be broken to $SU(5)$ as well as to $SO(10)$ by deleting the corresponding nodes.

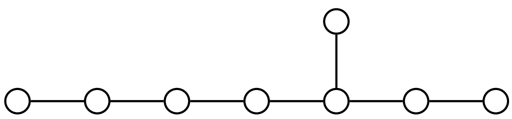
- $SO(12)$: 

- $E(6)$: 

For a sufficiently large mixing in the Cabibbo–Kobayashi–Maskawa (CKM) matrix, it is favorable to have the up- and down-type Yukawa couplings take place in one point, which would be a point of E_7, as can be seen by comparing the Dynkin diagrams of E_7, E_6 and $SO(12)$.

- E_7: 

Taking neutrinos into account together with the mixing in the Pontecorvo–Maki–Nakagawa–Sakata (PMNS) matrix pushes the symmetry further to E_8 [20].

- E_8: 

The fact that all the groups listed above can be obtained by deleting appropriate nodes from the Dynkin diagram of E_8 is a hint for a naturally underlying E_8 structure, which is also heavily explored in the heterotic $E_8 \times E_8$ string theory [21–24]. In the models which I will consider, the point of E_8 will give rise to all particles and interactions of the MSSM.

3. F-theory

F-theory can be thought of being the correct way to describe Type IIB compactifications with seven-branes in generic situations where one cannot assume the string coupling to be small. Seven-branes have a strong backreaction on the ambient space, which causes the string coupling to vary over the compact space and leave the perturbative regime somewhere. The problem of describing strong coupling effects is elegantly solved by encoding information in the geometry: A CY fourfold is constructed as an elliptic fibration over the three-complex-dimensional internal space, where a singularity of the fibration indicates the presence of seven-branes. From the phenomenological point of view F-theory is promising because it is still based on branes and thus admits a bottom-up approach, where some questions concerning particle physics can be answered comparatively easily in local models, though the branes in question are no longer exclusively D- and O-planes. The new branes exhibit complicated backreactions, and it is a remarkable property of the F-theory framework that consistency requirements like tadpole cancellation are automatically incorporated.

In section 3.1 I will start with a summary of the basic ingredients of Type IIB orientifolds, since many features of F-theory are already present in intersecting brane models, and give the proper definition of F-theory from the Type IIB perspective in section 3.2. In order to explain why exceptional gauge groups arise, I will make some use of the duality between M- and F-theory, which is presented in section 3.3. In section 3.4 I will continue to explain the geometry of F-theory compactifications as described by the Weierstraß model, which, apart from deducing the positions of seven-branes, in particular allows to read off which gauge group is realized on a given seven-brane. This is based on the Kodaira classification of singular fibers [25]. Afterwards, I will introduce the so called Tate model, which locally allows to determine the gauge group, the matter curves and interaction points of a given F-theory geometry. In section 3.5 an alternative description of the geometry, also valid on local patches of the internal space, will be given, which is based on eight-dimensional field theory and is crucial for chapters 4-6. The connection between the Tate model and the field theoretic description will be made in succession. In section 3.6 I will comment on the different levels of locality that can be considered in F-theory model building, thereby introducing the spectral cover formalism and presenting some relevant information for the later analysis.

3.1. Basics of intersecting brane models

Intersecting brane models are based on the fact that open strings end on D-branes. For a review, see e.g. [26]. A string which has both ends on the same brane is massless and can move along the brane, whereas strings that stretch between different branes become massive. However, if the branes intersect, the strings extending between them get localized at the intersection, and additional massless modes appear in the configuration.

There exist $U(N)$ gauge bosons on the worldvolume of N coincident D-branes, whose low-energy dynamics is governed by a Super–Yang–Mills (SYM)theory with gauge group $U(N)$. This can always be decomposed as $U(N) = SU(N) \times U(1)$. Along the intersection of the stack with another D-brane the symmetry gets enhanced by one rank to $U(N+1)$. Strings can now stretch between the stack and the extra brane and give rise to localized hypermultiplets. In general, this matter transforms in the bifundamental representation of the groups realized on the respective stacks. For example, if the gauge group of one stack is $SU(5)$, the transformation property of such matter under it can be inferred from the decomposition of the adjoint of the enhanced group. From

$$\begin{aligned} SU(6) &\to SU(5) \times U(1) \\ \mathbf{35} &\to \mathbf{24}_0 \oplus \mathbf{1}_0 \oplus \mathbf{5}_1 \oplus \bar{\mathbf{5}}_{-1} \end{aligned} \tag{3.1}$$

one can see that the $\mathbf{5}$ of $SU(5)$ localizes where the symmetry is enhanced to $SU(6)$.

An O-plane which is coincident with N D-branes leads to the group $SO(2N)$. From the decomposition of the adjoint of $SO(10)$

$$\begin{aligned} SO(10) &\to SU(5) \times U(1) \\ \mathbf{45} &\to \mathbf{24}_0 \oplus \mathbf{1}_0 \oplus \mathbf{10}_2 \oplus \overline{\mathbf{10}}_{-2} \end{aligned} \tag{3.2}$$

it follows that the $\mathbf{10}$ of $SU(5)$ is localized where an O-plane intersects a stack of five D-branes.

Finally, there can be intersections of multiple brane stacks corresponding to the meeting of several matter curves in a point. A point of $SO(12)$ symmetry is created where two $\mathbf{5}$ curves and a $\mathbf{10}$ curve meet. The cubic interaction of the adjoint

$$SO(12) \to SU(5) \times U(1)^a \times U(1)^b \tag{3.3}$$

$$\mathbf{66} \to \mathbf{24}_{0,0} \oplus \mathbf{1}_{0,0} \oplus \mathbf{1}_{0,0} \oplus \mathbf{5}_{-1,0} \oplus \bar{\mathbf{5}}_{1,0} \oplus \mathbf{5}_{1,1} \oplus \bar{\mathbf{5}}_{-1,-1} \oplus \mathbf{10}_{0,1} \oplus \overline{\mathbf{10}}_{0,-1} \tag{3.4}$$

allows for the coupling $\bar{\mathbf{5}}\,\bar{\mathbf{5}}\,\mathbf{10}$ that gives mass to the down-type quarks in an $SU(5)$ GUT.

For the up-type Yukawa coupling $\mathbf{5}\,\mathbf{10}\,\mathbf{10}$, in contrast, a point of E_6 enhancement is needed[1]:

$$\begin{aligned} &E_6 \to SU(5) \times U(1)^a \times U(1)^b \\ &\mathbf{78} \to \mathbf{24}_{0,0} \oplus \mathbf{1}_{0,0} \oplus \mathbf{1}_{0,0} \oplus \mathbf{1}_{-5,-3} \oplus \mathbf{1}_{5,3} \oplus \mathbf{5}_{-3,3} \oplus \bar{\mathbf{5}}_{3,-3} \oplus \mathbf{10}_{-1,-3} \\ &\qquad \oplus \overline{\mathbf{10}}_{1,3} \oplus \mathbf{10}_{4,0} \oplus \overline{\mathbf{10}}_{-4,0}\,. \end{aligned} \tag{3.5}$$

A shortcoming of Type IIB intersecting brane models is the lack of exceptional gauge groups – they cannot be engineered using exclusively D-branes and O-planes. There are, however, more general branes whose appearance can be explained when taking the $SL(2,\mathbb{Z})$ symmetry of Type IIB string theory seriously, as we will see now.

[1]The up-type Yukawa coupling in Type IIB models can only be generated by D-brane instantons [27].

3.2. F-theory from Type IIB orientifolds

In Type IIB theory there is a field, called the axio-dilaton:

$$\tau = C_0 + \frac{\mathrm{i}}{g_s} . \tag{3.6}$$

The D7-brane is magnetically charged under C_0 and g_s is the string coupling. Supersymmetry requires the axio-dilaton to be a holomorphic function in the complex coordinate z perpendicular to a seven-brane. When encircling the D7-brane, a monodromy acts on the axio-dilaton:

$$\tau \to \tau + 1 . \tag{3.7}$$

From this it is clear that close to the brane the dependence of the axio-dilation on the transverse coordinate is logarithmic:

$$\tau(z) = \tau_0 + \frac{1}{2\pi\mathrm{i}} \ln(z - z_0) + \text{regular terms in } z . \tag{3.8}$$

This implies that the imaginary part of τ becomes infinite at the position of the seven-brane and thus, due to (3.6), the string coupling g_s is zero. The gauge coupling, however, is independent of g_s and determined only by the volume of the internal four-cycle wrapped by the seven-branes.

To extend the set of branes from the well-known D-branes and O-planes to the aforesaid more exotic branes, it is important to note that the transformation (3.7) is just a special case of a general $SL(2,\mathbb{R})$ transformation

$$\tau \to \frac{a\tau + b}{c\tau + d} , \tag{3.9}$$

under which the classical Type IIB effective action is invariant. The classical symmetry is broken to $SL(2,\mathbb{Z})$ by a $D(-1)$ instanton at the non-perturbative level. The NS-NS field B_2, under which a fundamental string (F1-string) is electrically charged, and the RR two-form C_2 form an $SL(2,\mathbb{Z})$ doublet, which transforms as

$$\begin{pmatrix} C_2 \\ B_2 \end{pmatrix} \to \begin{pmatrix} aC_2 + bB_2 \\ cC_2 + dB_2 \end{pmatrix} = M \begin{pmatrix} C_2 \\ B_2 \end{pmatrix} . \tag{3.10}$$

The objects that are electrically charged under C_2 are D1-strings, and therefore F1- and D1-strings behave like a doublet under $SL(2,\mathbb{Z})$, too. Representing the F1-string by $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and the D1-string by $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$, one can define a general (p,q)-string as a string with B_2 charge p and C_2 charge q. The branes on which (p,q)-strings end, are (p,q)-branes [28]. An ordinary D7-brane is just a $(1,0)$-brane.

We have seen that a D7-brane has a characteristic monodromy:

$$M_{\mathrm{D}} = M_{1,0} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} . \tag{3.11}$$

In fact, any general (p, q)-brane can be characterized by its action on the background:

$$M_{p,q} = g_{p,q} M_{1,0} g_{p,q}^{-1} = \begin{pmatrix} 1 - pq & p^2 \\ -q^2 & 1 + pq \end{pmatrix} , \tag{3.12}$$

where

$$g_{p,q} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} p \\ q \end{pmatrix} . \tag{3.13}$$

The monodromy of an O7-plane can be expressed as the linear combination

$$M_{\mathrm{O}} = M_{3,-1} M_{1,-1} , \tag{3.14}$$

since this maps the vector $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ to minus itself and therefore corresponds to a reversal of the string orientation, which is precisely the effect of an O-plane. M_{D} and M_{O} do not generate the full $SL(2, \mathbb{Z})$ symmetry, so in general it is necessary to consider different (p, q)-branes as well. By means of (3.12) a single (p, q)-brane can always be mapped into a D7-brane, but this does not work if there are several (p, q)-branes of different type present in a compactification at the same time. Furthermore, it is a non-trivial problem to find configurations of (p, q)-branes that are self-consistent. The rescue is the celebrated geometrization of the monodromies induced by the different seven-branes: The axio-dilaton is identified with the complex structure of an auxiliary torus, which is fibered over the internal space B_3 and thus contains the information about the different seven-branes in the compactification. This works so well because $SL(2, \mathbb{Z})$ is also the symmetry group of a torus. The two additional dimensions of the torus are the reason why F-theory is said to be twelve-dimensional, although they are no physical dimensions.

It follows from (3.8) that the torus degenerates at the position of seven-branes, but later we will see that the fibration contains much more information then just the location of seven-branes. To summarize this discussion, let me conclude this section with the *F-theory conjecture*:

The physics of Type IIB orientifold models with seven-branes compactified on the complex threefold B_3 is encoded in the geometry of an elliptic fibration $T^2 \to Y_4 \to B_3$.

3.3. F-theory from M-theory

The duality between F- and M-theory, which is the topic of this section, clarifies why the complex structure τ of the elliptic curve plays a role in the F-theory framework whereas the volume does not. This is not apparent when only considering the relation to Type IIB string theory. Apart from that, the duality explains why the four-complex-dimensional space made up by the elliptic fibration has to be a CY manifold.

There is a conjecture stating that eleven-dimensional supergravity is the low-energy limit of M-theory. To approach F-theory, this eleven-dimensional theory is compactified on the space $\mathbb{R}^{1,8} \times T^2$, where $\mathbb{R}^{1,8}$ denotes a nine-dimensional Minkowski space and T^2 a torus spanned by the cycles S_A and S_B [29].

First of all, the duality between M- and Type IIA theory is explored by letting one cycle – here S_A is chosen – shrink to zero size. Since the coupling of Type IIA theory is given in terms of the string length l_s and the radius R_A of S_A by $g_\mathrm{IIA} \simeq \frac{R_\mathrm{A}}{l_s}$, one obtains the weak coupling limit. The remaining circle S_B can now be used to perform a T-duality transformation from Type IIA to Type IIB theory. The dual radius is given by $\tilde{R}_\mathrm{B} = \frac{l_s^2}{R_\mathrm{B}}$. Therefore, one obtains Type IIB theory on $\mathbb{R}^{1,9}$ if the radius R_B vanishes, too. The dimension that is needed for $\mathbb{R}^{1,8}$ to become the ten-dimensional Minkowski space is hence provided by one cycle of the torus, which grows large. The volume of the torus cannot be a physical field because it is a step of the duality transformation to let the volume of the torus approach zero.

Furthermore, the fact that the imaginary part of its complex structure is the string coupling can be seen by noting that after the T-dualization the Type IIB string coupling is $g_\mathrm{IIB} \simeq \frac{l_s}{R_\mathrm{B}} g_\mathrm{IIA} \simeq \frac{R_\mathrm{A}}{R_\mathrm{B}} \simeq \mathrm{Im}\,\tau^{-1}$. The simple argument presented here of course only applies to rectangular tori but can be generalized [7].

The upshot of the F/M- theory duality is that the elliptic fiber of F-theory is identified with the M-theory torus along which the M/Type IIB duality is performed. For realistic models, one must start with a compactification of the eleven-dimensional theory on the space $\mathbb{R}^{1,2} \times Y_4$, where $T^2 \to Y_4 \to B_3$. This elliptic fibration must be a CY fourfold to obtain $\mathcal{N} = 1$ supersymmetry in four dimensions, because in this case the M-theory compactification leads to a three-dimensional effective theory with four supercharges [30]. Following the same steps, one ends up with Type IIB on $B_3 \times \mathbb{R}^{1,3}$.

In addition, the F/M- theory duality provides a way to elucidate the appearance of exceptional gauge groups in F-theory. I will come back to this point later.

3.4. Geometrical description

Weierstrass Model

An elliptic curve can be obtained as a hypersurface in a weighted projective space. The projective space $W\mathbb{P}_{2,3,1}$, chosen here, is defined by three complex coordinates (x, y, z) with the additional identification

$$(x, y, z) \sim (\lambda^2 x, \lambda^3 y, \lambda z)\,, \quad \lambda \in \mathbb{C}^*\,. \tag{3.15}$$

A torus is Ricci-flat, and a necessary condition for this is that the polynomial cutting it out of $W\mathbb{P}_{2,3,1}$ is a homogeneous polynomial of degree six under rescaling by λ. Any such polynomial can be brought to the form

$$P_W = y^2 - x^3 - fxz^4 - gz^6 = 0\,. \tag{3.16}$$

This is the famous Weierstraß form, where the complex numbers f and g parameterize the shape of the torus. Promoting this single torus to an elliptic fibration over the base B_3 with coordinates u_i, amounts to declaring f and g to be functions of the u_i. This fibration is called a Weierstraß model. In contrast to other possible spaces, $W\mathbb{P}_{2,3,1}$ has the property of possessing an underlying E_8 structure, as will become clear later when examining the Kodaira classification. From section 3.1 we know that the fiber must become singular at the position of seven-branes. In terms of the above description this translates into the fact that the surface cut out by (3.16) degenerates, which is the case if the discriminant

$$\Delta = 27g^2 + 4f^3 \tag{3.17}$$

of (3.16) vanishes. The equation $\Delta = 0$ itself cuts out a codimension-one surface S of the base B_3, which is precisely the divisor wrapped by the seven-branes. The geometry of a general F-theory compactification is depicted in figure 3.1.

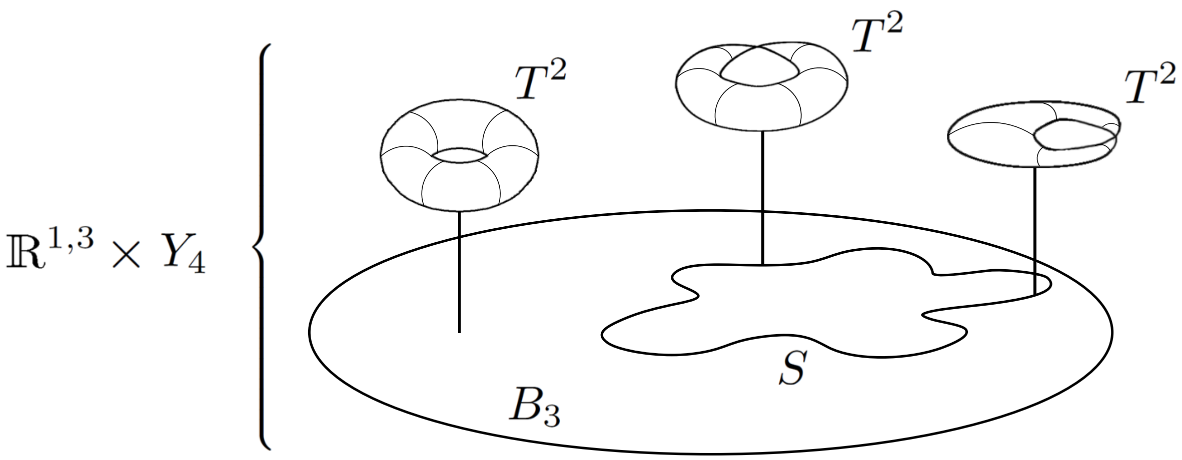


Figure 3.1.: *Schematic illustration of a general F-theory compactification: $\mathbb{R}^{1,3}$ denotes the four-dimensional Minkowski space and $T^2 \to Y_4 \to B_3$ the eight-dimensional CY manifold obtained as an elliptic fibration over the base B_3. Seven-branes locate where the torus T^2 degenerates.*

Emergence Of Gauge Groups

Even more important than the positions of the seven-branes is to know the precise way that the fiber degenerates over these locations, because this information tells us which gauge group is present on the branes. The Kodaira classification contains the different types of singularities and the corresponding gauge groups, as is displayed in table 3.1, taken from [31]. The mildest degeneration is the I_1 singularity, which does not create a true singularity in the CY fourfold and corresponds to a single seven-brane. It appears where the discriminant (3.17) vanishes to first order, whereas both f and g are non-vanishing. To obtain an actual singularity in the CY manifold, which allows

ord(f)	ord(g)	ord(Δ)	fiber type	singularity type
≥ 0	≥ 0	0	smooth	none
0	0	n	I_n	A_{n-1}
≥ 1	1	2	II	none
1	≥ 2	3	III	A_1
≥ 2	2	4	IV	A_2
2	≥ 3	$n+6$	I_n^*	D_{n+4}
≥ 2	3	$n+6$	I_n^*	D_{n+4}
≥ 3	4	8	IV^*	E_6
3	≥ 5	9	III^*	E_7
≥ 4	5	10	II^*	E_8

Table 3.1.: *Kodaira classification of singular fibers*

for non-Abelian and even exceptional gauge groups, Δ must develop a higher-order singularity.

The connection between a singularity in the CY fourfold and the appearance of gauge groups can be understood by blowing up the singularity and once more employing the F/M-theory duality. The resolution is performed by substituting the singular fiber with a tree of $\mathbb{P}^1$s such that the new space is still a CY manifold but non-singular. The number of $\mathbb{P}^1$s that is needed equals the rank of the group. The divisor which is formed in the new non-singular CY fourfold $\widetilde{Y}_4$ by the fibration of one $\mathbb{P}^1$ over S will be called D_i in the following. Furthermore, one can define the divisor $D_0 = T^2 - \sum_i a_i D_i$, where T^2 is the torus over a point of S and a_i are the Dynkin labels of the adjoint of the Lie algebra of G. These rank$(G) + 1$ divisors then satisfy the equation

$$\int_{\widetilde{Y}_4} [D_i] \wedge [D_j] \wedge \tilde{\omega} = -C_{ij} \int_S \tilde{\omega}\,, \quad i, j \in 0, \cdots, \mathrm{rank}(G)\,, \tag{3.18}$$

where $[D_i]$ denotes the Poincaré dual two-form to D_i, $\tilde{\omega} \in H^4(B_3)$ and C_{ij} are the entries of the Cartan matrix of the group G. One can visualize this in the following way:

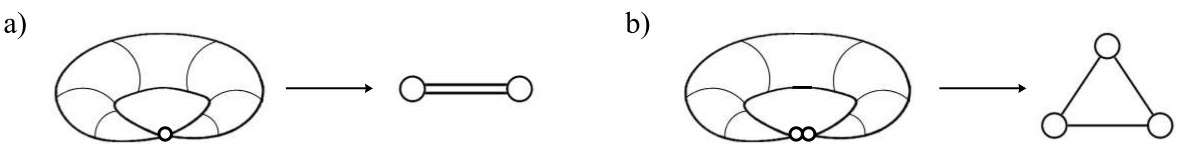


Figure 3.2.: *Two examples displaying the relation between a singularity in the elliptic fibration and a gauge group. The singularity in (a) is of type $A_1 \cong SU(2)$ and in (b) of type $A_2 \cong SU(3)$. The diagrams to the right of the arrow are the respective extended Dynkin diagrams of $SU(2)$ and $SU(3)$.*

In the F-theory limit all these $\mathbb{P}^1$s shrink to zero size and the gauge bosons in the Cartan subalgebra of G correspond to massless vector states obtained when reducing the M-theory three-form C_3 along the $\mathbb{P}^1$s. The other gauge bosons are due to the M-theory M2-brane wrapping chains of $\mathbb{P}^1$s.

The type of singularity is allowed to vary over a given divisor $S \subset B_3$. In Type IIB theory the symmetry group gets enhanced where other seven-branes intersect the GUT stack, leading to matter curves and interaction points. Something similar occurs in F-theory [32]: The divisors D_a, which are wrapped by seven-branes, can intersect and form complex codimension-one curves $C_{ab} = D_a \cap D_b$. In the blow-up of the singularities along D_a and D_b to a non-singular CY manifold, a tree of $\mathbb{P}^1$s had to be inserted for each divisor. These will now collide to form the extended Dynkin diagram of a group with rank

$$\mathrm{rank}(G_{ab}) = \mathrm{rank}(G_a) + \mathrm{rank}(G_b)\,. \tag{3.19}$$

This group is not really a gauge group because there is no vector multiplet to the corresponding $\mathcal{N} = 1$ gauge theory, but from the decomposition of the adjoint of G_{ab},

$$\begin{aligned} G_{ab} &\to G_a \times G_b \\ \mathrm{ad}_{G_{ab}} &\to (\mathrm{ad}_{G_a}, \mathbf{1}) \oplus (\mathbf{1}, \mathrm{ad}_{G_b}) \oplus \sum (R_{ax}, R_{bx})\,, \end{aligned} \tag{3.20}$$

one can deduce the representations (R_{ax}, R_{bx}) of the localized matter.

Tate Model

There exists a scheme based on the Tate algorithm which captures this additional information about the singularity structure of an elliptic fibration. The Weierstraß form can be brought into the Tate form by performing a local coordinate redefinition [31]:

$$P_W = x^3 - y^2 + xyza_1 + x^2z^2a_2 + yz^3a_3 + xz^4a_4 + z^6a_6 = 0\,. \tag{3.21}$$

Because of the scaling relations in (3.15), one can switch to inhomogeneous coordinates where $z = 1$. Instead of f and g, the shape of the torus is now parameterized by the a_i, which consequently are functions of the coordinates u_i of B_3 as well. The resulting symmetry group depends on the vanishing orders of the different a_i and the discriminant. The relation of the a_i to f and g is given by

$$\begin{aligned} &f = -\frac{1}{48}\left(\beta_2^2 - 24\beta_4\right)\,, \quad g = -\frac{1}{864}\left(-\beta_2^3 + 36\beta_2\beta_4 - 216\beta_6\right)\,, \\ &\beta_2 = a_1^2 + 4a_2\,, \quad \beta_4 = a_1a_3 + 2a_4\,, \quad \beta_6 = a_3^2 + 4a_6\,. \end{aligned} \tag{3.22}$$

The local validity of the Tate model is manifest in the non-linearity of these equations. Information about the intersection of branes can be obtained by examining the (rescaled) discriminant, which can now be written as

$$\Delta = -\frac{1}{4}\beta_2^2(\beta_2\beta_6 - \beta_4^2) - 8\beta_4^3 - 27\beta_6^2 + 9\beta_2\beta_4\beta_6 \tag{3.23}$$

and in general is a product of several factors, each of which corresponds to one seven-brane.

In the next paragraph an $SU(5)$ stack of seven-branes wrapping the twofold S will be considered which is given by a polynomial $w = 0$. For an $SU(5)$ singularity the required vanishing degrees are:

Δ	a_1	a_2	a_3	a_4	a_6
5	0	1	2	3	5

Denoting by b_k functions of the base coordinates which do not contain an overall factor of w, this translates into

$$a_1 = b_5 \, , \; a_2 = b_4 w \, , \; a_3 = b_3 w^2 \, , \; a_4 = b_2 w^3 \, , \; a_6 = b_0 w^5 \, . \tag{3.24}$$

The discriminant then reads:

$$\Delta = -w^5 \Big(b_5^4 P + w b_5^2 (8 b_4 P + b_5 R) + w^2 (16 b_3^2 b_4^2 + b_5 Q) + \mathcal{O}(w^3) \Big) \, , \tag{3.25}$$

$$P = b_3^2 b_4 - b_2 b_3 b_5 + b_0 b_5^2 \, . \tag{3.26}$$

Just as P, Q and R are also functions of the b_k. The factor w^5 corresponds to the five coincident seven-branes responsible for the gauge group $SU(5)$. The other factor does in general not vanish at $w = 0$ and consequently is associated with a brane that does not coincide with the $SU(5)$ stack. Furthermore, the expression in brackets does not factorize further and thus carries an I_1 singularity which is characterized by an order-one vanishing locus of the discriminant. Therefore, the extra brane does not carry a non-Abelian gauge symmetry. If, however, particular linear combinations of the b_k become zero at certain points of B_3, the overall vanishing order of the discriminant changes and this leads to symmetry enhancements. They correspond to intersections of the $SU(5)$ stack and the extra brane that is identified with the I_1 singularity and does not possess any non-Abelian gauge group. The rank of the local symmetry groups along the curves can therefore only differ from the rank of $SU(5)$ by one. The first possibility is an enhancement to $SO(10)$, which gives rise to the **10** of $SU(5)$, and the other is to $SU(6)$, corresponding to the **5**, as we have seen in (3.2) and (3.1), respectively. The former option leads to a D_5 singularity, for which the vanishing orders must be:

Δ	a_1	a_2	a_3	a_4	a_6
7	1	1	2	3	5

This can be achieved when

$$P_{\mathbf{10}} : \quad w = 0 \quad \cap \quad b_5 = 0 \, . \tag{3.27}$$

The A_5 singularity corresponding to $SU(6)$ only requires a vanishing of Δ to order six:

Δ	a_1	a_2	a_3	a_4	a_6
6	0	1	3	3	6

Therefore, one obtains

$$P_{\mathbf{5}} : \quad w = 0 \quad \cap \quad P = 0 \, . \tag{3.28}$$

In addition to the **5** and **10** one also needs GUT-singlets to generate masses for all quarks and leptons. GUT singlets localize where the I_1 brane intersects itself to form an A_1 singularity.

This discussion can be generalized to intersections of matter curves, where the symmetry enhancement will be at least of rank two. For the E_6 symmetry that is needed for the up-type Yukawa coupling, the condition is

$$\mathbf{10}\ \mathbf{10}\ \mathbf{5}: \quad b_5 = b_4 = 0 \tag{3.29}$$

and for the down-type Yukawa coupling located at a point of $SO(12)$ it is

$$\mathbf{10}\ \bar{\mathbf{5}}\ \bar{\mathbf{5}}: \quad b_5 = b_3 = 0 \tag{3.30}$$

in order to arrive at the required vanishing orders:

Group	Δ	a_1	a_2	a_3	a_4	a_6
E_6	8	1	2	2	3	5
$SO(12)$	8	1	1	3	3	5

3.5. Gauge theoretic description

Since the surface S locally carries symmetry groups larger than G_{GUT}, there is a different perspective to the one introduced in the previous section: One can (at least locally) think of the worldvolume theory on S as a gauge theory with a larger gauge group which is broken at generic points. The appropriate description for this is eight-dimensional $\mathcal{N}=1$ SYM field theory, which will be introduced in the following. Subsequently, I will make the connection to the Tate model and specialize the gauge theoretic description to what is needed for the study of chapter 4. A crucial ingredient is seven-brane monodromy. The gauge theoretic approach allows to explicitly deduce how brane configurations that exhibit monodromies are to be dealt with. Apart from providing a convenient formalism for the analysis of intersecting branes and monodromy, it has an important generalization to what is called T-branes, which are the content of chapter 6.

3.5.1. Field theory of seven-branes

A *stack*, i.e. a layer of coincident seven-branes, can be studied with an eight-dimensional gauge theory supporting an adjoint complex Higgs field Φ that parameterizes the normal motion of the branes, and a gauge field A, which is a one-form. In F-theory the branes extend through four-dimensional Minkowski space but wrap the compact four-cycle S in the internal space. To preserve unbroken $\mathcal{N}=1$ SUSY in four dimensions, the field theory must be twisted such that Φ is a $(2,0)$-form on S, see [15]. In order to investigate *configurations* of branes, i.e. more complicated setups where the branes can intersect, one must consider a situation where the Higgs field has a VEV $\langle\Phi\rangle$ which varies over the surface S. This VEV must be understood as a background and part of the definition of the field theory at hand. Since S is four-dimensional, whereas the the internal physical space of the compactification is six-dimensional, the most general Higgs field is one which allows the branes to move along one complex dimension:

$$\langle\Phi\rangle = c_a^1 T^a + \mathrm{i} c_a^2 T^a \,. \tag{3.31}$$

The T^a denote the Hermitean generators of the Lie algebra corresponding to the given gauge group, whereas c^1 and c^2 are arbitrary real fields such that the resulting Higgs field does not need to be Hermitean. In general,

$$[\langle\Phi\rangle^\dagger, \langle\Phi\rangle] \neq 0 \,. \tag{3.32}$$

The special case of an Hermitean background Higgs field permits to take $\langle\Phi\rangle$ to reside in the Cartan subalgebra. One can then diagonalize it and obtain the intersecting brane picture: Branes meet, and consequently the symmetry gets enhanced, where two or more eigenvalues of the Higgs field are coincident. For this case it is also clear from the string perspective how the matter emerges: It corresponds to strings stretching between different branes.

The gauge field A and the Higgs field Φ obey the following BPS equations, which are enforced by SUSY:

$$F_A^{0,2} = 0 \,, \tag{3.33}$$
$$\bar\partial_A \Phi = 0 \,, \tag{3.34}$$
$$\omega \wedge F_A + \frac{\mathrm{i}}{2}[\Phi^\dagger, \Phi] = 0 \,. \tag{3.35}$$

Here, ω denotes the Kähler form on S. The first two equations result from F-flatness conditions and are invariant under the complexified group of gauge transformations. This property is important because the space of solutions to the first two equations modulo complexified gauge transformations is the same as the space of solutions to all three equations modulo unitary gauge transformations. The third equation is a D-flatness condition. So far this theory describes a stack of branes.

More interesting configurations are obtained by switching on a background. A and Φ are decomposed into background and fluctuations as follows:

$$A = \langle A\rangle + a \,, \tag{3.36}$$
$$\Phi = \langle\Phi\rangle + \phi \,. \tag{3.37}$$

The linearized BPS equations, the solutions of which are matter fields, are

$$\bar\partial_A a = 0 \,, \tag{3.38}$$
$$\bar\partial_A \phi + [a, \Phi] = 0 \,, \tag{3.39}$$
$$\omega \wedge (\partial_A a - \bar\partial_A a^\dagger) + \frac{\mathrm{i}}{2}([\Phi^\dagger, \phi] + [\phi^\dagger, \Phi]) = 0 \,. \tag{3.40}$$

Here and from now on, A and Φ stand for $\langle A\rangle$ and $\langle\Phi\rangle$. The space of solutions to (3.38)–(3.40) has to be quotiented by the allowed gauge transformations. Under a gauge transformation

$$a \to a + \bar{\partial}_A \chi\,, \tag{3.41}$$

and from (3.39) one can see that

$$\phi \to \phi + [\Phi, \chi]\,. \tag{3.42}$$

How complicated it is to solve (3.38)–(3.40) modulo (3.41)–(3.42) depends on the background Higgs field . The simplest case is again the intersecting brane case where the commutator in (3.35) vanishes. It is only in this case that the equations for the gauge field A,

$$F_A^{0,2} = 0\,, \tag{3.43}$$

$$\omega \wedge F_A = 0\,, \tag{3.44}$$

decouple from the Higgs field. If, however, the commutator is not zero, A and Φ are entangled, which is the reason why such configurations are called *bound states.* This is always the case for adjoint Higgs fields whose matrices become upper triangular at some points of the brane worldvolume because these cannot be diagonalized everywhere. Configurations where a stack of seven-branes gets deformed by a holomorphic Higgs field with this property are called T-branes. They were first considered in [15], on which most of the discussion in this chapter is based. I will come back to non-diagonal Higgs fields later when introducing seven-brane monodromy.

3.5.2. Diagonal case

For now, I will consider an E_8 gauge theory on the worldvolume of S which is Higgsed down to $SU(5)_{\mathrm{GUT}}$ at generic points by a diagonal Higgs field. The reason why E_8 is a good starting point was explained at the end of chapter 2. The commutant of $SU(5)_{\mathrm{GUT}}$ in E_8 is also an $SU(5)$, which will be called $SU(5)_\perp$ from now on, and it is broken further down to $U(1)^4$ by the Higgs field. The extra $U(1)$'s correspond to the transverse branes that intersect the $SU(5)$ stack and are generically broken[2]. Still, they remain as global selection rules for the Lagrangean, which is the main reason why the gauge theoretic description is so useful. The matter curves are the loci where certain entries of the Higgs vanish or become degenerate such that the symmetry group is locally enhanced. One can determine the curves as well as the corresponding representations that localize on them by decomposing the adjoint of E_8:

$$\begin{aligned} E_8 &\longrightarrow SU(5)\times SU(5)_\perp \\ \mathbf{248} &\longrightarrow (\mathbf{24},\mathbf{1})\oplus(\mathbf{1},\mathbf{24})\oplus(\mathbf{10},\mathbf{5})\oplus\left(\bar{\mathbf{5}},\mathbf{10}\right)\oplus\left(\overline{\mathbf{10}},\bar{\mathbf{5}}\right)\oplus\left(\mathbf{5},\overline{\mathbf{10}}\right)\,. \end{aligned} \tag{3.45}$$

[2] This is a global issue that cannot be analyzed in a local model [33].

Denoting the diagonal entries of the Higgs field by t_i ($i = 1, \cdots, 5$), Φ acts on the basis e_i of the **5** of $SU(5)_\perp$ as $\Phi e_i = t_i e_i$. The eigenvalues have to satisfy the tracelessness condition

$$\sum_{i=1}^{5} t_i = 0\,. \tag{3.46}$$

The **10** and the non-Cartan elements of the **24** are spanned by the products $e_i \wedge e_j$ and $e_i \wedge e_j^*$, respectively, where $i \neq j$. These elements are again eigenvectors of Φ with eigenvalues $t_i + t_j$ and $t_i - t_j$. As mentioned above, the matter curves can be expressed in terms of the eigenvalues t_i of the Higgs. From the decomposition (3.45) it is apparent that the representations of $SU(5)$ and $SU(5)_\perp$ come in pairs. Focusing on the representations of the unbroken $SU(5)$, the equations for the matter curves read:

$$\mathbf{10}\colon t_i = 0\,, \qquad \mathbf{5}\colon -(t_i + t_j) = 0\,, \qquad \mathbf{1}\colon \pm(t_i - t_j) = 0\,. \tag{3.47}$$

For the $\overline{\mathbf{10}}$ and $\overline{\mathbf{5}}$ there are overall minus signs in the equations because they correspond to conjugate representations of $SU(5)_\perp$. Still, the matter curves are geometrically the same since the matter comes in hypermultiplets.

With (3.27) and (3.28) I have already shown how the **5** and **10** matter curves can be found using the Tate model defined by (3.21) and (3.24):

$$y^2 = x^3 + b_5 xy + b_4 x^2 w + b_3 y w^2 + b_2 x w^3 + b_0 w^5\,. \tag{3.48}$$

The connection between the two approaches is that the coefficients b_k are elementary symmetric polynomials of degree k in the t_i. If one rephrases the conditions (3.47) for the localization of matter representations in terms of the b_k, the resulting equations are precisely (3.27) and (3.28).

3.5.3. Seven-brane monodromy

An obvious consequence is that the b_k do not fully specify the t_i, since the relation is nonlinear. In particular it can happen that one encounters branch cuts when solving for the t_i, which leads to monodromies permuting some of them. There has been made use of this by *postulating* that the matter curves corresponding to t's that are related by a monodromy get identified and give rise to the same zero modes [20, 34–36]. This is necessary to have a tree-level top quark Yukawa coupling $\mathbf{10}_{\text{top}}\mathbf{10}_{\text{top}}\mathbf{5}_{H_u}$, which is favorable because of the large mass difference between the third and the first two SM families. For a term in the Lagrangean to be allowed by gauge symmetry, the t_i of the fields appearing in the operator have to sum up to zero, possibly using (3.46). Since the top and the anti-top quark are both in the same **10** representation, the desired coupling can only be achieved provided the $\mathbf{10}_M$'s participating in the mass term are the same. Keeping in mind that the up-type Higgs has a charge of the form $-t_i - t_j$, the $\mathbf{10}_M$'s must have charges t_i and t_j for the term to be gauge invariant. Since $t_i \neq t_j$, see (3.47), this would imply that the **10**'s are different and thus at least a $\mathbb{Z}_2$ monodromy is required in order to identify them. Its action is chosen to be $t_1 \leftrightarrow t_2$ so that the top quark generation is assigned to the curve $\mathbf{10}_1$ given by the weights $\{t_1, t_2\}$. Then there exists a tree-level up-type Yukawa coupling that leads to a heavy top quark provided the up-type Higgs curve $\mathbf{5}_{H_u}$ is fixed to be the curve with charge $-t_1 - t_2$. In terms of intersecting branes this is consistent with a transverse brane that intersects the GUT stack twice at the same point and therefore itself. Locally one cannot distinguish this setup from a configuration where the two branes are really distinct objects.

The T-brane formalism provides a way to explicitly confirm the correctness of the above postulate. To this end, I will explain how the diagonal background with branch cuts is obtained from a single-valued holomorphic background and deduce the consequences for the spectrum following [15].

The monodromy group is the Galois group of the spectral polynomial

$$P_\Phi(z) = \det(z\mathbb{1} - \Phi) \tag{3.49}$$

of the Higgs field, which is to be understood as a polynomial in z with coefficients themselves being polynomials in the worldvolume coordinates x, y on the brane[3]. For more information on Galois groups, see e.g. [37]. In the diagonal case

$$P_\Phi(x, y, z) = \prod_{i=1}^{5} (z - t_i(x, y)) \ , \tag{3.50}$$

and it is easy to see that the Higgs field deforms the brane stack at $z = 0$ to $P_\Phi(x, y, z) = 0$. This remains to be true for general non-diagonal Higgs fields that describe a configuration of a branes previously located at $z = 0$: The position of seven-branes in the space spanned by x, y and z is always given by the equation

$$P_\Phi(x, y, z) = 0 \, . \tag{3.51}$$

In general, there are several holomorphic matrices with the same spectral equation. It turns out that this equation alone does not contain sufficient information to deduce all aspects of the physics of a configuration as defined by a given Higgs field Φ, but that one needs to know its explicit matrix form. For explicit examples and more details on this, see chapter 6.

However, there is a special class of Higgs fields, for which the spectral equation carries in fact the complete information. Because of this property the Higgs fields belonging to this class are called *reconstructible*[4]. I will only consider the case of Higgs fields in the adjoint of $U(n)$ groups whose eigenvalues vanish at the origin. For a Higgs field to be reconstructible, it is a necessary and sufficient condition that the surface which is cut out of the space with the complex coordinates x, y, z by $P_\Phi(x, y, z) = 0$ is non-singular. When dealing with a local model, the only relevant singularities are those which reside at the origin $(x, y, z) = (0, 0, 0)$. A spectral polynomial always has the form

$$P_\Phi(x, y, z) = z^n + \sigma_2(x, y) z^{n-2} - \sigma_3(x, y) z^{n-3} + \cdots + (-1)^n \sigma_n(x, y) \, , \tag{3.52}$$

[3] These worldvolume coordinates x, y of the brane belong to the coordinates u_i of the base B_3 and are not to be confused with the coordinates x and y used above.

[4] This also includes Higgs fields that can be written in a block-diagonal form where each block itself is reconstructible.

where σ_k is the k-th symmetric polynomial in the eigenvalues of Φ. Because of the tracelessness condition σ_1 does not appear. At the origin the presence or absence of a singularity is therefore solely determined by the vanishing order of $\sigma_n = \det\Phi$. Thus, a Higgs field is reconstructible iff its determinant vanishes to exactly first order. Furthermore, it was shown in [15, 38] that a stack of n branes which is deformed by a reconstructible $n \times n$ Higgs field is nothing but a single smooth recombined brane. Now one can also intuitively understand why the surface cut out by $P_\Phi(x, y, z) = 0$ had to be non-singular.

The picture strongly resembles the action of a monodromy which can also identify several branes as an actually single object. It is helpful to consider an example of a Higgs field breaking $U(3) \to U(1)^2$:

$$\Phi = \begin{pmatrix} 0 & 1 & 0 \\ x & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} . \tag{3.53}$$

The spectral polynomial is $P_\Phi = z(z^2 - x)$ and thus exhibits a $\mathbb{Z}_2$ monodromy. If $x \neq 0$, this Higgs field can be diagonalized:

$$g\Phi g^{-1} = \begin{pmatrix} \sqrt{x} & 0 & 0 \\ 0 & -\sqrt{x} & 0 \\ 0 & 0 & 0 \end{pmatrix} , \quad g = \begin{pmatrix} \sqrt{x} & 1 & 0 \\ -\sqrt{x} & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} . \tag{3.54}$$

In this gauge the background Higgs field is not single-valued but exhibits branch cuts. Whenever the monodromy group is nonzero, a Higgs field cannot globally be diagonalized. The exchange of the eigenvalues when encircling the y-axis can also be expressed by acting with an element of the Weyl group of $U(3)$ on Φ:

$$W = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} , \quad \Phi(e^{2\pi \mathrm{i}}x, y) = W\Phi(x, y)W^{-1} . \tag{3.55}$$

Therefore, the monodromy group acts via its embedding in the Weyl group, and only the Weyl invariant data carries gauge independent information. This has consequences for the spectrum: The matter which is charged under the remaining unbroken gauge group is a doublet in the upper right corner of the perturbation[5] ϕ:

$$\phi = \begin{pmatrix} 0 & 0 & \phi_+ \\ 0 & 0 & \phi_- \\ 0 & 0 & 0 \end{pmatrix} . \tag{3.56}$$

Since ϕ is a perturbation of the background Higgs field Φ, an element W of the Weyl group acts on it:

$$\begin{pmatrix} \phi_+ \\ \phi_- \end{pmatrix} (e^{2\pi \mathrm{i}}x, y) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \phi_+ \\ \phi_- \end{pmatrix} (x, y) . \tag{3.57}$$

[5]The corresponding conjugate fields in the lower left corner are not displayed to simplify matters.

The action of the Weyl group therefore identifies the matter curves at $\sqrt{x}$ and $-\sqrt{x}$ as a single one at $x = 0$.

Every monodromy group $G = G_1 \times G_2 \times \cdots \times G_j$ can be realized by a Higgs field of the form

$$\begin{pmatrix} \Psi_1 & & & & \\ & \Psi_2 & & & \\ & & \ddots & & \\ & & & \Psi_j & \\ & & & & 0 \end{pmatrix}, \tag{3.58}$$

breaking $SU(k_1 + k_2 + ... + k_j + n) \to SU(n)$, where each Ψ_i is reconstructible. This corresponds to a stack of D7-branes intersected by j smooth D7-branes, which meet the $SU(n)$ stack, located at $z = 0$, along the curves where $\det \Psi_i = 0$. The charged matter then resides in blocks B_i,

$$\phi = \begin{pmatrix} & & & & B_1 \\ & & & & B_2 \\ & & & & \vdots \\ & & & & B_j \\ B_1^c & B_2^c & \cdots & B_j^c & 0 \end{pmatrix}, \tag{3.59}$$

each of which gives rise to exactly one localized field transforming as the antifundamental under the unbroken symmetry $SU(n)$. The blocks B_i^c comprise the corresponding conjugate fields.

What has been done in the literature so far and will also be done in chapter 4, is to use a diagonal Higgs field with branch cuts. This is equivalent to employing a single-valued reconstructible or block-reconstructible background, respectively, as we have seen. By gauge transforming from the T-brane background (3.53) to the diagonal background (3.54) it can be confirmed that the matter curves which are related by a monodromy have to be identified. Even the geometrical interpretation that several matter curves come from one brane intersecting itself has been validated by reinterpreting a brane configuration which is deformed by a reconstructible holomorphic background as a single recombined brane. The above discussion shows how T-brane configurations have been encountered implicitly in the literature. But besides giving new insights and a more elegant way to introduce the concept of monodromy, they open up many new options for model building once the assumption of block-reconstructability is dropped. In chapter 4 the important underlying assumption is that the diagonal Higgs field at hand is obtained from a reconstructible background via the procedure mentioned above, and therefore full information about the physics *can* be obtained by knowing only the spectral equation and the monodromy group. In chapter 6 in contrast, the full variety of single-valued Higgs fields breaking E_8 to $SU(5)$ will be explored.

3.6. Local, semilocal and global models

There exists a hierarchy in the dimensionalities of the models that one can examine within an F-theory compactification. A *global* model consists of an explicit Weierstraß model that specifies the CY fourfold Y_4 and with it the embedding of S in B_3. In a *semilocal* model the concern is with an eight-dimensional theory which is localized on S. To describe the local neighborhood of S, and in particular the fluxes turned on on S, a useful scheme is the spectral cover. Finally, a *local* model is a truly four-dimensional theory, in which the physics takes place on several, or in the extreme case, one point of S. In this chapter I will depict the advantages and drawbacks of the global, semilocal and local approach and clarify the relations between them, for example what assumptions can and must be made in order not to obviate the possibility of a more global embedding.

Decoupling Limit Of Gravity

For questions concerning particle physics gravity starts to play a role at the Planck scale, and it is therefore a reasonable step to analyze vacua that allow for a limit where gravity can be decoupled from the rest of particle physics. The idea is to first find a model which reproduces the MSSM and care about gravity afterwards. The four-dimensional Planck and GUT scales depend on the volumes of B_3 and the GUT surface S as follows:

$$M_{\mathrm{Pl}}^2 \propto \mathrm{Vol}(B)\,, \quad M_{\mathrm{GUT}}^4 \propto \mathrm{Vol}^{-1}(S)\,. \tag{3.60}$$

The decoupling limit of gravity corresponds to letting the volume of B_3 go to infinity whereas the volume of S stays finite. This is the same as to require that the ratio of the volumes goes to zero, from which follows that S must be a del Pezzo surface [39]. This has the important consequence that the only option for GUT breaking is via hypercharge flux: VEVs for adjoint valued chiral superfields cannot be used because they determine the position of S in B_3, but since S is decoupled there are no corresponding zero modes. In order to use Wilson lines to break $SU(5)$ a non-trivial fundamental group $\pi_1(S)$ is required. Del Pezzo surfaces, however, are simply connected.

Hypercharge Flux And Chiral Spectrum

The hypercharge flux takes values in the $U(1)_Y$ subgroup of $SU(5)$ and therefore breaks the GUT group to the commutant of $U(1)_Y$ in $SU(5)$, which is exactly the SM gauge group $SU(3)_C \times SU(2)_L \times U(1)_Y$. The Chern–Simons coupling

$$\int_{\mathbb{R}^{3,1}\times S} C_4 \wedge \mathrm{Tr}(F_{\mathbb{R}^{3,1}} \wedge F_S)\,, \tag{3.61}$$

where C_4 is the RR-four-form potential and the F's denote the field strengths, leads to a mass for the $U(1)_Y$ boson via the Stückelberg mechanism. This can be evaded by requiring that the two-cycles in S along which the hypercharge flux is switched on lift to homologically trivial two-cycles in the CY space [40]. To make sure that this constraint is fulfilled one would need a global model with an explicit embedding of S in a CY fourfold.

A matter field ψ which localizes along a curve Σ obeys the six-dimensional Dirac equation $\not{D}\psi = (\not{D}_4 + \not{D}_\Sigma)\psi = 0$. Since in the four-dimensional theory a Kaluza–Klein mass term is induced by $\bar{\psi}\not{D}_\Sigma\psi$, the massless modes are the zero modes of $\not{D}_\Sigma$. The chiral spectrum is therefore determined by an index theorem involving the fluxes on the seven-branes wrapping the divisors G_a and G_b that intersect to form the matter curve Σ. Denoting by q_a and q_b the charges of a field under the respective groups, the difference between the number of modes in the representation $\mathcal{R}$ and the conjugate representation $\bar{\mathcal{R}}$ is given by

$$n_{\mathcal{R}} - n_{\bar{\mathcal{R}}} = q_a \int_\Sigma \frac{F_a}{2\pi} + q_b \int_\Sigma \frac{F_b}{2\pi} \,. \tag{3.62}$$

Here, F_a and F_b denote the field strengths[6]. The models presented in chapter 4 are based on a point of E_8 enhancement, which finds itself where the four transverse $U(1)$ branes intersect the GUT stack. Whether or not these branes are really distinct objects depends on the monodromy group of the Higgs field, since it can lead to the identification of some branes. Hence, there are two kinds of fluxes which determine the chiralities of the curves: One of them is turned on along the $U(1)$'s and can only influence the chiralities of full GUT multiplets, whereas the other flux is the hypercharge flux, which splits the $SU(5)$ multiplets. Since the latter is confined to S_{GUT}, one can explicitly calculate how it restricts to the different matter curves within a semilocal framework. Denoting the restrictions of the $U(1)$ fluxes and the hypercharge flux to a curve by the integers M and N_Y, according to (3.62) the **5** curves get split in the following way:

$$\begin{aligned} n_{(3,1)_{-1/3}} - n_{(\bar{3},1)_{+1/3}} &= M_{\mathbf{5}} \,, \\ n_{(1,2)_{+1/2}} - n_{(1,2)_{-1/2}} &= M_{\mathbf{5}} + N_Y \,, \end{aligned} \tag{3.63}$$

and for the **10** curves we have

$$\begin{aligned} n_{(3,2)_{+1/6}} - n_{(\bar{3},2)_{-1/6}} &= M_{\mathbf{10}} \,, \\ n_{(\bar{3},1)_{-2/3}} - n_{(3,1)_{+2/3}} &= M_{\mathbf{10}} - N_Y \,, \\ n_{(1,1)_{+1}} - n_{(1,1)_{-1}} &= M_{\mathbf{10}} + N_Y \,. \end{aligned} \tag{3.64}$$

The $U(1)$ fluxes are turned on along the branes which intersect S_{GUT} transversally and thus cannot even be determined in the semilocal approach. The M's can therefore be treated as free parameters up to the constraint

$$\sum M_{\mathbf{10}} + \sum M_{\mathbf{5}} = 0 \,. \tag{3.65}$$

This equation follows from the tracelessness condition for the four $U(1)$ fluxes, $\sum_i F_{U(1)_i} = 0$, and implies anomaly cancellation (see also [41]). For the hypercharge

[6] The groups are restricted to be Abelian because of the implicit trace in (3.62).

flux, as anticipated earlier, there are more constraints because one must prevent it from receiving a Stückelberg mass [36]:

$$F_Y \cdot c_1 = 0\,, \quad F_Y \cdot \eta = 0\,. \tag{3.66}$$

The first Chern class of the tangent bundle of S is denoted by c_1 and $\eta = 6c_1 - t$, with $-t$ being the first Chern class of the normal bundle of S.

Semilocal Framework

Essentially following [42], the aim of this section is to calculate explicitly the flux restrictions N to the different curves. For this one needs to know the homology classes of the curves, which in turn can be calculated using the spectral cover approach [13].

The spectral surface is given by the constraint

$$C_{10} = b_0 U^5 + b_2 V^2 U^3 + b_3 V^3 U^2 + b_4 V^4 U + b_5 V^5 = 0\,, \tag{3.67}$$

where U and V are homogeneous coordinates of the projective threefold

$$X = \mathbb{P}(\mathcal{O}_{S_{\mathrm{GUT}}} \oplus K_{S_{\mathrm{GUT}}})\,, \tag{3.68}$$

where $\mathcal{O}_{S_{\mathrm{GUT}}}$ and $K_{S_{\mathrm{GUT}}}$ denote the trivial and the canonical bundle on S_{GUT}. The monodromy is encoded in the factorization of C_{10} as the number of $U(1)$'s that remain independent is in general one less than the number of factors of C_{10}. To visualize how this comes about one can picture all **10** curves to be one single **10** curve on the five-sheeted cover. The branch cuts connect the different layers causing the cover to break into slices, each of which is associated to a factor of C_{10} and corresponds to one **10** curve. Concretely, one can locally define a parameter $s = U/V$ and the five roots of (3.67), written as a polynomial in s, will then correspond to the five t_i. In order to realize the $\mathbb{Z}_2$ monodromy, a factorization into four parts is needed, where the curves $\mathbf{10}_1$ and $\mathbf{10}_2$ lift to a single curve on the spectral cover:

$$C_{10} = \left(a_1 V^2 + a_2 VU + a_3 U^2\right)(a_4 V + a_7 U)(a_5 V + a_8 U)(a_6 V + a_9 U) = 0\,. \tag{3.69}$$

It is possible to solve for the a_i in terms of the b_k and to calculate their homology classes. Note that the CY condition for Y_4 implies that the b_k are sections of line bundles with first Chern class $\eta - kc_1$, where

$$\eta = 6c_1 - t\,, \quad t = -c_1(N)\,, \quad c_1 = c_1(S_{\mathrm{GUT}}) \tag{3.70}$$

and N denotes the normal bundle of S_{GUT}. Since there are six b_k but nine a_i, three bundles remain unspecified. They are χ_7, χ_8 and χ_9, corresponding to a_7, a_8 and a_9.

The constraint one gets from

$$b_1 \sim t_1 + t_2 + t_3 + t_4 + t_5 = 0 \tag{3.71}$$

implies

$$a_2 a_7 a_8 a_9 + a_3 a_6 a_7 a_8 + a_3 a_5 a_7 a_9 + a_3 a_4 a_8 a_9 = 0 \tag{3.72}$$

Section	c_1(Bundle)
a_1	$\eta - 2c_1 - \tilde{\chi}$
a_2	$\eta - c_1 - \tilde{\chi}$
a_3	$\eta - \tilde{\chi}$
a_4	$-c_1 + \chi_7$
a_5	$-c_1 + \chi_8$
a_6	$-c_1 + \chi_9$
a_7	χ_7
a_8	χ_8
a_9	χ_9

Table 3.2.: *The first Chern classes of the line bundles corresponding to the* a_i.

and is nontrivial. It can be solved by the ansatz

$$\begin{aligned} a_2 &= -c\,(a_6a_7a_8 + a_5a_7a_9 + a_4a_8a_9)\ , \\ a_3 &= c\ a_7a_8a_9 \end{aligned} \tag{3.73}$$

without inducing non-Kodaira singularities, as was shown in [42]. This, however, does not need to be the only solution and might thus not constitute the most general one. The homology class $[c]$ introduced above is given by

$$[c] = \eta - 2\tilde{\chi}\,, \tag{3.74}$$

where $\tilde{\chi} = \chi_7 + \chi_8 + \chi_9$. Table 3.2 summarizes the Chern classes of the various bundles for all a_i.

The next step is to determine the matter curves in terms of the a_i. From (3.47) we know that the **10** curves are given by $t_i = 0$, which implies that $b_5 = t_1t_2t_3t_4t_5 = 0$. This, in turn, is the coefficient of V^5 and thus must also be equal to $a_1a_4a_5a_6$, as one can see from (3.69). Therefore, the **10** curves are given by $a_i = 0$, where $i = 1, 4, 5, 6$.

In order to determine the equations for the **5** curves, one must plug (3.73) into the defining polynomial for the **5** curves (3.26). This gives

$$\begin{aligned} P_5 = &\,(a_5a_7 + a_4a_8)\,(a_6a_7 + a_4a_9)\,(a_6a_8 + a_5a_9) \\ &\times (a_6a_7a_8 + a_5a_7a_9 + a_4a_8a_9) \\ &\times (a_1 - ca_5a_6a_7 - ca_4a_6a_8) \\ &\times (a_1 - ca_5a_6a_7 - ca_4a_5a_9) \\ &\times (a_1 - ca_4a_6a_8 - ca_4a_5a_9)\ , \end{aligned} \tag{3.75}$$

and we arrive at table 3.3, which displays the curves with their $SU(5)_\perp$ charges, their defining equations in terms of the a_i and the resulting homology classes.

After imposing the monodromy, only three independent $U(1)$ fluxes remain. If one denotes their restrictions to the curves $\mathbf{5}_1$, $\mathbf{5}_2$ and $\mathbf{5}_3$ by $M_{-t_1-t_3} = M_{\mathbf{5}_1}$, $M_{-t_1-t_4} = M_{\mathbf{5}_2}$ and $M_{-t_1-t_5} = M_{\mathbf{5}_3}$, it becomes apparent that the $U(1)$ flux restriction to $\mathbf{10}_1$ can be expressed as

Curve	$SU(5)_\perp$ charge	Equation	Homology
$\mathbf{5}_{H_u}$	$-2t_1$	$a_6a_7a_8 + a_5a_7a_9 + a_4a_8a_9$	$-c_1 + \tilde{\chi}$
$\mathbf{5}_1$	$-t_1 - t_3$	$a_1 - ca_4a_6a_8 - ca_4a_5a_9$	$\eta - 2c_1 - \tilde{\chi}$
$\mathbf{5}_2$	$-t_1 - t_4$	$a_1 - ca_5a_6a_7 - ca_4a_5a_9$	$\eta - 2c_1 - \tilde{\chi}$
$\mathbf{5}_3$	$-t_1 - t_5$	$a_1 - ca_5a_6a_7 - ca_4a_6a_8$	$\eta - 2c_1 - \tilde{\chi}$
$\mathbf{5}_4$	$-t_3 - t_4$	$a_5a_7 + a_4a_8$	$-c_1 + \chi_7 + \chi_8$
$\mathbf{5}_5$	$-t_3 - t_5$	$a_6a_7 + a_4a_9$	$-c_1 + \chi_7 + \chi_9$
$\mathbf{5}_6$	$-t_4 - t_5$	$a_6a_8 + a_5a_9$	$-c_1 + \chi_8 + \chi_9$
$\mathbf{10}_1$	t_1	a_1	$\eta - 2c_1 - \tilde{\chi}$
$\mathbf{10}_2$	t_3	a_4	$-c_1 + \chi_7$
$\mathbf{10}_3$	t_4	a_5	$-c_1 + \chi_8$
$\mathbf{10}_4$	t_5	a_6	$-c_1 + \chi_9$

Table 3.3.: *Matter curves and their homology classes.*

$$\begin{aligned} M_{\mathbf{10}_1} &= M_{t_1} = M_{-(-3t_1+2t_1)} = M_{-(-3t_1-t_3-t_4-t_5)} = M_{-(-t_1-t_3-t_1-t_4-t_1-t_5)} \\ &= -(M_{\mathbf{5}_1} + M_{\mathbf{5}_2} + M_{\mathbf{5}_3})\,. \end{aligned} \tag{3.76}$$

In addition to (3.65), this leads to the following constraint on the M's:

$$M_{\mathbf{10}_1} = -(M_{\mathbf{5}_1} + M_{\mathbf{5}_2} + M_{\mathbf{5}_3})\,. \tag{3.77}$$

It is interesting to note that (3.66) yields

$$\sum_{\mathbf{5}} N = \sum_{\mathbf{10}} N = 0\,. \tag{3.78}$$

The fields in the representations $n_{(1,2)_{+1/2}}$, $n_{(\bar{3},1)_{-2/3}}$ and $n_{(1,1)_{+1}}$ therefore have no net chirality.

From the column of table 3.3 that displays the homology classes one sees that the hypercharge restrictions to the curves are determined solely by N_7, N_8 and N_9. Table 3.4 summarizes the final values for N_Y and M, which will be needed for the calculation of the spectra that is to be performed in the next chapter.

It is important to note that there are strong correlations between the flux restrictions. For example, a split of the up-type Higgs curve inevitably leads to a split of the $\mathbf{10}_1$ curve.

The Point Of E_8

Working within the local framework, one can either build a model where the various interactions take place at different locations on the manifold or have all matter curves meet at one single point. This point is characterized by the vanishing of all t_i and thus all $b_k = 0$ except b_0. The singularity is then enhanced to E_8, as can be seen from (3.48), which becomes $y^2 = x^3 + w^5$. This point can in principle give rise to all matter and

	N_Y	M
10 Curves		
$\mathbf{10}_1$	$-\widetilde{N}$	$-(M_{\mathbf{5}_1} + M_{\mathbf{5}_2} + M_{\mathbf{5}_3})$
$\mathbf{10}_2$	N_7	$M_{\mathbf{10}_2}$
$\mathbf{10}_3$	N_8	$M_{\mathbf{10}_3}$
$\mathbf{10}_4$	N_9	$M_{\mathbf{10}_4}$
5 Curves		
$\mathbf{5}_{H_u}$	$\widetilde{N}$	$M_{\mathbf{5}_{H_u}}$
$\mathbf{5}_1$	$-\widetilde{N}$	$M_{\mathbf{5}_1}$
$\mathbf{5}_2$	$-\widetilde{N}$	$M_{\mathbf{5}_2}$
$\mathbf{5}_3$	$-\widetilde{N}$	$M_{\mathbf{5}_3}$
$\mathbf{5}_4$	$N_7 + N_8$	$M_{\mathbf{5}_4}$
$\mathbf{5}_5$	$N_7 + N_9$	$M_{\mathbf{5}_5}$
$\mathbf{5}_6$	$N_8 + N_9$	$M_{\mathbf{5}_6}$

Table 3.4.: *Restrictions of hypercharge and $U(1)$ fluxes to the curves,* $\widetilde{N} = N_7 + N_8 + N_9$.

interactions of the MSSM [20]. Apart from the phenomenological advantages of this setup there are some practical benefits: The allowed interactions are determined purely by the quantum numbers, i.e. the t_i, and there are no geometric suppression effects due to contingent separations of Yukawa points. This makes this setup very predictive. A selection of works that have so far analyzed it is given by [5, 6, 11, 35, 36, 43–45].

Let me summarize the assumptions that are made when considering just a point: Inter alia one has the freedom to choose the monodromy group as well as the zero modes on the matter curves by hand. Furthermore, arbitrary VEVs can be given to singlets, which is needed to generate mass textures. The mass terms that include singlet insertions are of the form

$$\mathbf{1}_a \cdots \mathbf{1}_b \mathbf{5}_{H_u} \mathbf{10}_M \mathbf{10}_M \,, \qquad \mathbf{1}_a \cdots \mathbf{1}_b \overline{\mathbf{5}}_{H_d} \overline{\mathbf{5}}_M \mathbf{10}_M \,. \tag{3.79}$$

Since all interactions take place at the same point in the compact space, it is assumed that all coupling that are allowed by gauge symmetry are present with order-one coefficients. Last but not least, a globally trivial but locally nontrivial hypercharge flux is presumed which achieves doublet-triplet splitting for the up- and down-type Higgses without introducing exotics on other curves.

In the following chapter local F-theory models at the point of E_8 are investigated which aim at reproducing the superpotential of the MSSM. In chapter 5 I will present the results of the efforts to take into account constraints from semilocal consistency, which is the first step towards a global completion. The spectral cover results obtained above will be applied to the local models and I will discuss the implications for the local construction.

4. Local F-theory models with a stable proton

As pointed out in chapter 2, the superpotential gives rise to the quark and lepton masses but may also contain potentially dangerous terms that cause fast proton decay. It is interesting to see whether the ingredients available at a *point* of the compact space alone are sufficient to realize a model that allows for proton stability and can reproduce the correct quark and lepton masses. We have seen that a successful way to prohibit most of the terms that violate B and L is to impose matter parity. Therefore, the aim will be to find an $SU(5)$ GUT model based on a point of E_8 symmetry enhancement, i.e. an assignment of fields to the matter curves displayed in (3.47), with the following properties:

- A matter parity $P_M \subset SU(5)_\perp$,
- a heavy top quark, i.e. a tree-level rank-one up-type Yukawa coupling involving the third generation **10** curve,
- absence of dimension-five proton decay via the W^1 operator, and
- masses for all quarks and leptons after switching on singlet VEVs.

As argued at the end of chapter 3, one must assume that all operators which are allowed by gauge symmetry and matter parity are present with order-one coefficients. This means that the operators W^3 and W^1 must be absent or generated at very high order with singlet insertions. We will later see that when splitting the Higgs curves, some of the **10** multiplets will be split as well. So one could wonder whether it is possible that the W^1 operator is present at the $SU(5)$ level while the split is such that there are no dangerous terms for the SM multiplets. To see that this is not possible, let me write out W^1 in terms of SM representations:

$$\begin{aligned} W^1_{ijkl}\mathbf{10}^i_M\mathbf{10}^j_M\mathbf{10}^k_M\overline{\mathbf{5}}^l_M = \; & W^1_{ijkl}\overline{e}^i\overline{u}^jQ^kL^l + W^1_{ijkl}\overline{e}^i\overline{u}^j\overline{u}^k\overline{d}^l \\ & + W^1_{ijkl}\overline{u}^iQ^jQ^k\overline{d}^l + W^1_{ijkl}Q^iQ^jQ^kL^l \, . \end{aligned} \tag{4.1}$$

The first term, for example, is only absent for all i, j and k when the $\overline{\mathbf{5}}^l_M$ is split such that there is no net-chiral L^l. For the second term, however, the same argument applies for $\overline{d}^l$. Since this reasoning works for all l, and because of the requirement that at least the sum of the chiralities over all l is minus three for both L and $\overline{d}$, the conclusion is that W^1 must be absent at the $SU(5)$ level.

The next section outlines the steps that lead to the two local models presented in this work. To specify a concrete model, one has to choose the monodromy, define the matter parity P_M, assign matter and Higgs fields to the curves and select a set of singlets that get a VEV. It will turn out that the number of options is drastically reduced by the few phenomenological requirements above. I will start with general arguments that must be valid in the local as well as in the semilocal framework by showing in section 4.1 that there are only two possible definitions of matter parity and essentially one choice of monodromy group. In the subsequent sections I will first consider Case I, demonstrating in section 4.2.1 that the way to assign matter fields to curves is very restricted when requiring the absence of proton decay. In section 4.2.2 the possibilities to choose the down-type Higgs curve will be discussed and we will see that this leads in fact to a unique local model for Case I, whose phenomenology will be examined in section 4.2.3. Afterwards, in section 4.3, a similar analysis will be performed for Case II, where there is the possibility to enlarge the monodromy group without introducing qualitatively new features.

4.1. Matter parity and monodromy

I would first like to motivate the choice of the monodromy group and the definition of matter parity. As argued in section 3.5.3, at least a $\mathbb{Z}_2$ monodromy is required to get a tree-level up-type Yukawa coupling leading to a heavy top quark. The matter parity emerges from $SU(5)_\perp$ and therefore must be defined in terms of the t_i. Each t_i can either contribute a factor of $+1$ or -1 and thus, a formula for P_M can be written in full generality as $P_M = (-1)^{\alpha_1 t_1 + \alpha_2 t_2 + \alpha_3 t_3 + \alpha_4 t_4 + \alpha_5 t_5}$, where $\alpha_i \in \{1, 2\}$. Note that the up-type Yukawa coupling will always be allowed by matter parity, because the requirement of gauge invariance alone leads to $P_M(\text{up-type Yukawa coupling}) = (-1)^0$ since the t's cancel, and this conclusion persists no matter how many singlets are inserted.

The requirement that the down-type Yukawa couplings $\mathbf{10}_M \overline{\mathbf{5}}_M \overline{\mathbf{5}}_{H_d}$ should be allowed *does* give a constraint on the matter parity definition though: Given that the $\mathbf{10}_M$ contributes a factor of t_i and the $\overline{\mathbf{5}}$'s account for $t_j + t_k$ and $t_l + t_m$, all of which have a positive sign, this operator can, in contrast to the up-type Yukawa coupling, only be gauge invariant if all t's are different. Remember that the sum $t_1 + t_2 + t_3 + t_4 + t_5$ is zero according to (3.46). At the same time the desired down-type Yukawa coupling must have matter parity $+1$ to be permitted, which can only be achieved provided the number of t's with a prefactor of two in the matter parity definition is odd. Note that this fact remains true with any number of singlet insertions because the singlets all have charge assignments of the form $t_i - t_j$ and thus do not change the number of t's in the operator. When setting all five α's to two, there will not be a single field left that could be identified with matter since matter must have $P_M = -1$. One option is to set only a single α to two, which I chose to be α_5:

$$\text{Case I}: \quad P_M = (-1)^{t_1 + t_2 + t_3 + t_4 + 2t_5} . \tag{4.2}$$

This model will be analyzed in section 4.2. Finally, having three prefactors of two in the matter parity definition would coerce three generations to come from a single $\mathbf{10}_M$ curve, namely the top curve. I will examine the model corresponding to the matter parity

$$\text{Case II}: \quad P_M = (-1)^{t_1+t_2+2t_3+2t_4+2t_5} \tag{4.3}$$

in section 4.3.

Now let me come back to the choice of the monodromy group. In Case I, for P_M to be well-defined, t_5 must not be related to any other t. So there are only two options left which maintain the chance of building a model where the three generations of the standard model emerge from at least two curves. The first one is to enlarge the monodromy group such that another t lies in the same orbit as t_1 and t_2 and the second one is to additionally relate t_3 and t_4 by a $\mathbb{Z}_2$ monodromy. Both ideas are to be discarded because they are accompanied by the occurrence of the operator $W^1_{ijkl}\mathbf{10}^i_M\mathbf{10}^j_M\mathbf{10}^k_M\overline{\mathbf{5}}^l_M$, which leads to proton decay. The origin of this issue is that a gauge invariant W^1 operator also needs a sum over all five distinct t's, as it is the case for the down-type Yukawa couplings. More precisely, since the three $\mathbf{10}$'s in W^1 each add a t with a prefactor of one, for an allowed W^1 the $\overline{\mathbf{5}}$ must provide the remaining two t's, in particular a factor of t_5 that cannot get in via the $\mathbf{10}_M$'s. A $\overline{\mathbf{5}}$ always has a charge assignment $t_i + t_j$, so for it to have $P_M = -1$ and from the definition of matter parity it is evident that one of these t's must in fact be t_5. Now the only chance to avoid W^1 is to not identify one of the odd matter parity $\mathbf{10}$'s with SM matter. Since already with the $\mathbb{Z}_2$ monodromy there are only three odd parity curves left, it is evident that the monodromy group must not be enlarged any further.

In Case II t_3, t_4 and t_5 appear symmetrically in the definition of P_M. Here it is possible to mutually relate them by an arbitrary monodromy, but we will see later that this does not affect the phenomenology of the resulting model and therefore one can as well leave it at the monodromy relating t_1 and t_2.

4.2. Matter parity Case I

4.2.1. Matter curves and singlet VEVs

Having fixed the monodromy group and a formula for matter parity,

$$P_M = (-1)^{t_1+t_2+t_3+t_4+2t_5} \,,$$

I proceed with the discussion which fields should be assigned to the different curves. The aim of this selection is to prevent the appearance of baryon and lepton number violating operators. The curves, their charges and matter parities are collected in table 4.1. As already mentioned in the previous section, all dimension-three, four and five baryon and lepton number violating operators apart from $W^3_{ij}\overline{\mathbf{5}}^i_M\overline{\mathbf{5}}^j_M\mathbf{5}_{H_u}\mathbf{5}_{H_u}$ and $W^1_{ijkl}\mathbf{10}^i_M\mathbf{10}^j_M\mathbf{10}^k_M\overline{\mathbf{5}}^l_M$ are forbidden by matter parity. Since the two up-type Higgs curves in W^3 contribute a factor $-2t_1 - 2t_2$ to the operator, W^3 is absent at tree level because these charges cannot be canceled by adding the t's for the other two curves. On the other hand, W^1 will appear at tree level, as argued above, unless some of the odd matter parity curves are declared to not carry SM matter.

From (4.4), which lists all gauge invariant combinations involving the **10**'s and **5**'s with $P_M = -1$, one can see that it is possible to evade W^1 when not assigning SM fields to the curves $\mathbf{10}_2$ and $\mathbf{5}_5$, because if these two curves do not carry SM fields, there is no operator left that contains SM fields only[1].

$$\begin{aligned} &\mathbf{10}_1\mathbf{10}_1\mathbf{10}_2\overline{\mathbf{5}}_6 \\ &\mathbf{10}_1\mathbf{10}_1\mathbf{10}_3\overline{\mathbf{5}}_5 \\ &\mathbf{10}_1\mathbf{10}_2\mathbf{10}_3\overline{\mathbf{5}}_3 \end{aligned} \tag{4.4}$$

Hence, the **10** and **5** curves carrying SM matter are fixed to be $\mathbf{10}_1$ and $\mathbf{10}_3$ as well as $\mathbf{5}_3$ and $\mathbf{5}_6$, respectively, because one can see from table 4.1 that these are the only remaining fields with odd matter parity. Because of this three fields must emerge from two curves.

Addressing the even matter parity **5** curves, there are four possibilities to assign the down-type Higgs to one of the Higgs-like curves. The choice must be made such that after turning on VEVs for selected singlets the down quarks become massive without reintroducing operators that lead to proton decay. Odd matter parity singlets must not receive a VEV because this would break matter parity and reintroduce the successfully eliminated baryon and lepton number violating operators. W^1 will be generated immediately when VEVs are given to the singlets $\mathbf{1}_1$ or $\mathbf{1}_4$ because the matter **10**'s have the charges t_1, t_2 and t_4 and the matter **5**'s have the charges $\{t_1 + t_5, t_2 + t_5\}$ and $t_4 + t_5$. $\mathbf{1}_1$ and $\mathbf{1}_4$ have the charges $\pm\{t_1 - t_3, t_2 - t_3\}$ and $\pm(t_3 - t_4)$ and thus both contain a t_3, which is needed for W^1 to be gauge invariant. Therefore, they must not get a VEV. Summing up, the aim is to select a down-type Higgs curve that gives masses to the down-type quarks using only VEVs for the singlets $\mathbf{1}_2$ and $\mathbf{1}_7$. The assignment of fields to the matter curves is summarized in the last column of table 4.1.

4.2.2. Higgs curves

In this section we will see that, when working in the purely local framework, requiring no proton decay and down-type masses at the same time singles out a unique model for Case I. The main assumption is that the chiralities of the curves can be chosen freely while simultaneously the Higgs curves can be split correctly, that is, only the Higgs doublets remain light. In particular, it is assumed that all **10** and **5** curves apart from $\mathbf{10}_1$, $\mathbf{10}_3$, $\mathbf{5}_{H_u}$, $\overline{\mathbf{5}}_{H_d}$, $\overline{\mathbf{5}}_3$ and $\overline{\mathbf{5}}_6$ appear as vector-like pairs and can be given a high-scale mass. In chapter 5 this point will be analyzed in more detail.

The previous section tells us that, since the SM matter curves are fixed, the next important question is to which of the four possible even matter parity **5** curves $\mathbf{5}_{H_u}$, $\mathbf{5}_1$, $\mathbf{5}_2$ and $\mathbf{5}_4$ the down-type Higgs field is assigned. Table 4.2 lists all gauge invariant tree-level down-type Yukawa couplings for the different choices of the down-type Higgs curve.

[1] One could also pick $\mathbf{10}_3$ and $\mathbf{5}_6$ instead but this choice just amounts to a relabeling.

	Charge	Matter Parity	Assigned Fields
10 Curves			
$\mathbf{10}_1$	$\{t_1, t_2\}$	$-$	$\mathbf{10}_{\text{top}}$, possibly more
$\mathbf{10}_2$	t_3	$-$	no SM matter
$\mathbf{10}_3$	t_4	$-$	possible SM matter
$\mathbf{10}_4$	t_5	$+$	no SM matter
5 Curves			
$\mathbf{5}_{H_u}$	$-t_1 - t_2$	$+$	up-type Higgs
$\mathbf{5}_1$	$\{-t_1 - t_3, -t_2 - t_3\}$	$+$	Higgs-like
$\mathbf{5}_2$	$\{-t_1 - t_4, -t_2 - t_4\}$	$+$	Higgs-like
$\mathbf{5}_3$	$\{-t_1 - t_5, -t_2 - t_5\}$	$-$	possible SM matter
$\mathbf{5}_4$	$-t_3 - t_4$	$+$	Higgs-like
$\mathbf{5}_5$	$-t_3 - t_5$	$-$	no SM matter
$\mathbf{5}_6$	$-t_4 - t_5$	$-$	possible SM matter
Singlets			
$\mathbf{1}_1$	$\pm\{t_1 - t_3, t_2 - t_3\}$	$+$	no VEV
$\mathbf{1}_2$	$\pm\{t_1 - t_4, t_2 - t_4\}$	$+$	VEV possible
$\mathbf{1}_3$	$\pm\{t_1 - t_5, t_2 - t_5\}$	$-$	no VEV
$\mathbf{1}_4$	$\pm(t_3 - t_4)$	$+$	no VEV
$\mathbf{1}_5$	$\pm(t_3 - t_5)$	$-$	no VEV
$\mathbf{1}_6$	$\pm(t_4 - t_5)$	$-$	no VEV
$\mathbf{1}_7$	$\{t_1 - t_2, t_2 - t_1\}$	$+$	VEV possible

Table 4.1.: *List of curves, their charges and their matter parity values with the field assignment for Case I.*

Down-type Higgs curve	Gauge invariant couplings
$\overline{\mathbf{5}}_{H_u}$	—
$\overline{\mathbf{5}}_1$	$\overline{\mathbf{5}}_{H_d}\mathbf{10}_1\overline{\mathbf{5}}_6$ $\overline{\mathbf{5}}_{H_d}\mathbf{10}_3\overline{\mathbf{5}}_3$
$\overline{\mathbf{5}}_2$	—
$\overline{\mathbf{5}}_4$	$\overline{\mathbf{5}}_{H_d}\mathbf{10}_1\overline{\mathbf{5}}_3$

Table 4.2.: *Gauge invariant down-type Yukawa terms for all possible choices of down-type Higgs curves for Case I.*

Taking the down-type Higgs curve to be $\overline{\mathbf{5}}_1$ leads to a rank-two Yukawa matrix at tree level resulting in two heavy and one light generation, which is phenomenologically problematic. One can check that a particular split of the curves reduces the rank of the matrix to one or zero, but since with the spectral cover formalism it is not possible to realize the $\overline{\mathbf{5}}_1$ as the down-type Higgs curve anyway, as we will see later, I dismiss this option here and relegate more details to appendix A.1.

Next, the choice of $\overline{\mathbf{5}}_{H_u}$ or $\overline{\mathbf{5}}_2$ with charges $\{t_1+t_2\}$ and $\{t_1+t_4, t_2+t_4\}$ is considered. It is easy to see that in both cases there cannot be any down-type masses because the matter $\mathbf{10}$'s, $\mathbf{10}_1$ and $\mathbf{10}_3$, have charges $\{t_1, t_2\}$ and t_4, while the matter $\overline{\mathbf{5}}$'s, $\overline{\mathbf{5}}_3$ and $\overline{\mathbf{5}}_6$, have charges $\{t_1+t_5, t_2+t_5\}$ and t_4+t_5, and the only two singlets that one is allowed to give a VEV to, as discussed in the previous section, are $\mathbf{1}_2$ and $\mathbf{1}_7$ with charges $\pm\{t_1-t_4, t_2-t_4\}$ and $\{t_1-t_2, t_2-t_1\}$. For the down-type mass term to be gauge invariant, a sum over all five distinct t's is needed and with this choice it is obvious that the sum will always lack a t_3 factor. Thus, the possibilities to select $\overline{\mathbf{5}}_{H_u}$ or $\overline{\mathbf{5}}_2$ as the down-type Higgs curve are excluded. Note that this conclusion holds even if the curves are eventually split.

This ultimately fixes the down-type Higgs curve to

$$\overline{\mathbf{5}}_{H_d} = \overline{\mathbf{5}}_4\,, \tag{4.5}$$

which is favored anyway because it leads to a rank-one down-type Yukawa matrix at tree level. Demanding that the quark which gets the large mass is the bottom quark then amounts to assigning the bottom quark generation to the curve $\overline{\mathbf{5}}_3$ and the light generations to the curve $\overline{\mathbf{5}}_6$:

$$\overline{\mathbf{5}}_{\text{bottom}} = \overline{\mathbf{5}}_3\,, \quad \overline{\mathbf{5}}_{\text{down/strange}} = \overline{\mathbf{5}}_6\,. \tag{4.6}$$

Recapitulating, the requirements of no proton decay, a heavy top quark and a down-type Yukawa matrix which has rank zero or one have lead to a unique model. The next step is to explore its phenomenology.

4.2.3. Masses and mixings

Let us now examine the Yukawa textures and the CKM matrix in the model that was just specified to see whether reasonable mass hierarchies and mixings in the quark sector can be achieved.

Choosing $\mathbf{10}_1$ to carry only the top generation and $\mathbf{10}_3$ to accommodate the light generations, the up-type Yukawa matrix including insertions of the singlets $\mathbf{1}_2$ and $\mathbf{1}_7$ reads, at leading order,

$$Y^u_{ij} \sim \begin{pmatrix} \epsilon^2 & \epsilon^2 & \epsilon \\ \epsilon^2 & \epsilon^2 & \epsilon \\ \epsilon & \epsilon & 1 \end{pmatrix}\,, \tag{4.7}$$

where

$$\epsilon = \frac{\langle X_2 \rangle}{M_*} \,. \tag{4.8}$$

Here, $\langle X_2 \rangle$ is the VEV for the field assigned to the curve $\mathbf{1}_2$, which is suppressed by the winding scale M_*. This scale has been shown to coincide which the GUT scale for local models [46]. The VEV for $\mathbf{1}_7$ will only appear at higher order and can just as well be completely omitted.

It is important to note that there are order-one prefactors in front of each entry, which depend on the geometry of the different curves and also come from integrating out heavy states in the case of elements with singlet insertions. Within this framework these prefactors cannot be determined. Nevertheless, it is possible to see if the masses and mixings show acceptable patterns.

For the down-type Yukawa matrix the result is similar:

$$Y^d_{ij} \sim \begin{pmatrix} \epsilon^2 & \epsilon^2 & \epsilon \\ \epsilon^2 & \epsilon^2 & \epsilon \\ \epsilon & \epsilon & 1 \end{pmatrix} \,. \tag{4.9}$$

Since both matrices are approximately diagonal, I used the simplified formulae for the mixing angles in the CKM matrix [47]

$$s^{\text{CKM}}_{ij} \simeq s^d_{ij} - s^u_{ij} \,, \tag{4.10}$$

where

$$s^{u/d}_{12} = \frac{Y^{u/d}_{12}}{Y^{u/d}_{22}} \,, \quad s^{u/d}_{13} = \frac{Y^{u/d}_{13}}{Y^{u/d}_{33}} \,, \quad s^{u/d}_{23} = \frac{Y^{u/d}_{23}}{Y^{u/d}_{33}} \,, \tag{4.11}$$

in order to arrive at

$$V_{\text{CKM}} \sim \begin{pmatrix} 1 & 1 & \epsilon \\ 1 & 1 & \epsilon \\ \epsilon & \epsilon & 1 \end{pmatrix} \,. \tag{4.12}$$

These results show that in this model, where the Yukawa matrices and the CKM matrix contain only a single parameter $\langle X_2 \rangle$, a mass can be given to all generations and mixing is achieved. Neither the Yukawa matrices nor the CKM matrix match the SM data very well, but this was not expected because in the setup at hand three generations come from only two curves.

4.3. Matter parity Case II

In this section I present an analysis of the matter parity Case II along the lines of the previous section: I will first clarify the possible field assignment and then discuss in section 4.3.2 the down-type Higgs sector and the resulting Yukawa textures.

The matter parity in Case II is defined as

$$P_M = (-1)^{t_1+t_2+2t_3+2t_4+2t_5} .$$

It will turn out that one generally has more freedom in this model because the $SU(5)_\perp$ charges split into two decoupled groups,

$$t_{\text{even}} = \{t_3, t_4, t_5\} \text{ and } t_{\text{odd}} = \{t_1, t_2\} \tag{4.13}$$

such that the possible Higgs **5** curves involve two t_{even}'s and the matter **5** curves involve one t_{even} and one t_{odd}. Furthermore, the positive matter parity singlets do not mix t_{even} and t_{odd}.

As in Case I, a basically unique model will emerge. There is only one matter **10** curve, and both Higgses are unique (up to a relabeling of the t_{even}). The only freedom is in the choice of matter $\overline{\mathbf{5}}$ curves: One can choose one, two or three curves for the three generations, or alternatively identify some of these by an extended monodromy. However, this will not give qualitatively new features.

4.3.1. Matter assignment

In the case at hand the different curves have even or odd matter parity as displayed in table 4.3. Since there is only one odd matter parity **10** curve in Case II, all three generations of SM fields that belong to the **10** representation have to be assigned to $\mathbf{10}_1$. Note, however, that one can choose the matter in the $\overline{\mathbf{5}}$ representation to emerge from one, two or three $\overline{\mathbf{5}}$ curves.

Concerning the W^1 operator, the situation has improved compared to Case I: Using t_{even} and t_{odd} as defined above, the operator $\mathbf{10}_M\mathbf{10}_M\mathbf{10}_M\overline{\mathbf{5}}_M$ has the charge $4t_{\text{odd}}+t_{\text{even}}$. Therefore, with this choice of matter parity and under the assumption that VEVs are given only to even matter parity singlets, W^1 cannot be generated regardless of which and how many singlets are inserted. This statement is also completely independent of the assignment of fields to the curves, which is only constrained by the matter parity and shown in table 4.3.

4.3.2. Higgs assignment and flavor

Since there is only one **10** matter curve, there is only one up-type Yukawa coupling, $\mathbf{5}_{H_u}\mathbf{10}_1\mathbf{10}_1$, which at tree level leads to the Yukawa matrix

$$Y^u_{ij} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} . \tag{4.14}$$

It is possible to get singlet contributions from e.g. the VEV of $\mathbf{1}_7$, or various higher powers of other singlet VEVs. The generic form of the Yukawa matrix is then

$$Y^u_{ij} \sim \begin{pmatrix} \epsilon & \epsilon & \epsilon \\ \epsilon & \epsilon & \epsilon \\ \epsilon & \epsilon & 1 \end{pmatrix} . \tag{4.15}$$

	Charge	Matter Parity	Assigned Fields
10 Curves			
$\mathbf{10}_1$	$\{t_1, t_2\}$	$-$	all families
$\mathbf{10}_2$	t_3	$+$	no SM matter
$\mathbf{10}_3$	t_4	$+$	no SM matter
$\mathbf{10}_4$	t_5	$+$	no SM matter
5 Curves			
$\mathbf{5}_{H_u}$	$-t_1 - t_2$	$+$	up-type Higgs
$\mathbf{5}_1$	$\{-t_1 - t_3, -t_2 - t_3\}$	$-$	possible SM matter
$\mathbf{5}_2$	$\{-t_1 - t_4, -t_2 - t_4\}$	$-$	possible SM matter
$\mathbf{5}_3$	$\{-t_1 - t_5, -t_2 - t_5\}$	$-$	possible SM matter
$\mathbf{5}_4$	$-t_3 - t_4$	$+$	Higgs-like
$\mathbf{5}_5$	$-t_3 - t_5$	$+$	Higgs-like
$\mathbf{5}_6$	$-t_4 - t_5$	$+$	Higgs-like
Singlets			
$\mathbf{1}_1$	$\pm\{t_1 - t_3, t_2 - t_3\}$	$-$	no VEV
$\mathbf{1}_2$	$\pm\{t_1 - t_4, t_2 - t_4\}$	$-$	no VEV
$\mathbf{1}_3$	$\pm\{t_1 - t_5, t_2 - t_5\}$	$-$	no VEV
$\mathbf{1}_4$	$\pm(t_3 - t_4)$	$+$	VEV possible
$\mathbf{1}_5$	$\pm(t_3 - t_5)$	$+$	VEV possible
$\mathbf{1}_6$	$\pm(t_4 - t_5)$	$+$	VEV possible
$\mathbf{1}_7$	$\{t_1 - t_2, t_2 - t_1\}$	$+$	VEV possible

Table 4.3.: *List of curves, their charges and their matter parity values with the field assignment for Case II.*

Again, the ϵ's involve order-one coefficients.

Turning to the down-type Yukawa couplings, there are four Higgs-like $\overline{\mathbf{5}}$ curves. If the down-type Higgs is assigned to the same curve as the up-type Higgs, no down-type masses will be generated at any level, as can be seen from the charges. The reason for the absence is the same as for that of the W^1 operator: The coupling $\mathbf{10}_M \overline{\mathbf{5}}_M \overline{\mathbf{5}}_{H_u}$ involves $4t_{\text{odd}} + t_{\text{even}}$ which, as already stated, can never be made gauge invariant by singlet insertions. All other choices for the down-type Higgs curve are equivalent, given that this model is invariant under permutations of t_3, t_4 and t_5 and that the three remaining Higgs-like curves and possible $\overline{\mathbf{5}}$ matter curves only involve these charges. So the choice can be made as $\overline{\mathbf{5}}_{H_d} = \overline{\mathbf{5}}_4$. In order to have a tree-level coupling of the $\mathbf{10}_1$ curve to any down-type quark, one has to assign matter to the curve $\mathbf{5}_3$. It is also possible to start with a down-type Yukawa matrix of rank zero at tree level and generate all masses through singlet insertions. The allowed couplings to lowest order in the singlets between the down-type Higgs curve and the candidate SM matter $\overline{\mathbf{5}}$ curves are:

$$\mathbf{10}_1 \overline{\mathbf{5}}_3 \overline{\mathbf{5}}_{H_d}\,, \quad \mathbf{10}_1 \overline{\mathbf{5}}_1 \overline{\mathbf{5}}_{H_d} \mathbf{1}_5\,, \quad \mathbf{10}_1 \overline{\mathbf{5}}_1 \overline{\mathbf{5}}_{H_d} \mathbf{1}_4 \mathbf{1}_6\,, \quad \mathbf{10}_1 \overline{\mathbf{5}}_2 \overline{\mathbf{5}}_{H_d} \mathbf{1}_6\,, \quad \mathbf{10}_1 \overline{\mathbf{5}}_2 \overline{\mathbf{5}}_{H_d} \mathbf{1}_4 \mathbf{1}_5\,. \tag{4.16}$$

Depending on which singlets get a VEV and how the SM generations are assigned to the three $\overline{\mathbf{5}}$ matter curves, one can arrive at different down-type Yukawa matrices. Starting with a rank-one matrix at tree level, an option is to assign the bottom quark generation to $\mathbf{5}_3$ and the first and second generation to $\mathbf{5}_1$ and $\mathbf{5}_2$, respectively. After $\mathbf{1}_5$ and $\mathbf{1}_4$ received a VEV, the following down-type Yukawa matrix is obtained, where some entries are generated with only one singlet insertion and others with two:

$$Y^d_{ij} \sim \begin{pmatrix} \epsilon_5 & \epsilon_5 & \epsilon_5 \\ \epsilon_5\epsilon_4 & \epsilon_5\epsilon_4 & \epsilon_5\epsilon_4 \\ 0 & 0 & 1 \end{pmatrix} . \tag{4.17}$$

Switching on VEVs for $\mathbf{1}_5$ and $\mathbf{1}_6$, all entries involving $\overline{d}$ and $\overline{b}$ are generated with only one singlet insertion:

$$Y^d_{ij} \sim \begin{pmatrix} \epsilon_5 & \epsilon_5 & \epsilon_5 \\ \epsilon_6 & \epsilon_6 & \epsilon_6 \\ 0 & 0 & 1 \end{pmatrix} . \tag{4.18}$$

Of course, there also exists the option to assign both light generations to one curve, in which case one singlet insertion is sufficient to get all couplings.

If one starts with a rank-zero matrix at tree level, all matter must be assigned to $\mathbf{5}_1$ and $\mathbf{5}_2$. Choosing $\mathbf{5}_1$ to carry the bottom generation and $\mathbf{5}_2$ the other generations, a feasible matrix would be

$$Y^d_{ij} \sim \begin{pmatrix} \epsilon_5\epsilon_4 & \epsilon_5\epsilon_4 & \epsilon_5\epsilon_4 \\ \epsilon_5\epsilon_4 & \epsilon_5\epsilon_4 & \epsilon_5\epsilon_4 \\ \epsilon_5 & \epsilon_5 & \epsilon_5 \end{pmatrix} , \tag{4.19}$$

or the same matrix with $\epsilon_5\epsilon_4$ replaced by ϵ_6.

Since the aim of this work is not to present a detailed discussion of flavor, I will now move on to the question whether the models presented in this section can be realized in a more global framework.

5. Semilocal realization

This chapter is an attempt to extend the local models to semilocal models, which would be the first step in searching for a global realization. The important improvement of the semilocal framework, where the whole GUT surface S_{GUT} is considered, is that the chiral spectrum of a model can be calculated explicitly. I will apply the results for the flux restrictions to the curves, obtained in section 3.6, to the two local models, presented in chapter 4, and show that they unfortunately do not have a semilocal realization. One should, however, keep in mind that the approach used here is not the most general framework because it does not take into account the explicit form of the Higgs field but solely relies on the spectral equation. Therefore, it is implicitly based on reconstructible Higgs fields as introduced in section 3.5.3. In chapter 6 I will say more about the scope of the alternative approach, that is to not use a diagonal Higgs field but to employ general holomorphic single-valued Higgs fields as a background.

5.1. Semilocal embedding of Case I

The $\mathbf{10}$ curves that accommodate SM matter are fixed to be $\mathbf{10}_1$ and $\mathbf{10}_3$. It must be required that there are three net $\mathbf{10}$'s after splitting the curves. Using the results shown in table 3.4, this leads to the requirements

$$M_{\mathbf{10}_1} + M_{\mathbf{10}_3} = 3\,, \qquad\qquad N_7 + N_9 = 0\,. \tag{5.1}$$

Furthermore, it is necessary that $M_{\mathbf{10}_1} \geq 1$ and $M_{\mathbf{10}_1} + N_8 \geq 0$ to have a heavy top quark. Similarly, the conditions for the $\overline{\mathbf{5}}$ curves read:

$$M_{\mathbf{5}_3} + M_{\mathbf{5}_6} = -3\,, \qquad\qquad N_7 = 0\,. \tag{5.2}$$

Hence, also $N_9 = 0$, and the only remaining parameter that can be used to split some curves is N_8.

Let me continue with the other two $\mathbf{10}$ curves that are not associated with SM matter and should therefore better have no net chirality. This can be achieved easily by simply setting

$$M_{\mathbf{10}_2} = M_{\mathbf{10}_4} = 0\,. \tag{5.3}$$

Zero modes from the $\mathbf{5}_5$ curve can be evaded by also setting $M_{\mathbf{5}_5} = 0$. Thus, it is manageable to achieve a satisfactory matter sector.

	$n_{(3,1)_{-1/3}} - n_{(\bar{3},1)_{+1/3}}$	$n_{(1,2)_{+1/2}} - n_{(1,2)_{-1/2}}$
$\mathbf{5}_{H_u}$	$M_{\mathbf{5}_{H_u}}$	$M_{\mathbf{5}_{H_u}} + N_8$
$\mathbf{5}_1$	$M_{\mathbf{5}_1}$	$M_{\mathbf{5}_1} - N_8$
$\mathbf{5}_2$	$M_{\mathbf{5}_2}$	$M_{\mathbf{5}_2} - N_8$
$\mathbf{5}_4$	$M_{\mathbf{5}_4}$	$M_{\mathbf{5}_4} + N_8$

Table 5.1.: *Chiralities in terms of $U(1)$ and hypercharge flux restrictions for the Higgs-like* **5** *curves after imposing the matter sector constraints for Case I.*

I will now turn to the Higgs curves. The chiralities of the Higgs-like **5** curves are shown in table 5.1 in terms of the $U(1)$ and hypercharge flux restrictions. The M's are constrained by the condition (3.77) which now reads

$$M_{\mathbf{5}_{H_u}} + M_{\mathbf{5}_1} + M_{\mathbf{5}_2} + M_{\mathbf{5}_4} = 0\,. \tag{5.4}$$

The split of the Higgs-like **5** curves is controlled by the parameter N_8, as was already noted above. The goal is to obtain one down-type Higgs doublet and one up-type Higgs doublet. This is realized if the down-type Higgs doublet, being located on a $\overline{\mathbf{5}}$, has a chirality of -1, whereas the up-type Higgs doublet has a chirality of $+1$. Neither the down-type Higgs nor the up-type Higgs must have a light triplet. Since it is always assumed that fields which appear in vector-like pairs become massive, the simplest way to get rid of the triplets would be to set the corresponding M's to zero. However, they might be tolerated if they become heavy with the help of singlet insertions. Giving a VEV to the singlet $\mathbf{1}_2$, there are two allowed couplings between the four **5** curves of interest:

$$\overline{\mathbf{5}}_1\mathbf{5}_4\mathbf{1}_2\,, \tag{5.5}$$

$$\mathbf{5}_{H_u}\overline{\mathbf{5}}_2\mathbf{1}_2\,. \tag{5.6}$$

Hence, one can pairwise decouple a triplet from the $\mathbf{5}_{H_u}$ with an antitriplet from the $\mathbf{5}_2$ curve. At the same time, the up-type Higgs doublet must remain. In terms of the flux restrictions these requirements read

$$\begin{aligned} \text{doublets:} \quad & M_{\mathbf{5}_{H_u}} + N_8 + (M_{\mathbf{5}_2} - N_8) = 1\,, \\ \text{triplets:} \quad & M_{\mathbf{5}_{H_u}} + M_{\mathbf{5}_2} = 0\,, \end{aligned} \tag{5.7}$$

which is a contradiction. Therefore, one generically obtains a spectrum which exhibits additional exotic fields [36], or no up-type Higgs. Note that this result is independent of the hypercharge flux, since both the $\mathbf{5}_{H_u}$ and the $\mathbf{5}_2$ are split with $\widetilde{N}$.

A loophole would be to choose $\overline{\mathbf{5}}_2$ as the down-type Higgs curve. Then the spectrum is free of exotics, but the coupling (5.6) constitutes nothing but a μ-term. Furthermore, as discussed before, the $\overline{\mathbf{5}}_2$ curve has no t_3 factor, so there are no down-type masses in this model. One can nevertheless realize this spectrum with the parameter choice

$$M_{\mathbf{5}_{H_u}} = M_{\mathbf{5}_1} = M_{\mathbf{5}_2} = M_{\mathbf{5}_4} = 0\,, \quad N_8 = 1\,. \tag{5.8}$$

The remaining doublets from $\mathbf{5}_1$ and $\mathbf{5}_4$ can be decoupled by the term (5.5).

(a) Parameter Choice (5.8)

	Triplets	Doublets
$\mathbf{5}_5$	0	0
$\mathbf{5}_{H_u}$	0	1
$\mathbf{5}_1$	0	−1
$\mathbf{5}_2$	0	−1
$\mathbf{5}_4$	0	1

(b) Parameter Choice (5.10)

	Triplets	Doublets
$\mathbf{5}_5$	0	0
$\mathbf{5}_{H_u}$	0	1
$\mathbf{5}_1$	1	0
$\mathbf{5}_2$	1	0
$\mathbf{5}_4$	−2	−1

Table 5.2.: *Two possible splits of the Higgs curves for Case I.*

The embedding of the model from section 4.2.2, which has $\mathbf{5}_4$ as the down-type Higgs, fails for a similar argument as in (5.7) because one arrives at the mutually contradicting constraints

$$\begin{aligned} \text{doublets:} \quad & M_{\mathbf{5}_4} + N_8 + (M_{\mathbf{5}_1} - N_8) = -1\,, \\ \text{triplets:} \quad & M_{\mathbf{5}_4} + M_{\mathbf{5}_1} = 0\,. \end{aligned} \tag{5.9}$$

An example spectrum one can get using $\overline{\mathbf{5}}_4$ as the down-type Higgs curve is shown in table 5.1(b) with parameters

$$M_{\mathbf{5}_{H_u}} = 0\,, \quad M_{\mathbf{5}_1} = M_{\mathbf{5}_2} = 1\,, \quad M_{\mathbf{5}_4} = -2\,, \quad N_8 = 1\,. \tag{5.10}$$

The curve $\mathbf{5}_4$ has the desired doublet but also two triplets, one of which can be combined with the triplet of the $\mathbf{5}_1$ via the coupling $\overline{\mathbf{5}}_1\mathbf{5}_4\mathbf{1}_2$. The other one, however, will remain light and apart from that there is another unwanted triplet in the $\mathbf{5}_2$.

5.2. Semilocal embedding of Case II

Since there is only one **10** curve in this model that can carry SM matter, the requirements are

$$M_{\mathbf{10}_1} = 3\,, \quad N_{\mathbf{10}_1} = -\widetilde{N} = 0\,. \tag{5.11}$$

The first condition is unproblematic because it implies

$$M_{\mathbf{5}_1} + M_{\mathbf{5}_2} + M_{\mathbf{5}_3} = -3\,, \tag{5.12}$$

which is exactly what is needed since $\overline{\mathbf{5}}_1$, $\overline{\mathbf{5}}_2$ and $\overline{\mathbf{5}}_3$ are the possible matter curves. Furthermore, $\widetilde{N} = 0$ implies that the matter curves are not split, so again the matter sector works out reasonably. On the other hand, $\widetilde{N} = 0$ inhibits a split of the up-type Higgs because $N_{H_u} = \widetilde{N}$, see table 3.4. In addition, there is no way of coupling the up-type Higgs curve to some other even matter parity Higgs curve to make the triplet heavy, because the coupling has the charges $-2t_{\text{odd}} + 2t_{\text{even}}$ in terms of the notation introduced in section 4.3.1, which cannot be canceled with even matter parity singlets. Therefore, neither in Case I nor in Case II it is possible to arrive at a satisfying spectrum while giving masses to all quarks and leptons.

Note that it is due to the hypercharge flux restrictions to the Higgs-like curves that the semilocal embedding of the local models fails. This is true in both matter parity cases and even when allowing for exotics from the matter sector. Hence, it is ultimately the problem of doublet-triplet splitting which prohibits to extend the models globally.

6. T-branes

As opposed to the work presented so far, where the Higgs field breaking E_8 to $SU(5)$ was taken to be diagonal, this section deals with general holomorphic Higgs fields. I will start with a review of how localized matter is calculated in the T-brane framework following [15] and then motivate why such Higgs fields might be able to improve the models of chapter 4. Whereas in the original work [15] there are only a few examples given which show that the spectral equation does not capture all the properties of a T-brane background, the aim pursued in section 6.2 is to improve this situation towards concrete rules on what kind of matrices give rise to what kind of matter. I will propose an approach how to commence such a search starting with a computer-based scan of Higgs backgrounds. For each matrix the matter content is determined, and as an organizing principle for a classification of Higgs backgrounds with similar properties, the Galois group of the spectral equation is calculated in addition. In the end I will discuss the results and give an outlook on options for potential future analyses.

6.1. Localized matter

Holomorphic Gauge

From now on I will exclusively work in the holomorphic gauge, which makes use of complexified gauge transformations and is very convenient because the BPS equations (3.33)-(3.35) collapse to one purely algebraic one. As already mentioned in section 3.5.1, one can neglect the third BPS equation (3.35) when working with complexified gauge transformations such that the only concern is with equations (3.33) and (3.34). The holomorphic gauge is characterized by the vanishing of $A^{0,1}$. The linearized BPS equations then read

$$\bar{\partial} a = 0 \,, \tag{6.1}$$

$$\bar{\partial} \phi = [\Phi, a] \,. \tag{6.2}$$

Equation (6.1) can be solved by introducing a (0,0)-form ξ such that

$$a = \bar{\partial} \xi \,. \tag{6.3}$$

Note that this solution is only valid for trivial worldvolumes $V \subset \mathbb{C}^2 \subset S$ because on this patch the $\bar{\partial}$-operator is exact. As a consequence bulk modes cannot be studied. The second equation can then be integrated to

$$\phi = [\Phi, \xi] + h \,, \tag{6.4}$$

where h is a holomorphic (2,0)-form. Equation (3.41) implies that under a gauge transformation

$$\xi \to \xi + \chi \,. \tag{6.5}$$

Since χ is an element of the complexified Lie algebra $g_{\mathbb{C}}$ now, it is possible to set

$$\chi = -\xi \,. \tag{6.6}$$

Thus, a can be gauged to zero, from which follows that $\phi = h$ according to (6.2), which means that ϕ is now a holomorphic (2,0)-form. Keeping a gauge to zero, there is still freedom to perform gauge transformations with a holomorphic χ. Under these ϕ changes by $[\Phi, \chi]$: The space of gauge inequivalent modes is given by all holomorphic matrices modulo those which are commutators with the background Higgs field.

To see how this formalism reproduces the results of section 3.5.2, consider the diagonal Higgs field

$$\Phi = \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix} . \tag{6.7}$$

Under a gauge transformation a mode in the i-th row and j-th column of ϕ shifts as

$$\phi \to \phi + (\lambda_i - \lambda_j)\alpha \,, \tag{6.8}$$

where α is some arbitrary holomorphic function. Inequivalent modes in the direction $i - j$ can therefore be denoted by

$$\frac{\mathcal{O}}{I_{ij}} \,, \tag{6.9}$$

with $\mathcal{O}$ being the ring of holomorphic functions in the two brane worldvolume coordinates x and y and I_{ij} the ideal generated by $\lambda_i - \lambda_j$. This shows that it is very easy to study the existence of localized modes in the holomorphic gauge. For the computation of the explicit matter wave functions a change to the unitary gauge has to be made, but for all holomorphic quantities, like the superpotential, these wave functions are not needed anyway.

Six-Dimensional Matter Fields

As opposed to the diagonal background configurations, which clearly describe intersecting branes, there is no geometric illustration for general T-brane backgrounds known yet, see however [48]. The exception are the reconstructible backgrounds introduced in section 3.5.3. Given an arbitrary holomorphic background, it is nevertheless possible to calculate the corresponding matter spectrum and interactions.

For a background valid in the adjoint of some Lie algebra $\mathfrak{g}$,

$$\Phi = \mathfrak{g} \otimes \mathcal{O} \,, \tag{6.10}$$

(6.4) can be written as

$$\phi = \mathrm{ad}_\Phi(\xi) + h\,. \tag{6.11}$$

A mode which is gauge equivalent to zero can therefore be written as

$$\phi = \mathrm{ad}_\Phi(\chi)\,. \tag{6.12}$$

The space of physically distinct modes is then given by

$$\frac{\mathfrak{g}\otimes\mathcal{O}}{\mathrm{ad}_\Phi(\mathfrak{g}\otimes\mathcal{O})}\,. \tag{6.13}$$

This is the space of eight-dimensional solutions and thus comprises bulk[1] fields as well as localized modes. One can extract the localized modes by making use of their defining property: All the gauge invariant data is contained in an arbitrarily small neighborhood of the matter curve $f = 0$. This means that the mode should be gauge equivalent to zero everywhere but on f. Thus, one must have

$$\phi = \mathrm{ad}_\Phi\left(\frac{\eta}{f^m}\right) \tag{6.14}$$

with $\eta \in \mathfrak{g}\otimes\mathcal{O}$ and m a positive integer[2]. A mode which can be written in this way is gauge equivalent to zero away from the matter curve $f = 0$ but not globally trivial because $\frac{\eta}{f^m}$ is not defined over all of $\mathbb{C}^2$. The transition from the eight- to the six-dimensional field localized on the curves then amounts to mapping ϕ to η restricted to the curve.

Elements of $\mathfrak{g}\otimes\mathcal{O}$ which are in the kernel of the adjoint action can be added to η without changing (6.14). Therefore, the space of η's is given by

$$\frac{\mathfrak{g}\otimes\mathcal{O}}{\ker(\mathrm{ad}_\Phi)}\,. \tag{6.15}$$

The map to the six-dimensional fields is then

$$8D \to 6D : \phi \mapsto [\eta] = \eta|_\Sigma \in \frac{\mathfrak{g}\otimes\mathcal{O}/\langle f^m\rangle}{\ker(\mathrm{ad}_\Phi)}\,. \tag{6.16}$$

Taking the residue class $[\eta]$ is the formal way to restrict to the curve.

6.2. New options for model building

In section 3.5.3 we have seen that the T-brane framework collapses to the intersecting brane setup for the case of reconstructible Higgs fields, i.e. the whole configuration is determined by the spectral equation (3.49). If, however, one does not assume block

[1]Here, bulk refers to the eight-dimensional surface S.
[2]I will say more on the value of m later.

reconstructability, a configuration is not fully specified by the eigenvalues. The idea for future F-theory model building is *not* to work with reconstructible Higgs fields – or equivalently with diagonal Higgs fields that have branch cuts –, but to relax the assumption of reconstructability and employ backgrounds whose spectral equations do not give full information on the spectrum instead. I will start with two examples taken from [15] of backgrounds that possess new features. The first example shows that considering the spectral equation alone can be misleading for one would miss the existence of localized charged matter, and in the second example the converse happens.

Missing Charged Matter

The background

$$\Phi = \begin{pmatrix} 0 & x & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{6.17}$$

breaks $U(3)$ to $U(1)^2$. Judging from the spectral equation $P_\Phi(z) = z^3$, one would picture three coincident branes where the gauge symmetry is broken nowhere and therefore expect no localized matter. However, the charged doublet

$$\phi = \begin{pmatrix} 0 & 0 & \phi_+ \\ 0 & 0 & \phi_- \\ 0 & 0 & 0 \end{pmatrix} \tag{6.18}$$

gives rise to a localized mode:

$$\phi = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \eta = \begin{pmatrix} 0 \\ 1 \end{pmatrix} . \tag{6.19}$$

Too Much Matter

Also, it can happen that the spectral equation predicts localized matter where there is none. Both

$$\Phi_1 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & x & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{6.20}$$

and

$$\Phi_2 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ x & 0 & x & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{6.21}$$

have the spectral equation $P_\Phi(z) = z^2(z^2 - x)$. The charged localized field would now be a triplet

$$\phi = \begin{pmatrix} 0 & 0 & 0 & \phi_1 \\ 0 & 0 & 0 & \phi_2 \\ 0 & 0 & 0 & \phi_3 \\ 0 & 0 & 0 & 0 \end{pmatrix} . \tag{6.22}$$

The gauge freedom allows to set $\phi_1 = \phi_2 = 0$ in the first case and $\phi_1 = 0$ in the second case. The equation

$$x \begin{pmatrix} 0 \\ 0 \\ \phi_3 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & x & 0 \end{pmatrix} \begin{pmatrix} \eta_1 \\ \eta_2 \\ \eta_3 \end{pmatrix} \tag{6.23}$$

has no solution, which means that for the background Φ_1 there is no localized matter at $x = 0$. The equation

$$x \begin{pmatrix} 0 \\ \phi_2 \\ \phi_3 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ x & 0 & x \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \eta_1 \\ \eta_2 \\ \eta_3 \end{pmatrix} \tag{6.24}$$

on the other hand is solved by

$$\phi = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} , \quad \eta = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} . \tag{6.25}$$

The two examples show that inferring the matter content of a T-brane configuration from the spectral equation can be misleading. For potential future model building it would be great to know what a background must look like to support matter. More precisely, being interested in $SU(5)$ GUTs, one would like to know under which conditions matter is present in the different **5** and **10** representations. A priori, it is not clear whether if a spectral equations fails in the prediction of a **5** curve, it will also lead to a wrong detection of a **10** curve. In view of the discussion in chapter 5 one can hope that there is no such correlation – and in fact, that will turn out to be the case. There is a good chance that the homology classes of the different curves at not related, at least not as calculated by the spectral cover formalism, which might abrogate the no-go result of chapter 5. It should in principle be possible to realize models with a specific monodromy group whose spectra differ considerably from the ones of the corresponding diagonal backgrounds in the sense that for example supernumerary exotic fields can be disposed of.

6.3. Systematic scan

I performed a computer based analysis of 968 toy models where the background Higgs fields break E_8 to $SU(5)$. The following discussion will make it apparent that the properties of a background are hardly visible to the naked eye. A statement in [15] is that the spectral equation correctly predicts the existence of matter whenever the monodromy groups is a transitive subgroup of the non-vanishing Jordan block. The example Too much matter of section 6.2 supports this assumption. This suggests an analysis of the Galois group of the spectral polynomial as a promising starting point.

The simplest possible background one can consider is one with a 5×5 Jordan block structure:

$$\Phi^{(5)} = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} . \tag{6.26}$$

This background, where up to three x's get inserted in the different slots which are not filled by one[3], will be subject to the scan. One cannot expect to find an unambiguous relation between the Galois group and the presence of matter, since we already know that this does not exist. But eventually the complexity of the problem to tell which backgrounds give rise to which representation is reduced by arranging the Higgs fields in certain classes, given by the Galois group of the spectral polynomial.

Since an efficient calculation of the Galois group requires a factorization of the characteristic polynomial, for which there exists no pre-assembled algorithm, this step is very technical and therefore explained in appendix A.2. Given the factorized spectral polynomial, a few mathematical statements, summarized in appendix A.3, make it straightforward to calculate the Galois group.

For the toy models at hand the only possible matter curve is a power of x. Hence, these models are phenomenologically not interesting because they do not possess any Yukawa interactions, which would require different curves to meet in a point.

6.3.1. Calculation of localized modes

In this section I will show how to make use of the theory introduced in section 6.1 for practical calculations of localized fields. In the same order the computer-based scan is performed.

For the **5** and **10** of $SU(5)$ the equations (6.14) read

$$\phi^{\mathbf{5}} = \mathrm{ad}_\Phi \left(\frac{\eta^{\mathbf{5}}}{f_{\mathbf{5}}^m} \right) \quad \text{and} \quad \phi^{\mathbf{10}} = \mathrm{ad}_\Phi \left(\frac{\eta^{\mathbf{10}}}{f_{\mathbf{10}}^n} \right) . \tag{6.27}$$

The first step is to make use of the gauge freedom to set to zero as many entries of $\phi^{\mathbf{5}}$ and $\phi^{\mathbf{10}}$ as possible. It is convenient to think of the Higgs field Φ as being embedded

[3] There are exactly 968 distinct possibilities to do this when taking into account the tracelessness condition.

in the adjoint representation of E_8 such that under the decomposition

$$\begin{aligned} E_8 &\longrightarrow SU(5) \times SU(5)_\perp \\ \mathbf{248} &\longrightarrow (\mathbf{24},\mathbf{1}) \oplus (\mathbf{1},\mathbf{24}) \oplus (\mathbf{10},\mathbf{5}) \oplus \left(\bar{\mathbf{5}},\mathbf{10}\right) \oplus \left(\overline{\mathbf{10}},\bar{\mathbf{5}}\right) \oplus \left(\mathbf{5},\overline{\mathbf{10}}\right) \end{aligned}$$

the nonzero values of Φ reside in the adjoint representation of $SU(5)_\perp$: $\Phi \sim (\mathbf{1},\mathbf{24})$. The holomorphic gauge parameter χ is also in the adjoint representation of E_8. Writing the eight-dimensional matter fields ϕ as a fluctuation around the background as in (3.36) and using (3.42), one obtains

$$\begin{aligned} \delta\phi &\in \mathbf{248} \\ &= \mathbf{248} \otimes \mathbf{248}|_{\mathbf{248}} \\ &\supset (\mathbf{1},\mathbf{24}) \otimes \left[(\mathbf{24},\mathbf{1}) \oplus (\mathbf{1},\mathbf{24}) \oplus (\mathbf{10},\mathbf{5}) \oplus \left(\bar{\mathbf{5}},\mathbf{10}\right) \oplus \left(\overline{\mathbf{10}},\bar{\mathbf{5}}\right) \oplus \left(\mathbf{5},\overline{\mathbf{10}}\right)\right]|_{\mathbf{248}} \\ &= (\mathbf{1},[\mathbf{24},\mathbf{24}]) \oplus (\mathbf{10},\mathbf{24}\times\mathbf{5}|_{\mathbf{5}}) \oplus \cdots \oplus \left(\mathbf{5},\mathbf{24}\times\overline{\mathbf{10}}|_{\overline{\mathbf{10}}}\right) . \end{aligned} \tag{6.28}$$

It follows that the shift of $\phi^{\mathbf{5}}$ and $\phi^{\mathbf{10}}$ under a gauge transformation is given by

$$\begin{aligned} \delta\phi^{\mathbf{5}} &= \Phi\chi^{\mathbf{5}} \\ \delta\phi^{\mathbf{10}} &= \Phi\chi^{\mathbf{10}} + \chi^{\mathbf{10}}\Phi^T , \end{aligned} \tag{6.29}$$

where

$$\chi^{\mathbf{5}} = \begin{pmatrix} \chi_1 \\ \chi_2 \\ \chi_3 \\ \chi_4 \\ \chi_5 \end{pmatrix} \quad \text{and} \quad \chi^{\mathbf{10}} = \begin{pmatrix} 0 & \chi_{12} & \chi_{13} & \chi_{14} & \chi_{15} \\ -\chi_{12} & 0 & \chi_{23} & \chi_{24} & \chi_{25} \\ -\chi_{13} & -\chi_{23} & 0 & \chi_{34} & \chi_{35} \\ -\chi_{14} & -\chi_{24} & -\chi_{34} & 0 & \chi_{45} \\ -\chi_{15} & -\chi_{25} & -\chi_{35} & -\chi_{45} & 0 \end{pmatrix} \tag{6.30}$$

are in the representations $\mathbf{5}$ and $\mathbf{10}$ of $SU(5)_\perp$, respectively.

For the right-hand side of (6.27) one must compute the adjoint action of Φ on the holomorphic quantities $\eta^{\mathbf{5}}$ and $\eta^{\mathbf{10}}$. The entries of

$$\eta^{\mathbf{5}} = \begin{pmatrix} \eta_1 \\ \eta_2 \\ \eta_3 \\ \eta_4 \\ \eta_5 \end{pmatrix} \quad \text{and} \quad \eta^{\mathbf{10}} = \begin{pmatrix} 0 & \eta_{12} & \eta_{13} & \eta_{14} & \eta_{15} \\ -\eta_{12} & 0 & \eta_{23} & \eta_{24} & \eta_{25} \\ -\eta_{13} & -\eta_{23} & 0 & \eta_{34} & \eta_{35} \\ -\eta_{14} & -\eta_{24} & -\eta_{34} & 0 & \eta_{45} \\ -\eta_{15} & -\eta_{25} & -\eta_{35} & -\eta_{45} & 0 \end{pmatrix} \tag{6.31}$$

will be determined by solving (6.27).

I solved these equations taking $m, n \in \{1,2,3,4,5,6\}$[4] for the matter curves $f^m_{\mathbf{5}} = x^m$ and $f^n_{\mathbf{10}} = x^n$. There are additional requirements on the quality of the solutions that have to be imposed: $\left(\frac{\eta}{f^m}\right)$ must be singular at $x = 0$, η must have some non-vanishing entries at $x = 0$ and I only want to keep those modes which do not differ by an element of the kernel of the adjoint map. For the last requirement I checked whether the difference of two modes is mapped to zero by the adjoint map. If there exists a $\mathbf{5}$

[4] No unambiguous new solutions appear at order six, so it seems to be sufficient to take $m, n \leq 6$.

$\eta^{(\text{at } x^a)}$ of $SU(5)$ at $f = x^a$ and another one $\eta^{(\text{at } x^b)}$ at $f = x^b$, I calculated the difference $\eta^{(\text{at } x^b)} \times x^a - \eta^{(\text{at } x^a)} \times x^b$. The multiplication by x^a and x^b makes up for the different residues taken in (6.16). I would have expected that removing all gauge equivalent solutions fixes m and n to a certain value so that there is one and only one localized mode of a given representation located on the curve which is geometrically defined by $x = 0$. Indeed, this removes many solutions, but some backgrounds still admit multiple gauge inequivalent solutions at different powers of x. The question is, whether these are really distinct modes or if they are to be treated as one and the same. This discussion will be continued after having looked at the results.

6.3.2. Results

The 968 matrices give rise to eleven different Galois groups:

$$\begin{aligned}
&e \times \mathbb{Z}_2^2\,,\ e \times \mathbb{Z}_4\,,\ e \times D_4\,,\\
&\mathbb{Z}_2 \times S_3\,,\ \mathbb{Z}_2 \times \mathbb{Z}_3\,,\\
&e \times \mathbb{Z}_2 \times \mathbb{Z}_2,\ e \times e \times e \times \mathbb{Z}_2\,,\\
&e \times e \times S_3\,,\ e \times e \times \mathbb{Z}_3\,,\\
&e \times e \times e \times e \times e\,,\\
&T\,.
\end{aligned} \tag{6.32}$$

The transitive groups of order five, permuting all eigenvalues of the Higgs field, are denoted by T. The tables displaying the matrices, spectral polynomials and localized modes with their matter curves, belonging to the respective groups, are shown and explained in appendix A.4. I will start with a group-by-group description of the results.

- None of the $e \times \mathbb{Z}_2^2$ or $e \times \mathbb{Z}_4$ backgrounds possesses any localized matter.
- The localized matter in the $e \times D_4$ class is always a **5** of $SU(5)$.
- There are exactly three groups of the form $e \times G_{\text{rank } 4}$: $e \times \mathbb{Z}_2^2$, $e \times \mathbb{Z}_4$ and $e \times D_4$. For all of them a sufficient condition for the absence of localized matter is that the remainder $P^{e \times G_{\text{rank } 4}}/z$ is a sum of the z^4-, a z^2- and z^0-term, where the coefficients of z^2 and z^0 are allowed to be zero. Stated in another way, a necessary condition for the presence of a **5** is that $P^{e \times G_{\text{rank } 4}}/z$ contains a term linear in z.

The polynomials with the Galois groups $\mathbb{Z}_2 \times S_3$ and $\mathbb{Z}_2 \times \mathbb{Z}_3$ have features which distinguish them from all remaining groups:

- There are no polynomials in $\mathbb{Z}_2 \times S_3$ and $\mathbb{Z}_2 \times \mathbb{Z}_3$ which are not associated with at least one matrix that gives rise to matter.
- For $n = 1$ their matrices often give rise to a **10** but never to a **5**, whereas for larger powers $(n = 2, 4)$ of x **5**'s appear. All other groups, exclusive of T, possess either no matter or a **5** of $SU(5)$.

Just as with any good rule, there is one exception. The last polynomial of the group $e \times e \times \mathbb{Z}_3$ hints at what might be an underlying pattern:

- The polynomials in all groups – T is still excluded – apart from the ones in $\mathbb{Z}_2 \times S_3$, $\mathbb{Z}_2 \times \mathbb{Z}_3$ and the exception contain at least one factor of z.

In the $e \times \mathbb{Z}_2 \times \mathbb{Z}_2$ and $e \times e \times e \times \mathbb{Z}_2$ sector, respectively, there is only one polynomial with brane configurations that support matter.

- Both are exclusively made up of the factors $z^2 - x$ and z.

For $e \times e \times S_3$ and $e \times e \times \mathbb{Z}_3$ one can state that

- no matter is present if there appears a summand $-3x$ or $-x^3$ in P/z^2,

and

- they are the only groups with a matrix that has a non-gauge equivalent mode at $n = 6$. Except for these two matrices, $n = 6$ never shows up.

Three polynomials factor completely, thus belonging to the Galois group $e\times e\times e\times e\times e$. Two of them can give rise to a **5** and one does not.

Finally, for the transitive Galois groups I only listed the results for the non-reconstructible Higgs fields. There are too many reconstructible matrices to show them all, so I will only give a summary of the interesting observations:

- $n = 1$ for all reconstructible Higgs fields that have matter. There exist matrices which give rise to no matter, a **5**, a **10**, and both.

Focusing on polynomials, instead of single matrices, one can ask the question whether the matrices with the same polynomial support the same kind of matter. The statement in [15] that the spectral equations contained full information in case of reconstructible backgrounds, suggests this. However,

- 19 out of 39 polynomials in the reconstructible category have matrices leading to different kinds of matter. To be specific, denoting by "-", "5", "10" and "B" the absence or presence of a **5**, a **10** and both, the combinations which appear are: -, 10, B, {B,-}, {B,5}, {B,10}, {5,-}, {B,10,-}, {B,10,5}[5]. The combinations which do not appear are 5, {10,-}, {10,5}, {B,5,-}, {10,5,-}, {B,10,5,-}. So 9 of 15 possibilities arise.

For the non-reconstructible polynomials n and m are not always equal to one.

- For the **10**'s, $m = 1, 3$ and for the **5**'s, $n = 1, 2, 3, 5$. Whenever a matrix supports a **5** it also supports a **10**.
- There are no non-reconstructible polynomials in the transitive group which do not have at least one matrix that gives rise to matter.

All in all, the results for the transitive group completely contradict the statement in [15] that in case of a transitive monodromy group, the spectral polynomial made correct

[5] To clarify the notation: If four matrices had the same polynomial, two of which gave rise to a **5**, one to a **10** and one to both, the polynomial would belong to the class {B,10,5}.

predictions. As we have seen, one polynomial is in general associated with matrices that give rise to different kinds of matter. So this can evidently not be true.

Now let me come back to the discussion at the end of section 6.3.1. The localized **10** modes are never ambiguous whereas the **5**'s often are. There are multiple gauge inequivalent modes at different n in the groups T, $e \times D_4$, $e \times e \times \mathbb{Z}_3$, $e \times e \times S_3$ and $e \times e \times e \times e \times e$. The question is whether they are one and the same mode or different ones. I am not able to answer this question but I would like to give one argument for each side:

Taking a look at the $\eta^{\mathbf{10}}$'s, it is apparent that the entries which are zero remain zero when n changes[6]. The difference between two modes is mostly a combination of a change of sign somewhere or a multiplication of an entry or a summand of an entry by x. However, in some cases new summands are added. The upshot is that for each matrix there is an obvious pattern how $\eta^{\mathbf{10}}$ is modified. This can be taken as a hint that the different modes should be treated as one and the same mode. A necessary condition for this interpretation is that a fictitious Yukawa coupling is not changed when using one or the other mode. Since Yukawa couplings are calculated by residue integrals, in which the matter curves appear in the denominator and the entries of the localized modes show up in the numerator, see [15], this might be fulfilled.

An argument for treating the modes as distinct modes and thus being allowed to assign different matter fields to them can be formulated as follows: It should not be surprising that for matrices supporting both a **10** and a **5** $m \neq n$, because the equations (6.27) are very different and there is no reason at all why the **10** and the **5** matter curves should coincide. The matrices at hand, having only x-entries, however, only admit the option $f^{\mathbf{10}} = f^{\mathbf{5}} = x$, but this does not imply $n = m$. Extending this argument to declaring the **5**'s to be different modes is now trivial. If there are for example two non-gauge equivalent $\eta^{\mathbf{10}}$'s at $n = 1$ and $n = 2$, one could assign the field α to the one and the field β to the other. In the fictitious Yukawa residue one would then either use the combination $\frac{\alpha}{x}$ or $\frac{\beta}{x^2}$.

The scan was performed over a set of matrices with a comparatively simple structure. Nevertheless, it is already quite challenging to calculate quantities – here, the monodromy group – which help to categorize the infinite number of possible matrices and to put the results into a sufficiently well-arranged form that permits do draw conclusions. The fact that the results do not allow for many concrete statements is inter alia due to the Galois group depending only on the spectral polynomial. It is thus clearly not the best variable to look for structures in the landscape of T-brane configurations. It would be better to find a way to directly analyze the Higgs matrices since this might lead to waterproof rules. Yet, the examination of the monodromy group is a good starting point, because the above discussion makes it apparent that there are patterns to discover even when solely focusing on the spectral equation.

[6] The **10** in $\eta^{\mathbf{10}}$ refers to the representation of $SU(5)_\perp$.

6.4. Towards phenomenologically interesting models

Concerning phenomenology none of the models examined above is useful, because realistic setups require several distinct matter curves which intersect each other to produce the needed Yukawa couplings. In this last section I suggest a way how to continue the research on T-brane configurations that might reproduce the MSSM. The results obtained so far are not strong enough to dictate the course of action for finding viable backgrounds. Hence, the next reasonable step could be to perform another scan that has the prerequisites to produce phenomenologically interesting configurations. I propose the inspection of matrices with a $2 \oplus 3$ Jordan structure, for example

$$\Phi = \begin{pmatrix} a_{1,1} & 1 & 0 & 0 & 0 \\ a_{2,1} & a_{2,2} & 0 & 0 & 0 \\ 0 & 0 & a_{3,3} & 1 & 0 \\ 0 & 0 & a_{4,3} & a_{4,4} & 1 \\ 0 & 0 & a_{5,3} & a_{5,4} & -a_{1,1} - a_{2,2} - a_{3,3} - a_{4,4} \end{pmatrix}, \tag{6.33}$$

where $a_{i,j}$ denote holomorphic functions in the seven-brane worldvolume coordinates x and y. These backgrounds create more complicated configurations, in which, based on the matter parity definition of Case II, it is possible to distinguish between up- and down-type Higgs curves, **5** and **10** matter curves and **10** matter curves which cannot carry SM matter because of the wrong matter parity. It is convenient to look at Case II because the block-diagonal structure of Φ manifestly reproduces the weight-classes defined in (4.13).

Denoting the basis vectors of $SU(5)_\perp$ by e_i, as in section 3.5.2, the subspaces for the different curves are spanned by:

$$\mathbf{5}_{H_u} : (e_1^* \wedge e_2^*)\,, \quad \bar{\mathbf{5}}_{\text{Higgs-like}} : \begin{pmatrix} e_3 \wedge e_4 \\ e_3 \wedge e_5 \\ e_4 \wedge e_5 \end{pmatrix}, \quad \bar{\mathbf{5}}_M : \begin{pmatrix} e_1 \wedge e_3 \\ e_1 \wedge e_4 \\ e_1 \wedge e_5 \\ e_2 \wedge e_3 \\ e_2 \wedge e_4 \\ e_2 \wedge e_5 \end{pmatrix}, \tag{6.34}$$

$$\mathbf{10}_M : \begin{pmatrix} e_1 \\ e_2 \end{pmatrix}, \quad \mathbf{10}_{\text{Exo.}} : \begin{pmatrix} e_3 \\ e_4 \\ e_5 \end{pmatrix}. \tag{6.35}$$

Here, the matter parity definition of Case II is used. It is straightforward to calculate how Φ acts on each of these multiplets:

$$\begin{aligned}
\Phi_{\mathbf{5}_{H_u}} &= a_{1,1} + a_{2,2}\,,\\
\Phi_{\bar{\mathbf{5}}_{\text{Higgs-like}}} &= \begin{pmatrix} a_{3,3}+a_{4,4} & 1 & 0 \\ a_{5,4} & a_{3,3}+a_{5,5} & 1 \\ -a_{5,3} & a_{4,3} & a_{4,4}+a_{5,5} \end{pmatrix},\\
\Phi_{\bar{\mathbf{5}}_M} &= \begin{pmatrix} a_{1,1}+a_{3,3} & 1 & 0 & 1 & 0 & 0 \\ a_{4,3} & a_{1,1}+a_{4,4} & 1 & 0 & 1 & 0 \\ a_{5,3} & a_{5,4} & a_{1,1}+a_{5,5} & 0 & 0 & 1 \\ x & 0 & 0 & a_{2,2}+a_{3,3} & 1 & 0 \\ 0 & x & 0 & a_{4,3} & a_{2,2}+a_{4,4} & 1 \\ 0 & 0 & x & a_{5,3} & a_{5,4} & a_{2,2}+a_{5,5} \end{pmatrix},\\
\Phi_{\mathbf{10}_M} &= \begin{pmatrix} a_{1,1} & 1 \\ a_{2,1} & a_{2,2} \end{pmatrix},\\
\Phi_{\mathbf{10}_{\text{Exo.}}} &= \begin{pmatrix} a_{3,3} & 1 & 0 \\ a_{4,3} & a_{4,4} & 1 \\ a_{5,3} & a_{5,4} & -a_{1,1}-a_{2,2}-a_{3,3}-a_{4,4} \end{pmatrix}.
\end{aligned} \tag{6.36}$$

The equation for localized modes (6.14) now splits up into five distinct equations,

$$f_R\phi_R = \Phi_R\eta_R\,, \tag{6.37}$$

each of which must be solved independently.

In case the determinants of the above Φ_R are not zero, it is particularly easy to determine the localized modes η_R by using the *adjugate* matrix to Φ_R, which exists even if Φ_R is not invertible. Given the spectral equation for Φ_R,

$$P_{\Phi_R}(z) = z^{k_i} - \sigma_1 z^{k_i-1} + \ldots + (-1)^{k_i}\sigma_{k_i}\,, \tag{6.38}$$

the adjugate matrix is defined by

$$\mathcal{A}_R = (-1)^{k_i+1}\left(\Phi_R^{k_i-1} - \sigma_1\Phi_R^{k_i-2} + \ldots + (-1)^{k_i-1}\sigma_{k_i-1}\mathbb{1}\right)\,, \tag{6.39}$$

and fulfills the equation

$$\mathcal{A}_R\Phi_R = \Phi_R\mathcal{A}_R = \det(\Phi_R)\mathbb{1}\,. \tag{6.40}$$

Therefore,

$$\mathcal{A}_R\phi_R = \eta_R \tag{6.41}$$

and $f_R = \text{Det}\ \Phi_R$. This technique is for instance employed in [38]. The example Missing Charged Matter in section 6.2, however, already shows that this does not work in general.

Furthermore, the background suggested here has the advantage that the entries in ϕ_R which can be set to zero by a gauge transformation are independent of the entries of Φ as long as it is of the form (6.33). The result is:

$$\begin{aligned}
\phi_{\mathbf{5}_{H_u}} &= \{\phi\}\,, \\
\phi_{\bar{\mathbf{5}}_{\text{Higgs-like}}} &= \{0,0,\phi\}\,, \\
\phi_{\bar{\mathbf{5}}_M} &= \{0,0,\phi_1,0,0,\phi_2\}\,, \\
\phi_{\mathbf{10}_M} &= \{0,\phi\}\,, \\
\phi_{\mathbf{10}_{\text{Exo.}}} &= \{0,0,\phi\}\,.
\end{aligned} \tag{6.42}$$

These can plugged in for the ϕ_R in (6.37).

7. Conclusion and outlook

The aim of this thesis was to study how many properties of particle physics can be explained within local models. The basic principle is to take advantage of the large freedom in model building present at the point of E_8 in F-theory. The obvious problem with such a bottom-up approach is that there is no guarantee for the global existence of such a model, because some assumptions that are made in the four-dimensional theory of the point may contradict the conditions that must be imposed to attain a consistent global construction. It is not until the achievement of this objective that we have an ultraviolet complete description of all fundamental interactions including gravity.

In the search for local $SU(5)$ models at the point of E_8 with appealing phenomenological features, i.e. a stable proton and realistic quark and lepton masses and mixings, exactly two models were found, which differ in their definition of matter parity. Both of them feature semi-realistic quark masses and in addition exhibit reasonable mixing patterns in the CKM matrix. It is possible to extend these models by addressing also neutrinos, but we only concerned ourselves with the quark sector. The fact that one can define matter parity in local models is a non-trivial result and a very beneficial one with regard to the bottom-up idea, because it drastically reduces the number of viable models and thus renders the local model highly predictive.

In chapter 5 I have shown that a global embedding of the two models unfortunately already fails in the first step, where constraints from the eight-dimensional theory taking place on the worldvolume of the GUT seven-brane are included. One can assess this result in different ways: The most conservative one is to conclude that a potential string realization of the MSSM needs some nonlocal ingredients. For example, it is assumed in [11, 43, 44, 49, 50] that the matter parity originates from mechanisms that can only be described in a global picture. I tried to tackle another route by following the proposal in [15] of using an eight-dimensional field theory with general holomorphic Higgs fields as the framework for a local description instead of a diagonal multivalued Higgs field. As argued in chapter 6, this opens up new possibilities for model building and raises hope that the local models of chapter 4 might be validated or that better models might be found. The main point is that this so called T-brane framework, as opposed to the spectral cover approach, is not based on the spectral equation and could therefore remedy the correlations between the homology classes of the matter curves, which cause the problems with the spectra. However, this framework is still very little developed and there are no rules yet how to make use of it for practical intentions.

In chapter 6 I presented my first steps towards a better understanding of the relation between the structure of non-diagonal Higgs fields and the kind of matter they support. I performed a scan over 968 toy models, analyzing Higgs fields that break E_8 to $SU(5)$, and it is indeed possible to find some regularities when sorting the Higgs matrices by the Galois group of their spectral polynomials and comparing the spectra they give rise to. Since the transitive monodromy groups offer such a rich pool of possibilities, one might want to disentangle them into concrete Galois groups in future studies.

As an unsolved problem, it remains to clarify how to assess the phenomenon of multiple gauge inequivalent modes which localize on the same geometric curve. It is interesting so see whether this also happens for the scan over phenomenologically viable backgrounds which I have proposed. This framework might be more appropriate to answer the question because the MSSM Yukawa couplings can be calculated explicitly.

An analogous scan, adapted to matter parity Case I, may then be performed in succession. In the long run one can hope to find simple model building rules which can be proven by hand and might also be understood from a geometric point of view, that is in terms of string boundary conditions.

I believe that this approach leads to promising models which can generalize the local models of chapter 4. One must not forget, however, that what we are really looking for are at least semilocal models obeying some basic consistency conditions. In this respect, the statements of [41] have to be taken into account in order to see what constraints anomaly cancellation puts on the spectra. It is necessary to calculate the homology classes of the different matter curves and perform analogous calculations to the ones of the spectral cover approach presented in chapter 5 to obtain the flux restrictions and the chiral spectrum. This should be possible given the functional dependence of the matter curves on the brane worldvolume coordinates. This approach seems to be straightforward, but there is a caveat: As mentioned in section 6.1 and used in (6.3), so far all calculations have been performed on a local patch $V \in \mathbb{C}^2 \in S$. The formalism becomes more involved when working on a compact S, but it is outlined in the appendix of [15] how one must proceed in that case.

It will be exciting to observe the future developments in F-theory model building, because the fate of local model building does not seem to be definite yet.

A. Appendix

A.1. $\overline{\mathbf{5}}_1$ as the down-type Higgs curve

Choosing the curve $\overline{\mathbf{5}}_1$ as the down-type Higgs curve, the gauge invariant couplings are

$$\overline{\mathbf{5}}_{H_d}\mathbf{10}_1\overline{\mathbf{5}}_6\,,\quad \overline{\mathbf{5}}_{H_d}\mathbf{10}_3\overline{\mathbf{5}}_3\,, \tag{A.1}$$

which lead to a rank-two down-type Yukawa matrix if the curves are not split and three generations come from two curves. The question is whether there exists a split such that the rank is reduced to zero or one.

The relevant couplings in $\overline{\mathbf{5}}_{H_d}\mathbf{10}_M\overline{\mathbf{5}}_M$ in terms of SM representations are the ones which involve the Higgs doublet D:

$$D\bar{e}L\,,\quad D\bar{d}Q\,. \tag{A.2}$$

The chiralities of the fields are given by (3.63),

$$\begin{aligned} n_{(3,1)_{-1/3}} - n_{(\bar{3},1)_{+1/3}} &= M_{\mathbf{5}}\,,\\ n_{(1,2)_{+1/2}} - n_{(1,2)_{-1/2}} &= M_{\mathbf{5}} + N_Y\,, \end{aligned} \tag{A.3}$$

for the $\mathbf{5}$ curves and (3.64) for the $\mathbf{10}$ curves:

$$\begin{aligned} n_{(3,2)_{+1/6}} - n_{(\bar{3},2)_{-1/6}} &= M_{\mathbf{10}}\,,\\ n_{(\bar{3},1)_{-2/3}} - n_{(3,1)_{+2/3}} &= M_{\mathbf{10}} - N_Y\,,\\ n_{(1,1)_{+1}} - n_{(1,1)_{-1}} &= M_{\mathbf{10}} + N_Y\,. \end{aligned} \tag{A.4}$$

The primary concern is the quark Yukawa matrix. Since the anti-down quark belongs to the representation $n_{(\bar{3},1)_{+1/3}}$ and the down quark belongs to $n_{(3,2)_{+1/6}}$, their chiralities are fully determined by $M_{\mathbf{5}_3}$, $M_{\mathbf{5}_6}$, $M_{\mathbf{10}_1}$ and $M_{\mathbf{10}_3}$. Overall, three generations must come from the $\overline{\mathbf{5}}$ matter curves and three generations from the $\mathbf{10}$ matter curves, which leads to the conditions

$$M_{\mathbf{5}_3} + M_{\mathbf{5}_6} = -3\,,\quad M_{\mathbf{10}_1} + M_{\mathbf{10}_3} = 3\,. \tag{A.5}$$

Setting one of the $M_{\mathbf{10}}$'s equal to one and the other one equal to two, yields nothing new. Explicitly, choosing $M_{\mathbf{10}_1} = 1$ and $M_{\mathbf{10}_3} = 2$ amounts to demanding that a heavy bottom quark is generated through the coupling $\overline{\mathbf{5}}_{H_d}\mathbf{10}_1\overline{\mathbf{5}}_6$ and thus setting $M_{\mathbf{5}_6} = -1$

and $M_{\mathbf{5}_3} = -2$. In this case the other term, $\overline{\mathbf{5}}_{H_d}\mathbf{10}_1\overline{\mathbf{5}}_6$, also exists and the matrix has rank two, which is the old situation.

Therefore, the only new option is given by $M_{\mathbf{10}_1} = 3$ and $M_{\mathbf{10}_3} = 0$ such that the remaining relevant coupling is $\overline{\mathbf{5}}_{H_d}\mathbf{10}_1\overline{\mathbf{5}}_6$. $M_{\mathbf{5}_6} = 0$ leads to a rank-zero matrix and all other values for $M_{\mathbf{5}_6}$ yield a rank-one matrix. In the rank-one or -zero case it is additionally necessary that $N_{\mathbf{10}_1} = N_{\mathbf{10}_3} = 0$ in order not to introduce chiralities larger than three. One can see this from

$$n_{(\bar{3},1)_{-2/3}} - n_{(3,1)_{+2/3}} = M_{\mathbf{10}} - N_Y \text{ and } n_{(1,1)_{+1}} - n_{(1,1)_{-1}} = M_{\mathbf{10}} + N_Y\,. \tag{A.6}$$

This solution is a rather trivial one.

It is important to note that nowhere in the above argumentation any use was made of the homology classes and the corresponding correlations between the different M's and N's as determined by the spectral cover approach.

A.2. Polynomial factorization

If the spectral polynomial factorizes, the Galois group will be the product group of the Galois groups of the different factors. Therefore, the first problem consists in factorizing the spectral equation. The spectral equations of the Higgs fields examined in the scan are polynomials of order five in z. The coefficients are polynomials in x of order three, which themselves have integer coefficients and possess no constant term. The z^5 term always has a prefactor of one.

The factorization must be performed over $\mathbb{C}[x]$, i.e. over arbitrary polynomials in x. Since there exists no built-in command in `mathematica` to perform such a factorization, we invented an algorithm which is optimized for the specific properties of the polynomials at hand and is very efficient compared to manually extending `mathematica`'s algorithms[1]. From the partitions of 5 it is apparent that for a factorization of a polynomial of degree five it is sufficient to find a way to pull out polynomials of order one and two of a polynomial of larger order. If $P_\Phi(z,x)$ possesses a linear factor it can be written as $(z - ax + bx^2 + cx^3)P_\Phi^{(4)}(z,x)$. As a polynomial in x, $\Phi^{(5)}(z = ax + bx^2 + cx^3, x)$ is a polynomial of order at most 15 and must vanish. The system of linear equations obtained by setting to zero the coefficients of all powers of x determines the parameters a, b and c. Having found the linear factor, `mathematica` can calculate the remaining polynomial and one can apply the algorithm to find linear factors until all of them are pulled out.

A factor of order two has the form $P_\Phi(z,x)^{(2)} = (z^2 - \beta_1 z - \beta_0)$, where $\beta_1 = ax + bx^2 + cx^3$ and $\beta_0 = dx + ex^2 + fx^3$. This factor vanishes for

$$z = \frac{\beta_1}{2} \pm \sqrt{\frac{\beta_1^2}{4} + \beta_0} \equiv \frac{\beta_1}{2} \pm Q\,. \tag{A.7}$$

[1] The option to use `Extension` to upgrade the command `Factor` to use the imaginary unit and irrational numbers instead of integers only turns out to prolongate the machine time to such an extent that the program would never terminate.

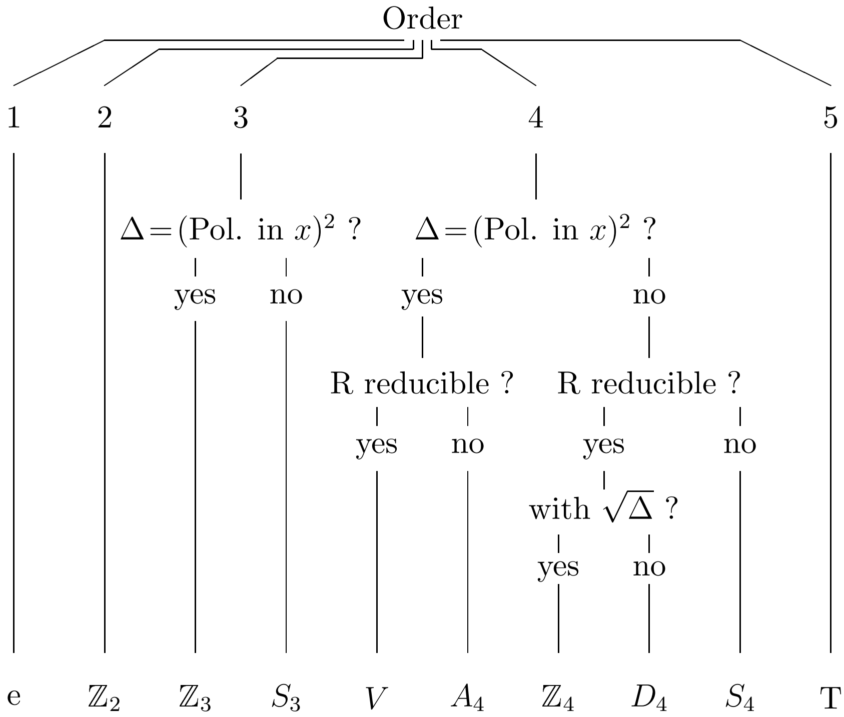


Figure A.1.: *Scheme on how to determine the Galois group of a fully factorized polynomial of a given order.*

Proceeding as in the linear case, one must insert this into $\Phi^{(5)}(z,x)$ and solve for a, b, c, d, e and f. This time it has to be taken into account that Q is not a polynomial. If it was, the quadratic term would factorize into two linear terms, which is excluded because the search for linear factors is performed in advance. Inserting (A.7) into $\Phi^{(5)}(z,x)$, one therefore gets $\Phi^{(5)}(x) = \Delta_1 + \Delta_2 Q = 0$, where Δ_1 and Δ_2 are polynomials in x which have to vanish separately. From this it is also clear that the sign in front of Q in (A.7) is irrelevant.

A.3. Galois groups

The Galois group G_P of $\Phi^{(5)}(z,x)$ is the monodromy group of the T-brane configuration. For a polynomial P^k of order k it is a subgroup of the symmetric group S_k and a subgroup of the alternating group A_k if and only if the discriminant $\Delta_P = \prod_{i<j}(z_i - z_j)^2$, where z_i denote the roots of P, is the square of a polynomial in x [51]. For irreducible polynomials G_P is transitive and since the polynomials at hand are already factorized, I will only consider irreducible polynomials of order one to five in the following.

The order-one case is trivial: The Galois group is the identity element e.

For order two the symmetric group S_2 is $\mathbb{Z}_2$ and the alternating group A_2 is e, which is not transitive and therefore cannot appear. Hence, the monodromy group must be $\mathbb{Z}_2$.

The possible transitive groups of order three are S_3 and $A_3 = \mathbb{Z}_3$. Here, the discriminant condition must be checked.

Order four is slightly more complicated. The transitive subgroups of S_4 are S_4, A_4, $V = \mathbb{Z}_2 \times \mathbb{Z}_2$, $C_4 = \mathbb{Z}_4$ and the dihedral group D_4. V is a subgroup of A_4, and thus, the discriminant-condition divides the groups into the classes A_4, V and S_4, C_4, D_4. One can further differentiate between A_4 and V as well as between S_4 and C_4, D_4 by testing if the cubic resolvent R of P is reducible. Finally, C_4 and D_4 can be distinguished by checking whether P factorizes over the polynomials in x together with $\sqrt{\Delta}$, see figure A.1.

For order five the calculation of the Galois group becomes very complicated, so I will leave it at the information that it must be a transitive group and denote it by T.

A.4. Systematic scan: Tables

Each of the following tables is associated with one of the eleven Galois groups. $GG : e \times \mathbb{Z}_2^2$ means for example that all matrices or polynomials, respectively, which are listed in the table have the Galois group $e \times \mathbb{Z}_2^2$. The rows are ordered by polynomials such that matrices with equal spectral polynomials are placed directly below each other. In many cases I do not show the matrices themselves but only a number: N.o.m. stands for the number of matrices which have the properties shown to the right of it. These are the spectral polynomial in the second column, displayed in its fully factorized form by listing the factors, and the matter spectrum in the third and fourth column. The integers m and n denote the powers of x in $f^{(\mathbf{5})m} = x^m$ and $f^{(\mathbf{10})n} = x^n$ for which the equations (6.27) have a solution. If $n = 1$ for example, this means that the corresponding matrix gives rise to a **5** of $SU(5)$. Remember that the **5** and **10** in (6.27) denote representations of $SU(5)_\perp$. For the Galois groups whose matrices never give rise to a **10**, the m-column is omitted.

I only list those matrices which *do* give rise to matter. There may be more which do not support any matter at all. They are not accounted for. Furthermore, there are Higgs fields for which n is not unique. In all these cases I listed the explicit solutions for the localized mode $\eta^{(\mathbf{10})}$ in vector form. This is to be understood as

$$\{\eta^1, \eta^2, \eta^3, \eta^4, \eta^5, \eta^6, \eta^7, \eta^8, \eta^9, \eta^{10}\} \cong \begin{pmatrix} 0 & \eta^1 & \eta^2 & \eta^3 & \eta^4 \\ -\eta^1 & 0 & \eta^5 & \eta^6 & \eta^7 \\ -\eta^2 & -\eta^5 & 0 & \eta^8 & \eta^9 \\ -\eta^3 & -\eta^6 & -\eta^8 & 0 & \eta^{10} \\ -\eta^4 & -\eta^7 & -\eta^9 & -\eta^{10} & 0 \end{pmatrix}.$$

Let me give an example:

2	$\left\{1, x^2, \frac{1}{2}x\left(-1+x+2x^3\right), -2x, \frac{1}{2}x\left(1+x+2x^3\right), x, -2x^3, 2x^3, x^2, 2x^2\right\}$
3	$\left\{1, x^2, \frac{1}{2}x^2\left(-1+x+2x^2\right), -2x, \frac{1}{2}x^2\left(1+x+2x^2\right), x, -2x^3, 2x^3, x^3, 2x^2\right\}$
4	$\left\{-\frac{1}{2}, -\frac{x^2}{2}, -\frac{x^3}{2}, x, \frac{x^3}{2}, -\frac{x}{2}, x^3, -x^3, x^4, -x^2\right\}$

means that a **5** of $SU(5)$ is present at $n = 2, 3, 4$. The η's listed with them are gauge inequivalent since all gauge equivalent solutions have been sorted out. From all the gauge equivalent modes I kept those with the lowest n values. Finally, the polynomials listed under "No matter" do not have any matrices that give rise to matter. If this part of a table is absent, such polynomials do not exist.

For the sake of readability the information for the group $e \times D_4$ is scattered over three tables.

GG: Higgs	$e \times e \times e \times e \times e$	m	n
$\begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & x & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & x^2 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$	$\{z, z, z, -x+z, x+z\}$	–	3
$\begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & x^3 & 0 & 0 \end{pmatrix}$	$\{z, z, -x+z, ax+z, bx+z\}$	–	1 $\{1, 0, -1+x, 0, 1, 0, 0, 0, 0, 0\}$ 2 $\{1, 0, -1+x^2, 0, 1, 0, 0, 0, 0, 0\}$ 3 $\{1, 0, -1+x^3, 0, 1, 0, 0, 0, 0, 0\}$
$\begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & x^3 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$	$\{z, z, -x+z, ax+z, bx+z\}$	–	1 $\{1, 0, -1+x, -x^3, 1, 0, 0, 0, x^3, 0\}$ 2 $\{1, 0, -1+x^2, -x^3, 1, 0, 0, 0, x^3, 0\}$ 3 $\{1, 0, -1+x^3, -x^3, 1, 0, 0, 0, x^3, 0\}$
No matter: $\{z, z, z, z, z\}$			

In the above, $a = -\frac{1}{2}\left(-1 - \mathrm{i}\sqrt{3}\right)$ and $b = \frac{x}{2} - \frac{1}{2}\mathrm{i}\sqrt{3}$.

GG: Higgs	$e \times e \times e \times \mathbb{Z}_2$	n
$\begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & x & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & x & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 & 0 & x \\ 0 & 0 & 1 & x & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & x & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & x & 0 & 0 \\ 0 & 0 & 1 & x & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & x & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$	$\{z, z, z, -x+z^2\}$	2
$\begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & x^2 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & x & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$	$\{z, z, z, -x+z^2\}$	3
No matter: $\{z, z, z, -2x+z^2\}$ $\{z, z, z, -x-x^2+z^2\}$ $\{z, z, z, -x^3+z^2\}$ $\{z, -x+z, x+z, -x+z^2\}$		

GG: N.o.m.	$e \times \mathbb{Z}_2 \times \mathbb{Z}_2$	n
4	$\{z, -x+z^2, -x+z^2\}$	2
No matter: $\{z, -2x+z^2, -x+z^2\}$ $\{z, -x+z^2, x+z^2\}$ $\{z, -\frac{1}{2}\left(1-\sqrt{5}\right)x+z^2, -\frac{x}{2}-\frac{\sqrt{5}x}{2}+z^2\}$ $\{z, -\frac{1}{2}\left(3-\sqrt{5}\right)x+z^2, -\frac{3x}{2}-\frac{\sqrt{5}x}{2}+z^2\}$		

GG: $e \times \mathbb{Z}_2^2$
No matter: $\{z, -x^2-x^3z^2+z^4\}$

GG: $e \times \mathbb{Z}_4$
No matter: $\{z, -2x+z^4\}$ $\{z, -x+z^4\}$ $\{z, -x-x^2+z^4\}$ $\{z, -x^3+z^4\}$

GG: N.o.m.	$e \times D_4$	n
2	$\{z, -x-2xz+z^4\}$	1
2	$\{z, -x-2xz+z^4\}$	2
2	$\{z, -x^3-2xz+z^4\}$	4
1	$\{z, -x^3-2xz+z^4\}$	2 $\{1, x^2, \frac{1}{2}x\left(-1+x+2x^3\right), -2x, \frac{1}{2}x\left(1+x+2x^3\right), x, -2x^3, 2x^3, x^2, 2x^2\}$ 3 $\{1, x^2, \frac{1}{2}x^2\left(-1+x+2x^2\right), -2x, \frac{1}{2}x^2\left(1+x+2x^2\right), x, -2x^3, 2x^3, x^3, 2x^2\}$ 4 $\left\{-\frac{1}{2}, -\frac{x^2}{2}, -\frac{x^3}{2}, x, \frac{x^3}{2}, -\frac{x}{2}, x^3, -x^3, x^4, -x^2\right\}$
2	$\{z, -2x-xz+z^4\}$	1
1	$\{z, -2x-xz+z^4\}$	2
1	$\{z, -x-xz+z^4\}$	1 $\{1, 0, -1+x, 0, 1, 0, 0, x, 0, 0\}$ 2 $\{-1, 0, (-1+x)x, 0, x, 0, 0, -x, 0, 0\}$
1	$\{z, -x-xz+z^4\}$	1 $\{1, 0, -1+x, -x, 1, 0, 0, x, x, 0\}$ 2 $\{1, 0, (-1+x)x, -x, x, 0, 0, x, x^2, 0\}$
1	$\{z, -x-xz+z^4\}$	1 $\{1, 0, -1+x+x^2, -x, 1, 0, 0, x, x, 0\}$ 2 $\{1, 0, x(-1+2x), -x, x, 0, 0, x, x^2, 0\}$
1	$\{z, -x-xz+z^4\}$	2
1	$\{z, -x-xz+z^4\}$	1 $\{1, 0, 1, -x, -1+x, x, 0, x, -x, x^2\}$ 2 $\{1, 0, -x, -x, x(1+x), x, 0, x, x^2, x^2\}$
4	$\{z, -x-xz+z^4\}$	1
2	$\{z, -x^2-xz+z^4\}$	1
1	$\{z, -x^2-xz+z^4\}$	1 $\{1, 0, -1+x, 0, 1, 0, 0, x^2, 0, 0\}$ 2 $\{1, 0, (-1+x)x, 0, x, 0, 0, x^2, 0, 0\}$ 3 $\{-1, 0, (-1+x)x^2, 0, x^2, 0, 0, -x^2, 0, 0\}$
1	$\{z, -x^2-xz+z^4\}$	1 $\{1, 0, -1+x, -x, 1, 0, 0, x^2, x, 0\}$ 2 $\{1, 0, (-1+x)x, -x, x, 0, 0, x^2, x^2, 0\}$ 3 $\{1, 0, (-1+x)x^2, -x, x^2, 0, 0, x^2, x^3, 0\}$
1	$\{z, -x^2-xz+z^4\}$	3
1	$\{z, -x^2-xz+z^4\}$	1 $\{1, 0, 1, -x, -1+x, x, 0, x^2, -x, x^2\}$ 2 $\{1, 0, -x, -x, x(1+x), x, 0, x^2, x^2, x^2\}$ 3 $\{1, 0, -x^2, -x, x^2(1+x), x, 0, x^2, x^3, x^2\}$
1	$\{z, -2x-2x^2z+z^4\}$	2 $\{1, \frac{1}{2}\left(x-x^3\right), \frac{1}{2}\left(-1+x^2\right), -2x^2, \frac{1}{2}\left(1+x^2\right), 2x^2, -x, 2x, \frac{1}{2}\left(x^2-x^4\right), \frac{1}{2}x\left(1+x^2\right)\}$ 3 $\{1, \frac{1}{2}\left(x^2-x^4\right), \frac{1}{2}x\left(-1+x^2\right), -2x^2, \frac{1}{2}x\left(1+x^2\right), 2x^2, -x, 2x, \frac{1}{2}\left(x^3-x^5\right), \frac{1}{2}x^2\left(1+x^2\right)\}$
1	$\{z, -2x-x^2z+z^4\}$	1 $\{1, 0, -1+x+x^3, -x^2, 1, x^2, -x, 2x, x^2, x\}$ 2 $\{1, 0, -1+x^2+x^3, -x^2, 1, x^2, -x, 2x, x^2, x\}$ 3 $\{1, 0, -x+2x^3, -x^2, x, x^2, -x, 2x, x^3, x^2\}$
1	$\{z, -2x-x^2z+z^4\}$	1 $\{1, x, -1+x, -2x^2-x^4, 1, 0, -x-x^3, 2x+x^3, x^2, x\}$ 2 $\{1, x, -1+x^2, -2x^2-x^4, 1, 0, -x-x^3, 2x+x^3, x^2, x\}$
1	$\{z, -x-x^2z+z^4\}$	1 $\{1, 0, -1+x, -x^2, 1, 0, -x, x, x^2, x\}$ 2 $\{1, 0, -1+x^2, -x^2, 1, 0, -x, x, x^2, x\}$
1	$\{z, -x-x^2z+z^4\}$	1 $\{1, 0, -1+x, -x^2, 1, 0, 0, x, x^2, 0\}$ 2 $\{1, 0, -1+x^2, -x^2, 1, 0, 0, x, x^2, 0\}$ 3 $\{1, 0, -x+x^3, -x^2, x, 0, 0, x, x^3, 0\}$
1	$\{z, -x-x^2z+z^4\}$	2 $\{1, 0, 1, -x^2, -1+x^2, x^2, -x, x, -x^2, -x+x^3+x^4\}$ 3 $\{1, 0, -x, -x^2, x\left(1+x^2\right), x^2, -x, x, x^3, x^2\left(1+2x^2\right)\}$
1	$\{z, -x-x^2z+z^4\}$	1 $\{1, 0, 1, -x^2, -1+x, x^2, 0, x, -x^2, x^4\}$ 2 $\{1, 0, 1, -x^2, -1+x^2, x^2, 0, x, -x^2, x^4\}$ 3 $\{1, 0, -x, -x^2, x\left(1+x^2\right), x^2, 0, x, x^3, x^4\}$
5	$\{z, -x-x^2z+z^4\}$	3
3	$\{z, -x-x^3z+z^4\}$	4
1	$\{z, -x^2-x^3z+z^4\}$	5
1	$\{z, -x^2-x^3z+z^4\}$	2 $\{1, x-x^3, -1+x^2, -x^3, 1, x^3, -x^2, x^2, 0, x^2\}$ 3 $\{1, x-x^4, -1+x^3, -x^3, 1, x^3, -x^2, x^2, 0, x^2\}$ 5 $\{-1, x^3-x^6, -x^2+x^5, x^3, x^2, -x^3, x^2, -x^2, 0, x^4\}$
1	$\{z, -x^2-x^3z+z^4\}$	3 $\{1, x, -1, -x^3, 1+x^3, x^3, 0, x^2, x^3, 0\}$ 5 $\{-1, x^3, -x^2, x^3, x^2\left(1+x^3\right), -x^3, 0, -x^2, x^5, 0\}$
1	$\{z, -x^2-x^3z+z^4\}$	3 $\{1, x, -1+x^3, -x^3-x^5, 1, 0, -x^4, x^2\left(1+x^2\right), x^3, 0\}$ 5 $\{-1-x^4, x^3, -x^2+x^5, x^3, x^2, 0, -x^6, -x^2, x^5, 0\}$
2	$\{z, -x+\left(-x-x^2\right)z+z^4\}$	1
6	$\{z, -x-xz-xz^2+z^4\}$	1
1	$\{z, -x-xz-xz^2+z^4\}$	1 $\{1, 0, -1, 0, 1, 0, 0, x, 0, 0\}$ 2 $\{1, 0, -2x+x^2, 0, x, 0, 0, x, 0, 0\}$
1	$\{z, -x-xz-xz^2+z^4\}$	1 $\{1, 0, -1, -x, 1, 0, 0, x, x, 0\}$ 2 $\{1, 0, -2x+x^2, -x, x, 0, 0, x, x^2, 0\}$
1	$\{z, -x-xz-xz^2+z^4\}$	1 $\{1, 0, 1, -x, -1, x, 0, x, -x-x^2, x^2\}$ 2 $\{1, 0, -2x, -x, x(1+x), x, 0, x, x^2, x^2\}$
4	$\{z, -x-xz-xz^2+z^4\}$	2
1	$\{z, -x^3-xz-xz^2+z^4\}$	4

GG:	$e \times D_4$	
N.o.m.		n
1	$\{z, -x - x^2z - xz^2 + z^4\}$	1 $\{1, -x, 1, -x^2, -1, x^2, 0, x, -x^2 - x^3, 0\}$ 2 $\{1, x(1+x), -1-x, -x^2, 1+x^2, x^2, 0, x, x^2, 0\}$ 3 $\{1, 2x^2, -2x, -x^2, x\left(1+x^2\right), x^2, 0, x, x^3, 0\}$
1	$\{z, -x - x^2z - xz^2 + z^4\}$	3
2	$\{z, -x - x^3z - xz^2 + z^4\}$	4
1	$\{z, -x - xz - x^2z^2 + z^4\}$	1 $\{1, 0, -1+x, 0, 1, 0, 0, x, 0, 0\}$ 2 $\{-1, 0, (-1+x)x, 0, x, 0, 0, -x, 0, 0\}$
2	$\{z, -x - xz - x^2z^2 + z^4\}$	1
1	$\{z, -x - xz - x^2z^2 + z^4\}$	1 $\{1, 0, -1+x+x^2, -x, 1, 0, 0, x, x, 0\}$ 2 $\{1, 0, x(-1+2x), -x, x, 0, 0, x, x^2, 0\}$
1	$\{z, -x - xz - x^2z^2 + z^4\}$	1 $\{1, x^2, 1, -x, -1+x+x^2+x^4, x, -x^3, x\left(1+x^2\right), -x+x^3, x^2\}$ 2 $\{1, x^2, (-1+x)x, -x, x\left(1+x+x^3\right), x, -x^3, x\left(1+x^2\right), x^2, x^2\}$

GG: $e \times D_4$
No matter:
$\{z, -x - x^3 - xz + z^4\}$
$\{z, -x + x^2 - 2xz^2 + z^4\}$
$\{z, -2x - xz^2 + z^4\}$
$\{z, -x - xz^2 + z^4\}$
$\{z, -x - x^3 - xz^2 + z^4\}$
$\{z, -2x - x^2z^2 + z^4\}$
$\{z, -x - x^2z^2 + z^4\}$
$\{z, -x - 2x^2z + z^4\}$
$\{z, -x - x^3z^2 + z^4\}$
$\{z, -x + \left(-x - x^2\right)z^2 + z^4\}$
$\{z, -x + x^3 + \left(-x - x^2\right)z^2 + z^4\}$
$\{z, -x - 2xz^2 + z^4\}$
$\{z, -x^2 - xz - 2xz^2 + z^4\}$
$\{z, -x - x^2z - x^2z^2 + z^4\}$
$\{z, -x + (-x - x^2)z - x^2z^2 + z^4\}$

GG:	$\mathbb{Z}_2 \times \mathbb{Z}_3$		
N.o.m.		m	n
7	$\{-x + z^2, -x + z^3\}$	1	–
2	$\{-x + z^2, -x + z^3\}$	1	2
1	$\{-x + z^2, -x^2 + z^3\}$	1	–
1	$\{-x + z^2, -x^2 + z^3\}$	2	2

GG:	$\mathbb{Z}_2 \times S_3$		
Higgs		m	n
$\begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ x & x & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & x & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ x & 0 & 1 & 0 & 0 \\ x & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & x & 0 \end{pmatrix}$	$\{-x + z^2, -x - xz + z^3\}$	1	–
$\begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ x & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & x & x & 0 \end{pmatrix}$	$\{-x + z^2, -x - xz + z^3\}$	1	4
$\begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ x & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & x & 0 & 1 \\ 0 & 0 & x & 0 & 0 \end{pmatrix}$	$\{-x + z^2, -x - xz + z^3\}$	–	4
$\begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ x & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & x \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & x & 0 & 0 \end{pmatrix}$	$\{-x + z^2, -x - x^2z + z^3\}$	1	2
$\begin{pmatrix} 0 & 1 & x & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ x & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & x & 0 \end{pmatrix}$	$\{-x + z^2, -x - x^2z + z^3\}$	1	–

GG: Higgs/N.o.m.	$e \times e \times \mathbb{Z}_3$	m	n
$\begin{pmatrix} 0&1&0&0&0\\ 0&0&1&0&0\\ 0&0&0&1&0\\ 0&x&0&0&1\\ 0&0&x&0&0 \end{pmatrix}, \begin{pmatrix} 0&1&0&x&0\\ 0&0&1&0&0\\ 0&0&0&1&0\\ 0&x&0&0&1\\ 0&0&x&0&0 \end{pmatrix}$	$\{z, z, -2x + z^3\}$	–	1 $\{1, 0, -1 + x, -x, 1, 0, 0, 0, x, 0\}$ 4 $\left\{1, 0, \frac{1}{2}x^3(-1 + 2x), -x, \frac{x^3}{2}, 0, 0, 0, \frac{x^4}{2}, 0\right\}$
$\begin{pmatrix} 0&1&0&0&0\\ 0&0&1&0&0\\ x&0&0&1&0\\ 0&x&0&0&1\\ 0&0&0&0&0 \end{pmatrix}$	$\{z, z, -2x + z^3\}$	–	2
$\begin{pmatrix} 0&1&x&0&0\\ 0&0&1&0&0\\ 0&0&0&1&0\\ 0&x&0&0&1\\ 0&0&x&0&0 \end{pmatrix}$	$\{z, z, -2x + z^3\}$	–	1 $\{1, x, -1 + x, -x - 2x^3, 1, 0, -2x^2, 2x^2, x, 0\}$ 6 $\left\{1, \frac{x^6}{2}, \frac{1}{2}x^5(-1 + 2x), -x - x^8, \frac{x^5}{2}, 0, -x^7, x^7, \frac{x^6}{2}, 0\right\}$
14	$\{z, z, -x + z^3\}$	–	1
3	$\{z, z, -x + z^3\}$	–	3
2	$\{z, z, -x + z^3\}$	–	4
4	$\{z, z, -x + z^3\}$	–	5
$\begin{pmatrix} 0&1&0&0&0\\ 0&0&1&0&0\\ 0&0&0&1&0\\ 0&0&0&0&1\\ 0&0&x^2&0&0 \end{pmatrix}, \begin{pmatrix} 0&1&0&0&0\\ 0&0&1&0&x\\ 0&0&0&1&0\\ 0&0&0&0&1\\ 0&0&x^2&0&0 \end{pmatrix}$ $\begin{pmatrix} 0&1&0&x&0\\ 0&0&1&0&0\\ 0&0&0&1&0\\ 0&0&0&0&1\\ 0&0&x^2&0&0 \end{pmatrix}$	$\{z, z, -x^2 + z^3\}$	–	1 $\{1, 0, -1 + x, 0, 1, 0, 0, 0, 0, 0\}$ 2 $\{1, 0, -1 + x^2, 0, 1, 0, 0, 0, 0, 0\}$
$\begin{pmatrix} 0&1&0&0&0\\ 0&0&1&0&0\\ 0&0&0&1&0\\ 0&x^2&0&0&1\\ 0&0&0&0&0 \end{pmatrix}, \begin{pmatrix} 0&1&0&x&0\\ 0&0&1&0&0\\ 0&0&0&1&0\\ 0&x^2&0&0&1\\ 0&0&0&0&0 \end{pmatrix}$	$\{z, z, -x^2 + z^3\}$	–	1 $\{1, 0, -1 + x, -x^2, 1, 0, 0, 0, x^2, 0\}$ 2 $\{1, 0, -1 + x^2, -x^2, 1, 0, 0, 0, x^2, 0\}$
$\begin{pmatrix} 0&1&0&0&0\\ 0&0&1&0&0\\ 0&0&0&1&x\\ 0&x^2&0&0&1\\ 0&0&0&0&0 \end{pmatrix}, \begin{pmatrix} 0&1&0&0&x\\ 0&0&1&0&0\\ 0&0&0&1&0\\ 0&x^2&0&0&1\\ 0&0&0&0&0 \end{pmatrix}$	$\{z, z, -x^2 + z^3\}$	–	1 $\{1, 0, -1 + x + x^3, -x^2, 1, 0, 0, 0, x^2, 0\}$ 2 $\{1, 0, -1 + x^2 + x^3, -x^2, 1, 0, 0, 0, x^2, 0\}$
$\begin{pmatrix} 0&1&0&0&0\\ 0&0&1&0&x\\ 0&0&0&1&0\\ 0&x^2&0&0&1\\ 0&0&0&0&0 \end{pmatrix}$	$\{z, z, -x^2 + z^3\}$	–	1 $\{1, x^3(1 + x), -1 + x, -x^2 - x^3, 1, x^3, 0, 0, x^2, 0\}$ 2 $\{1, x^3(1 + x), -1 + x^2, -x^2 - x^3, 1, x^3, 0, 0, x^2, 0\}$
1	$\{z, z, -x^2 + z^3\}$	–	3
$\begin{pmatrix} 0&1&x&0&0\\ 0&0&1&0&0\\ 0&0&0&1&0\\ 0&0&0&0&1\\ 0&0&x^2&0&0 \end{pmatrix}$	$\{z, z, -x^2 + z^3\}$	–	1 $\{1, x, -1 + x, -x^4, 1, 0, -x^3, x^3, 0, 0\}$ 2 $\{1, x, -1 + x^2, -x^4, 1, 0, -x^3, x^3, 0, 0\}$
$\begin{pmatrix} 0&1&x&0&0\\ 0&0&1&0&0\\ 0&0&0&1&0\\ 0&x^2&0&0&1\\ 0&0&0&0&0 \end{pmatrix}$	$\{z, z, -x^2 + z^3\}$	–	1 $\{1, x, -1 + x, -x^2 - x^4, 1, 0, -x^3, x^3, x^2, 0\}$ 2 $\{1, x, -1 + x^2, -x^2 - x^4, 1, 0, -x^3, x^3, x^2, 0\}$
2	$\{z, z, -x - x^2 + z^3\}$	–	1
$\begin{pmatrix} 0&1&0&0&0\\ 0&0&1&0&0\\ x&0&0&1&0\\ 0&0&0&0&1\\ 0&0&0&x^2&0 \end{pmatrix}$	$\{-x + z, x + z, -x + z^3\}$	2	–
1	$\{-x + z, x + z, -x + z^3\}$	1	3
<u>No matter:</u> $\{z, z, -3x + z^3\}$ $\{z, z, -2x - x^3 + z^3\}$			

GG: N.o.m.	$e \times e \times S_3$	n
2	$\{z, z, -x - 2xz + z^3\}$	4
2	$\{z, z, -x - 2xz + z^3\}$	1
1	$\{z, z, -x - 2xz + z^3\}$	2
2	$\{z, z, -2x - xz + z^3\}$	4
1	$\{z, z, -2x - xz + z^3\}$	1 $\{1, 0, -1 + x, -x, 1, 0, 0, -x, x, 0\}$ 6 $\left\{1, 0, \frac{1}{2}x^5(-1 + 2x), -x, \frac{x^5}{2}, 0, 0, -\frac{x^6}{2}, \frac{x^6}{2}, 0\right\}$
1	$\{z, z, -2x - xz + z^3\}$	1 $\{1, 0, -1, -x, 1, 0, 0, 0, x, 0\}$ 4 $\left\{1, 0, \frac{1}{2}\left(-2x - x^3 + 2x^4\right), -x, \frac{x^3}{2}, 0, 0, 0, \frac{x^4}{2}, 0\right\}$
1	$\{z, z, -2x - xz + z^3\}$	2
3	$\{z, z, -x - xz + z^3\}$	3
14	$\{z, z, -x - xz + z^3\}$	1
1	$\{z, z, -x - xz + z^3\}$	2
1	$\{z, z, -x - xz + z^3\}$	1 $\{1, 0, 1, -x, -1, x, 0, 0, -x - x^2, x^2\}$ 2 $\{1, 0, -2x, -x, x(1 + x), x, 0, 0, x^2, x^2\}$
3	$\{z, z, -x - xz + z^3\}$	5
4	$\{z, z, -x - xz + z^3\}$	4
1	$\{z, z, -x^2 - xz + z^3\}$	3
1	$\{z, z, -x^2 - xz + z^3\}$	1 $\{1, 0, -1 + x, 0, 1, 0, 0, -x, 0, 0\}$ 2 $\{1, 0, -1 + x^2, 0, 1, 0, 0, -x, 0, 0\}$
1	$\{z, z, -x^2 - xz + z^3\}$	2
1	$\{z, z, -x^2 - xz + z^3\}$	1 $\{1, 0, -1 + x, -x^2, 1, 0, 0, -x, x^2, 0\}$ 2 $\{1, 0, -1 + x^2, -x^2, 1, 0, 0, -x, x^2, 0\}$
1	$\{z, z, -x^2 - xz + z^3\}$	1 $\{1, 0, -1, 0, 1, 0, 0, 0, 0, 0\}$ 2 $\{1, 0, -1 - x + x^2, 0, 1, 0, 0, 0, 0, 0\}$
1	$\{z, z, -x^2 - xz + z^3\}$	1 $\{1, 0, -1, -x^2, 1, 0, 0, 0, x^2, 0\}$ 2 $\{1, 0, -1 - x + x^2, -x^2, 1, 0, 0, 0, x^2, 0\}$
1	$\{z, z, -x^2 - xz + z^3\}$	1 $\{1, 0, 1, -x^2, -1, x^2, 0, 0, -x^2 - x^3, x^4\}$ 2 $\{1, 0, 1, -x^2, -1 - x + x^2, x^2, 0, 0, -x^2 - x^3, x^4\}$ 3 $\{1, 0, -2x, -x^2, x\left(1 + x^2\right), x^2, 0, 0, x^3, x^4\}$
2	$\{z, z, -2x - x^2z + z^3\}$	4
2	$\{z, z, -x - x^2z + z^3\}$	4
7	$\{z, z, -x - x^2z + z^3\}$	1
2	$\{z, z, -x - x^2z + z^3\}$	3
1	$\{z, z, -x - x^2z + z^3\}$	1 $\{1, 0, 1, -x, -1 + x - x^2, x, 0, 0, -x - x^3, x^2\}$ 2 $\{1, 0, -x - x^2, -x, x(1 + x), x, 0, 0, x^2, x^2\}$
2	$\{z, z, -x - x^2z + z^3\}$	2
1	$\{z, z, -x - x^3z + z^3\}$	1
1	$\{z, z, -x - x^3z + z^3\}$	3
1	$\{z, z, -x^2 - x^3z + z^3\}$	1 $\{1, 0, -1 + x, 0, 1, 0, 0, 0, 0, 0\}$ 2 $\{1, 0, -1 + x^2, 0, 1, 0, 0, 0, 0, 0\}$
1	$\{z, z, -x^2 - x^3z + z^3\}$	2
2	$\{z, z, -x + \left(-x - x^2\right)z + z^3\}$	1
2	$\{z, z, -x + \left(-x - x^2\right)z + z^3\}$	4
1	$\{z, z, -x + \left(-x - x^2\right)z + z^3\}$	2
1	$\{z, z, -x + \left(-x - x^2\right)z + z^3\}$	3
No matter: $\{z, z, -x^3 - 2xz + z^3\}$ $\{z, z, -x - x^3 - xz + z^3\}$		

In the following table I only listed the non-reconstructible Higgs fields.

GG: N.o.m.	T	m	n
1	$\{-x^2+z^5\}$	1	1 $\{1,0,1,0,-1+x,0,0,0,0,0\}$ 2 $\{-1,0,1,0,-1+x^2,0,0,0,0,0\}$
1	$\{-x^3+z^5\}$	1	1 $\{1,0,1,0,-1+x,0,0,0,0,0\}$ 2 $\{1,0,1,0,-1+x^2,0,0,0,0,0\}$ 3 $\{-1,0,1,0,-1+x^3,0,0,0,0,0\}$
1	$\{-x^3-2xz+z^5\}$	3	1 $\{1,-x^2,-1+x-x^4,x^2,1,0,-x,2x,-x^3,x\}$ 2 $\{1,-x^2,-1+x^2-x^4,x^2,1,0,-x,2x,-x^3,x\}$ 3 $\{-1,x^2,-1+x^3+x^4,x^2,1,0,x,-2x,x^3,x\}$
1	$\{-x^2-xz+z^5\}$	1	2
1	$\{-x^2-xz+z^5\}$	1	1 $\{1,0,1,0,-1+x,0,0,x,0,0\}$ 2 $\{1,0,1,0,-1+x^2,0,0,x,0,0\}$
1	$\{-x^2-x^3z+z^5\}$	1	1 $\{1,0,1,0,-1+x,0,0,0,0,0\}$ 2 $\{-1,0,1,0,-1+x^2,0,0,0,0,0\}$
1	$\{-x^2-x^3z+z^5\}$	1	1
1	$\{-x^2-x^3z+z^5\}$	1	–
1	$\{-x^3-2xz^2+z^5\}$	3	5
1	$\{-x^2-xz^2+z^5\}$	1	2 $\{1,0,-2x+x^2,0,2x,0,0,0,0,0\}$ 3 $\{1,0,-x-x^2+x^3,0,x(1+x),0,0,0,0,0\}$
1	$\{-x^2-xz^2+z^5\}$	1	2 $\{1,0,-2x+x^2,-x,2x,0,0,0,2x^2,0\}$ 3 $\{1,0,-x-x^2+x^3,-x,x(1+x),0,0,0,x^2(1+x),0\}$
1	$\{-x^2-xz^2+z^5\}$	1	1 $\{1,0,1,-x,-1+x,x,0,0,-x,x^2\}$ 3 $\{1,0,-x-x^2,-x,x\left(1+x+x^2\right),x,0,0,x^2(1+x),x^2\}$
1	$\{-x^3-xz-xz^2+z^5\}$	3	1 $\{1,0,-1+x,0,1,0,0,x,0,0\}$ 2 $\{-1,0,(-1+x)x,0,x,0,0,-x,0,0\}$
1	$\{-x^3-xz-xz^2+z^5\}$	3	–
2	$\{-x^2-x^3z^2+z^5\}$	1	1 $\{1,0,1,0,-1+x,0,0,0,0,0\}$ 2 $\{-1,0,1,0,-1+x^2,0,0,0,0,0\}$
1	$\{x^2-xz^2-2xz^3+z^5\}$	1	–
1	$\{x^2-xz^2-2xz^3+z^5\}$	1	3
1	$\{-x^2-xz^3+z^5\}$	1	1 $\{1,0,1,0,-1+x,0,-x,x,0,-x^2\}$ 2 $\{-1,0,1,0,-1+x^2,0,-x,x,0,-x^2\}$
1	$\{-x^2-xz^3+z^5\}$	1	2
1	$\{-x^2-xz^3+z^5\}$	1	1 $\{1,0,1,0,-1,0,0,0,0,0\}$ 2 $\{1,0,1,0,-1-x+x^2,0,0,0,0,0\}$
1	$\{-x^2-xz^3+z^5\}$	1	1 $\{1,0,1,0,-1+x,0,0,-x,0,0\}$ 2 $\{-1,0,1,0,-1+x^2,0,0,-x,0,0\}$
1	$\{-x^3-xz-xz^3+z^5\}$	3	–
1	$\{-x^3-xz-xz^3+z^5\}$	3	3
4	$\{x^2-2xz^2-xz^3+z^5\}$	1	–
2	$\{x^2-xz-xz^2-xz^3+z^5\}$	1	–
1	$\{x^2-xz-xz^2-xz^3+z^5\}$	1	1
1	$\{x^2-xz-xz^2-xz^3+z^5\}$	1	2
1	$\{-x^2-x^3z^3+z^5\}$	1	1 $\{1,0,1,0,-1+x,0,0,0,0,0\}$ 2 $\{-1,0,1,0,-1+x^2,0,0,0,0,0\}$

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