Orbifolds and Kaluza-Klein-Monopoles in Heterotic $E_8 \times E_8$ String Theory Preserving Eight Supercharges

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Abstract

We give a detailed account on heterotic $E_8 \times E_8$ orbifold models with $\mathcal{N} = (1,0)$, $D = 6$ supersymmetry using the operator approach. It is shown that there is a weak and a strong form of the level matching condition. The weak form, which is the only form known in previous literature, allows for classification of orbifold models whereas the strong form is needed to fix transformation phases of twisted states under the orbifold twist. It is shown that only the weak form of the level matching conditions translates into a local matching of fractional parts of gravitational and gauge instanton numbers from a ten-dimensional viewpoint. This, in turn, is used to show that all orbifold models considered in this work can be classified by flat $E_8 \times E_8$ bundles on orbifolds with the fixed points taken out, under the only constraint that the fractional parts of gravitational and gauge instanton numbers match. This directly carries over to M-theory on $S^1/\mathbb{Z}_2$.

We construct multiple Kaluza-Klein-monopole solutions in Wilson line backgrounds and verify the result by computing the correct quantum numbers and comparing to the charge spectrum of toroidal compactifications of the heterotic string. We propose that the moduli space of a single SU(2) non-abelian instanton on a Kaluza-Klein-Monopole background is given by the t'Hooft ansatz as in flat space and explicitly show that instantons can become pointlike at orbifold singularities even though their scale parameter remains finite. The case of N non-abelian instantons on N Kaluza-Klein-monopoles is analyzed in the limit where all instantons and Kaluza-Klein-monopoles are located within a region of scale much smaller than the size of the Kaluza-Klein direction. Based on index calculations for manifolds with boundary, we argue that this situation is locally identical to heterotic string theory on $\mathbb{Z}_N$ orbifold singularities for general N in the standard embedding. This implies, that heterotic $E_8 \times E_8$ orbifolds with standard embedding allow for small instantons singularities in their moduli spaces.

Applying these results to orbifold models, we give new evidence that higgsing of the models leads to smooth K3 compactifications of heterotic $E_8 \times E_8$ string theory. Based on index calculations as well as details of the Higgs mechanism in $\mathcal{N} = (0,1)$, $D = 6$ supersymmetry, we argue that massless modes ascribed to the supergravity multiplet in ten dimensions, such as the geometric moduli of K3, have to appear in twisted sectors as massless modes charged with respect to gauge groups of the orbifold models. Turning to M-theory on $S^1/\mathbb{Z}_2$, we give further evidence that these modes are eleven-dimensional bulk modes which have to live in the interior of the interval. Especially, these modes cannot be localized on the ends of the eleven-dimensional interval as suggested in the previous literature.

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Contents

Chapter 1

Introduction

In our current understanding of the fundamental laws of nature we are faced with two rivaling theories. At length or energy scales from everyday life up to the highest energies accessible in laboratory experiments, the standard model of elementary particle physics with its underlying theory of quantum fields provide us with physical laws that are established at a remarkable experimental accuracy. From everyday life length scales on, up to cosmological scales, however, Einstein's theory of general relativity provides us with seemingly different fundamental laws of nature which are also in perfect agreement with all available observations and experiments and moreover have played a predominant role in modern cosmology.

From the early days of quantum theory and general relativity on, of all the approaches made by theorists to bridge the gap between those two theories, only one approach — string theory — reached a level at which it became possible to build models that are sufficiently similar to the standard model. This theory, which was initially invented to describe strong interactions and was later found to include gravitational interactions as a byproduct, is an inherently quantum theory, that is, its results can be stated in terms of unitary scattering matrices or correlation functions. For a brilliant introduction to string theory, the reader is referred to [76].

In its old formulation, string theory was understood as quantizing the embedding a two-dimensional surface, the string world-sheet, into a target space-time. Depending on the field content in two dimensions, there are many different versions of string theory, leading to space-times of various dimensions and symmetries. As it turned out, there are in general two types of theories where the quantization can be carried out in a consistent manner: either a theory contains tachyons in its space-time spectrum or it shows some amount of space-time supersymmetry.

Therefore, string theory models of phenomenological relevance either have to describe how to get rid of tachyonic excitations, for example by tachyon condensation, or how to break supersymmetry in a consistent manner. Even though there has been progress in the field of tachyon condensation over the last years, in this work, we will focus on the much more evolved field of supersymmetric theories.

Using the powerful methods of supersymmetry, strong evidence was given over the last six or seven years that all supersymmetric string theories are in fact deeply related and can be viewed as different instances of one and the same theory. Even more, this whole of a theory contains in its various limits supersymmetric Yang-Mills theories as well as other, more exotic theories, which are still lacking a proper understanding.

In this modern viewpoint, the link between the different theories can be made by the common feature of many supersymmetric models to develop flat directions in the effective potential of its scalars. The key issue here is, that these flat directions, if enough supersymmetry is present, can be shown not to be lifted, neither by perturbative nor by nonperturbative effects. In turn, this implies that the corresponding scalars remain massless. Therefore, all observables of a model, like coupling constants or symmetry breaking scales, depend on the vacuum expectation values of those scalars parameterizing the space of flat directions. This space, which is in many ways characteristic of a given model, is called the moduli space.

The importance of the moduli space lies in the fact that the string coupling constant, which enters string theory as a simple parameter controlling the perturbative expansion of scattering amplitudes, in fact has to be understood as a vacuum expectation value. All string theories contain in their space-time spectrum a scalar, the dilaton, the expectation value of which controls the string coupling. From the modern viewpoint, therefore, all supersymmetric string theories share one moduli space, and the string theories of the old formulation are simply perturbative expansions around special points, which correspond to zero coupling, that is, zero expectation value of the dilaton of the respective theory.

Of course, this implies, that to relate different points in the moduli space one has to rely on very powerful symmetry arguments, since, generically, if the coupling of one theory is small, the couplings of other theories will be strong and perturbative calculations are mostly impossible.

Starting from the maximal amount of thirtytwo supersymmetries, there are two string theories in ten dimensions: type IIA string theory with $\mathcal{N} = (1,1)$, $D = 10$ supersymmetry and type IIB string theory with $\mathcal{N} = (2,0)$, $D = 10$ supersymmetry. Here $\mathcal{N} = (a,b)$ means that the supercharges are comprised by a positive and b negative chirality D-dimensional spinors. For supergravities, this implies that there are *a negative* and *b positive* chirality gravitinos. Since a spinor of definite chirality in ten dimensions has sixteen components and transforms as a real representation of $\mathrm{SO}(9,1)$, there are indeed thirtytwo real supersymmetries. That type IIA and type IIB theories are deeply related can be shown exactly by compactifying them on a d-dimensional torus T^d. In that case, the results are simply identical, up to a discrete transformation called T-duality. In light of this, toroidal compactifications of type IIA or type IIB are called type II theories. To pass from type IIA to type IIB and vice versa, one simply compactifies on a circle $S^1 = T^1$, applies T-duality and then decompactifies by sending the size of the S^1 (which is different from the original one, of course) to infinity. That this is a well defined operation including nonperturbative effects heavily relies on the thirtytwo supersymmetries and comprises the difficult part of the argument.

However, there is one more theory, which is only known as a classical supergravity (SUGRA): eleven-dimensional $\mathcal{N} = 1$ supergravity [30]. This theory only allows for a single massless multiplet containing a graviton, an antisymmetric three-index tensor and a single gravitino. Indeed, there can be no other supersymmetric field theory in eleven or higher dimensions, since this theory would contain massless fields of spin higher than two, for which interacting field theories could not be constructed in a consistent manner. Upon compactification on a circle, eleven-dimensional supergravity yields $\mathcal{N} = (1,1)$, $D = 10$ supergravity, which happens to be the low energy description of type IIA string theory. Therefore, by the same argument as in the type II case, there has to be a well defined theory in eleven dimensions. Especially, the graviton mode corresponding to the size of the circle becomes the ten-dimensional dilaton, which relates the string coupling constant in ten dimensions to the size of the eleventh dimension such that strong coupling translates into a large size of the circle.

This theory, however, which was termed M-theory, cannot be a string theory from the old viewpoint, because there is no massless scalar in the spectrum for eleven non-compact dimensions and, therefore, there is no field the expectation value of which could serve as a coupling constant to define a perturbative expansion. Because of this problem, there is no perturbative expansion at all, except for a long wavelength expansion which can only be applied at the classical level and the definition of the theory as a quantum theory is still obscure.

Now turning to sixteen supersymmetries, there is already a whole plethora of possibilities. In general, there are states in the spectra of theories with thirtytwo supersymmetries that break exactly half of the supersymmetries. These so called Bogomolnyi-Prasad-Sommerfield (BPS) states play a dominant role in the arguments used above, since supersymmetry relates their masses to their charges and protects that relation against all corrections. This is deeply linked to the fact that BPS multiplets in general have only the square root of the number of degrees of freedom of an ordinary massive multiplet and, therefore, are very similar to massless multiplets. Indeed, one of the easiest examples of a BPS multiplet is given by a massless multiplet propagating on a circle or torus with nonzero compact momentum.

However, there are three other theories with sixteen supercharges which are not that easy to relate to M-theory and to the type II theories. All of these have $\mathcal{N} = (1,0)$, $D = 10$ supersymmetry and are heavily constrained by anomaly cancellation. The first two are type I and heterotic $\mathrm{Spin}(32)/\mathbb{Z}_2$ string theory (sometimes called SO(32) heterotic string theory) which both have Spin(32)/$\mathbb{Z}_2$ gauge symmetry. The third one is heterotic $E_8 \times E_8$ string theory with $E_8 \times E_8$ gauge symmetry. Of course, as in the type II case, all these can be compactified on tori T^d without reducing the amount of supersymmetry. As it turns out, the heterotic theories are T-dual to each other, where the discrete transformation involves gauge degrees of freedom and is somewhat more involved as in the type II case.

As became evident over the last ten years, heterotic compactifications on T^6 are indeed equivalent to type IIA theory compactified on $K3 \times T^2$. K3 is a very special four-dimensional manifold which is curved in such a way that exactly sixteen of the thirtytwo supersymmetries of type IIA theory remain unbroken. This "heterotic-type II" duality, in fact, is a highly nontrivial statement, which, nevertheless, has passed all thinkable consistency checks and, moreover, has led the way to the discovery of many other dualities and theories such as M-theory.

Our main interest, however, will be in those theories derived from heterotic $E_8 \times E_8$ string theory in ten dimensions. This theory can be compactified on six-dimensional Calabi-Yau manifolds, which (similar to K3) are curved in such a way that exactly four of the sixteen supersymmetries of the ten-dimensional theory remain unbroken. Therefore, the resulting four-dimensional theory has $\mathcal{N} = 1$, $D = 4$ supersymmetry, the generic starting point of supersymmetric phenomenology in elementary particle physics. Furthermore, the group E_8 is well suited for model building and the presence of the second E_8 serves as a natural candidate for a hidden sector which might trigger supersymmetry breaking.

By applying the power of dualities, Hořava and Witten in [57, 56], based on earlier work relating heterotic and type I theories by Polchinski and Witten [75], gave strong evidence that the dynamics of heterotic and type I theories can be understood from M-theory compactified on an interval $S^1/\mathbb{Z}_2$. Here the S^1 is parameterized by $x^{11} = x^{11} + 2\pi R$ and the $\mathbb{Z}_2$ is generated[1] by the reflection $x^{11} \mapsto -x^{11}$ leading to fixed points located at $x^{11} = 0$ and $x^{11} = \pi R$ which comprise the ends of the interval $0 \leq x^{11} \leq \pi R$. The low energy effective action in the interior of the interval is given by the action of eleven-dimensional supergravity

$$S_M = \frac{1}{2\kappa^2} \int_{M^{11}} R\Omega - \frac{1}{2} K_4 \wedge *K_4 + \frac{1}{6} C_3 \wedge K_4 \wedge K_4 \qquad (1.0.1)$$

where we have left out fermionic degrees of freedom. The first term is the standard action of the gravitational field with gravitational coupling constant κ. The second and third terms comprise the action of the three-index tensor

[1] In general, discrete transformations on string theories or M-theory include non-trivial transformations of space-time or world-sheet fields which often can not be understood from a simple geometric viewpoint. The precise definition of these transformations is absolutely crucial in the arguments given above. For example, the $\mathbb{Z}_2$ generator used in M-theory on $S^1/\mathbb{Z}_2$ reverses the sign of the three-index tensor C_3.

C_3 with its four-index tensor field strenght $K_4 = \mathrm{d}C_3$. In the case of ten non-compact dimensions, the eleven-dimensional manifold with boundary M^{11} is just $\mathbb{R}^{10} \times S^1/\mathbb{Z}_2$.

This action is augmented by Super-Yang-Mills actions confined to the two ten-dimensional "walls" M^{10} and M'^{10}, located at the ends of the interval

$$\begin{aligned} S_{YM} &= -\frac{1}{4\lambda^2}\int_{M^{10}} \mathrm{d}^{10}x\sqrt{g}\,\mathrm{tr}\left(F_{AB}F^{AB}\right) \\ S'_{YM} &= -\frac{1}{4\lambda^2}\int_{M'^{10}} \mathrm{d}^{10}x\sqrt{g}\,\mathrm{tr}\left(F'_{AB}F'^{AB}\right) \end{aligned} \tag{1.0.2}$$

where we again have left out the fermionic terms. A crucial point of this action is that, by anomaly cancellation, the Yang-Mills coupling constant λ is completely fixed in terms of the gravitational coupling [57, 56, 26]: $\lambda^2 = 4\pi(4\pi\kappa^2)^{2/3}$. Therefore κ is the only dimensionful parameter of the theory. The relation of the size of the eleventh dimension to the coupling of heterotic $\mathrm{E}_8\times\mathrm{E}_8$ string theory works as in the case of M-theory on S^1: strong coupling corresponds to a large interval.

From this, the relation to type IIA theory is clear (see the discussion in section 4 of [57]): we compactify the heterotic $\mathrm{E}_8\times\mathrm{E}_8$ string theory on S^1 which is equivalent to M-theory on $S^1/\mathbb{Z}_2 \times S^1$. However, this is nothing but type IIA theory on S^1/mZ_2 since type IIA theory is equivalent to M-theory on S^1. Of course, all these equivalences from the new perspective or dualities from the old perspective relate weakly coupled theories to strongly coupled ones and, since only sixteen supersymmetries are preserved, that statement is as non-trivial as heterotic-type II duality.

Using those results, Witten could show in [106], that already in the simplest Calabi-Yau compactifications to four dimensions one can adjust the parameters of the model to meet the observed strength of Newton's constant, which was not possible in weakly coupled $\mathrm{E}_8\times\mathrm{E}_8$ models. In fact, such models provide the first example of brane world models, which have gained much attention in the recent literature. Because of all this, heterotic $\mathrm{E}_8\times\mathrm{E}_8$ string theory to date is still the best suited string theory for string phenomenology.

However, the details of the model depend on the chosen six-dimensional Calabi-Yau manifold and the computation of the four-dimensional spectrum together together with the interactions pose enormous mathematical problems. To overcome these difficulties, people have studied orbifold compactifications, which can be understood as compactifications on Calabi-Yau manifolds in special limits that make more detailed computations possible. But, when applying orbifolds to the theory of Hořava and Witten, a general problem arises. Many orbifold models contain fields which are charged simultaneously under the remnants of both E_8 groups. From an eleven-dimensional viewpoint, these fields are therefore simultaneously charged with respect to gauge groups located at different ends of the interval. The study of this problem, which has been analyzed in previous publications [62, 49], comprises the main theme of this work. However, by arguments given in the following, we have to turn to compactifications to six dimensions.

The reason for this can be easily understood from simple arguments about supersymmetric field theories (an introduction to supersymmetric field theories in various dimensions can be found in appendix B of [76]; [96] contains very useful tables of multiplets; all details can be found in [79]). Even though $\mathcal{N} = 1$, $D = 4$ supersymmetry usually is introduced as the simplest example of supersymmetric field theories, this is only true at a mere technical level. In terms of control over the properties of a given model, $\mathcal{N} = 1$, $D = 4$ supersymmetric field theories are by far the most complicated and most difficult to understand supersymmetric theories, especially when gravity is included: as there are only four supercharges, half of which become inactive for massless particles, there remain only two supercharges to generate a minimal multiplet. Treating one as a creation operator and the other as a anihilation operator, the minimal multiplet seemingly consists of only two degrees of freedom, one bosonic and one fermionic. However, since the CPT conjugates must also be present, it actually consists of four degrees of freedom. By the same argument, in a massive multiplet, where we have all four supercharges at our disposal, there are two creation operators and therefore $2^2 = 4$ states from the beginning. Those, in addition, happen to be their own CPT conjugates. This implies, that a massless multiplet containing scalars and fermions (the chiral multiplet) can easily be deformed into a massive one and there is no reason why flat directions in the effective potential should not be lifted, even though this might happen only because of nonperturbative effects.

Furthermore, the inclusion of gravity has a peculiar effect on the moduli space [14]: the scalar curvature of the moduli space is negative and proportional to the gravitational coupling constant. This means, that by switching on gravitational interactions, the moduli space generically will loose some amount of symmetry. For instance, in $\mathcal{N} = 1$, $D = 6$ supersymmetry the moduli space of hypermultiplets in general is a quaternionic Kähler manifold, whereas for zero gravitational interactions it becomes a hyperkähler manifold which is much more restricted (see [4]).

In the case of eight supercharges, as for example in $\mathcal{N} = 2$, $D = 4$ Super-Yang-Mills theory or $\mathcal{N} = (1, 0)$, $D = 6$ supersymmetric theories, things are already under much better control. With eight supersymmetries, a massless multiplet, by the same reasoning as above, consists of at least eight degrees of freedom whereas a massive multiplet already contains sixteen degrees of freedom. Therefore, a massless multiplet can not be deformed into a massive one. However, in case of extended ($\mathcal{N} > 1$) supersymmetry with eight supercharges, that is, in dimensions below six, there is the possibility of BPS multiplets which preserve half of the supersymmetries and therefore have only eight degrees of freedom.

These multiplets are called hypermultiplets (massless or BPS) and contain four scalar and four spinor degrees of freedom. Since the BPS condition relates mass and charge, these multiplets are under good control. Furthermore, in dimensions below six, multiplets containing a vector will necessarily contain scalars which remain massless as long as the vector remains massless. Hence, there are two possible sources for scalars parameterizing the moduli space: vector multiplets and hypermultiplets. In fact, except for certain singularities of the moduli space (see [4] for a good introduction), the moduli space splits into a product of vector multiplet moduli space and hypermultiplet moduli space.

In six dimensions, however, we have $\mathcal{N} = (1, 0)$ supersymmetry and vector multiplets contain four vector degrees of freedom and four spinor degrees of freedom. In addition to the hypermultiplet as in four dimensions, there also is a tensor multiplet containing a single scalar together with a self-dual tensor plus fermionic partners. Hence, there is no vector multiplet moduli space at all and often singular points in hypermultiplet moduli space correspond to a hypermultiplet being eaten up by a vector multiplet due to the Higgs effect (see [105] for a thorough account on this). There is, of course, also a moduli space corresponding to the tensor multiplet, but we will not attempt to consider models with more than one tensor multiplet and, hence, this moduli space will not be too important to this work.

Since the problem of fields charged under both E_8 factors simultaneously already appears in heterotic orbifolds in $D = 6$, it is clear that it is much easier to analyze the problem in six-dimensional models. In fact, in four-dimensional models superconformal theories appear which are rather out of reach of present methods (see [49]). The analysis will be carried out in the following steps.

In chapter 2 we give a detailed and even to some extent pedagogical account of the studied orbifold models using the operator approach of string theory. We will focus on abelian supersymmetric symmetric orbifolds and calculate some examples of six-dimensional models. Special attention will be paid to the level matching condition, which is the main consistency condition for heterotic orbifolds in general. By a detailed and very technical mathematical calculation, we prove the relation of this condition to fractional instanton numbers in heterotic $E_8 \times E_8$ string theory. This relation allows us to show that the above orbifolds indeed provide the most general perturbative orbifold models to discuss the problem of states charged under both E_8 groups at the same time.

Chapter 3 is devoted to a detailed study of Kaluza-Klein-monopoles in heterotic compactifications. These BPS states develop orbifold singularities in certain limits. By the arguments given above, this can be used to derive moduli spaces of heterotic string theories on orbifold singularities. We will study Kaluza-Klein-monopole solutions as Wilson line backgrounds are switched on and calculate their correct quantum numbers. To make contact to orbifold models, we will look at gauge instantons sitting on Kaluza-Klein-monopoles.

In chapter 4, collecting the results of previous chapters, we shed some light on the low energy effective descriptions of general heterotic orbifold models in six dimensions. By higgsing the massless spectrum, we give strong arguments on where the massless modes appearing in orbifold models have to be located from the viewpoint of M-theory on $S^1/\mathbb{Z}_2$.

We end by giving conclusions and an outlook in chapter 5.

Some of the results of this work haven been published in

- J. O. Conrad, "On fractional instanton numbers in six dimensional heterotic E(8) x E(8) orbifolds," JHEP **0011** (2000) 022 [arXiv:hep-th/0009251].
- J. O. Conrad, "On fractional instanton numbers in six dimensional heterotic E(8) x E(8) orbifolds," Fortsch. Phys. **49** (2001) 455 [arXiv:hep-th/0101023].

Chapter 2

Supersymmetric Orbifolds

In this chapter we shall be mainly concerned with instances of superstring theory which, in a sense to be investigated in this work, can be understood as superstring theory on tori divided by a discrete $\mathbb{Z}_n$ symmetry of the tori. To do so we will consider a special case of the so called "Orbifold Construction" in which a discrete symmetry of a string theory as a whole is divided out in a consistent manner.

Our focus will be on such constructions in which the orbifolded theory in turn can be interpreted as the original string theory on an "Orbifold", that is, somewhat loosely speaking, on a manifold divided by discrete symmetries. This enables us, if supersymmetry is preserved, to apply the full power of supersymmetric effective field theory to the problem.

2.1 Geometric Orbifolds and the Orbifold Construction

Before we begin, we have to make clear the distinction between orbifolds as geometric objects and the so called orbifold construction of string theory.

In the orbifold construction, a discrete symmetry of a string theory is divided out in a consistent manner. This is a very general construction in which an interpretation in terms of geometry is often very obscure. This is the case, for example, in asymmetric orbifolds, which treat left and right movers of the closed string in an independent manner [70, 71].

Geometric orbifolds, on the other hand, are generalized manifolds. In the physics literature, orbifolds are mostly defined as manifolds divided out by a discrete symmetry group in a global way. At the points or higher dimensional planes where the symmetry does not act freely, the resulting object will have singularities, the so called orbifold singularities. In the mathematics literature, however, orbifolds are defined in a local way. Roughly speaking, they are defined like manifolds, except that every coordinate patch is diffeomorphic to $\mathbb{R}^d$ divided by some discrete symmetry group associated to the patch (which may be trivial, of course). So the physics orbifold is a special case of the mathematics one.

Since all this is standard by now we will not review all details here. The reader is referred to the introductions available ([74, 60], [76] chapter 16, [4]) or to the original literature [33, 34].

The Orbifold $T^d/\mathbb{Z}_n$

In this work we will consider the orbifold construction for string theory on the d-dimensional torus T^d, dividing by a discrete $\mathbb{Z}_n$ symmetry of the torus. This means that we will perform an orbifold construction that is associated to the geometric orbifold obtained from the torus by dividing out a $\mathbb{Z}_n$ symmetry. This is most easily accomplished by writing the torus itself as a quotient $T^d = \mathbb{R}^d/\mathbb{Z}^d$ where $\mathbb{Z}^d$ acts as the d-dimensional group of translations (see [34]).

Therefore, we define the (geometric) orbifold as $O = \mathbb{R}^d/S$, where S denotes the space group defined to consist of pairs $D = (\theta, v)$ of rotations $\theta \in \mathrm{SO}(d)$ and translations v acting like

$$Dx = (\theta, v)x := \theta x + v \tag{2.1.1}$$

on an element $x \in \mathbb{R}^d$. We note that we have equipped $\mathbb{R}^d$ with an euclidian metric in order to define the rotations. The subgroup Λ of S of pure translations consists of elements $(\mathbb{1}, v)$ and defines the torus $T^d = \mathbb{R}^d/\Lambda$. The lattice Γ of the torus is given as the image of the origin: $\Gamma = \Lambda 0$. It is often convenient to define a fundamental domain of the torus spawned by a set of generators of the lattice Γ.

Furthermore, we demand that S acts as a symmetry of the torus, that is, every element of S maps torus lattice points to other torus lattice points: $S\Gamma = \Gamma$.

The orbifold $O = \mathbb{R}^d/S$ is to a good extent characterized by the point group P, defined to be the group of rotations made up of all rotations θ appearing in the elements of S. The orbifold is called abelian, if the point group P is abelian. In this work we will only look at $\mathbb{Z}_n$-orbifolds, that is, abelian orbifolds with $P = \mathbb{Z}_n$. We note that, since $\Gamma = S\Gamma$, we can identify P with the subgroup of S consisting of all elements of the form $(\theta, 0)$.

Since we also will have to treat gauge fields, we can assign a gauge transformation to every element of S. As this has to be consistent with the group laws of S we actually have a group homomorphism from S to the group of gauge transformations.

Classifying the Elements of the Space Group S

In order to construct Hilbert spaces of string theory we will have to classify the elements D of the space group S (see [70]). The first level of classification of an element (θ, v) is given in terms of θ, the corresponding element of P.

In the following, let θ be any nontrivial rotation of P. To classify the translations v appearing in $D = (\theta, v)$, let us first look for fixed points x_0

$$Dx_0 = \theta x_0 + v = x_0 \tag{2.1.2}$$

Let I denote the elements of Γ left invariant by θ and let N denote the orthogonal complement of I. From (2.1.2) we have $v = (1-\theta)x_0$. This implies $v \in N$, since $\theta \in \mathrm{SO}(d)$ and $(1-\theta)w = 0$ for any $w \in I$. Therefore, for every element of N we have an element D of S together with a fixed point of D. On the other hand, shifting v by $(1-\theta)u$ for any lattice vector $u \in \Gamma$ just shifts x_0 by u. Thus, the number of fixed points in the fundamental domain of the torus is given by the index of the quotient of N by $(1-\theta)\Gamma$

$$\left|\frac{N}{(1-\theta)\Gamma}\right| \tag{2.1.3}$$

But if I is nontrivial, there will be elements of Γ which are not elements of N. To deal with this case we decompose a vector v of Γ into its parts parallel to N and I

$$v = \tilde{n} + \tilde{w} \qquad \text{with} \qquad \tilde{n} \perp I, \tilde{w} \perp N \tag{2.1.4}$$

Since $(1-\theta)$ is invertible on N we can define $x_0 = (1-\theta)^{-1}\tilde{n} + t$ where $t \perp N$ is otherwise arbitrary. This implies $\theta x_0 = x_0 - \tilde{n}$. We now get

$$Dx_0 = \theta x_0 + v = x_0 - \tilde{n} + \tilde{n} + \tilde{w} = x_0 + \tilde{w} \tag{2.1.5}$$

Shifting x_0 by u again corresponds to shifting v by $(1-\theta)u$.

In conclusion, given any nontrivial rotation θ together with any torus translation $v \in \Gamma$, we have either $v \in N$ corresponding to a fixed point x_0 or $v \notin N$ corresponding to a fixed plane $x_0 + t$, $\theta t = t$ and $\tilde{w}$ as in (2.1.5).

The Orbifold Construction

We will now describe how to apply the data of the orbifold to superstring theory on T^d, beginning with the bosonic worldsheet degrees of freedom $X^M(\sigma, \tau)$ with $M = 0, \ldots, 9$ (our conventions will mostly be those of [76]). The worldsheet coordinates are σ, τ or $(z, \bar{z}) = (\exp(-i\sigma + \tau), \exp(+i\sigma + \tau))$. The non-compact spacetime directions will be labeled by X^μ, $\mu = 0, \ldots, 9-d$ and the compact ones by X^m, $m = 9-d+1, \ldots, 9$.

The starting point is the natural action of (θ, v) on X:

$$(\theta, v)X(\sigma, \tau) = \theta X(\sigma, \tau) + v \tag{2.1.6}$$

Using this action, which is a symmetry of the ten-dimensional superstring theory since rotations and translations are part of the Super-Poincaré group, we can try to project onto states invariant under all elements D of S.

However, we have to be careful about modular invariance of the string theory, as we will see in the following. When calculating a one-loop partition function, that is, the partition function of a string theory on a 2-dimensional torus of modular parameter $\hat{\tau}$, we have the possibility of specifying two boundary conditions $G, H \in S$ (for convenience, we will work in the $\omega = \sigma + i\tau$ frame)

$$X_\omega(\omega + 2\pi) = GX_\omega(\omega) \qquad X_\omega(\omega + 2\pi\hat{\tau}) = HX_\omega(\omega) \tag{2.1.7}$$

This, of course, only works as long as G and H commute, which we will assume for now. In the operator formalism, a G boundary condition in σ direction corresponds to a sector of the Hilbert space H_G with G boundary condition whereas a boundary condition in τ direction is equivalent to the insertion of the operator corresponding to H into the expression of the partition function (see, for example, chapter 7 of [76]).

Therefore, the partition function of the original string theory, truncated to states invariant under all $H \in S$, is given by

$$Z(\hat{\tau}) = \frac{1}{|S|} \sum_{H \in S} Z_{\mathbb{1},H}(\hat{\tau}) \tag{2.1.8}$$

where $Z_{G,H}(\tau)$ denotes the partition function with G and H boundary conditions as in (2.1.7). All states are clearly invariant, as summing over all $H \in S$ and dividing by $|S|$ corresponds to inserting the complete projection operator onto S invariant states.

However, the theory has to be invariant under large reparametrizations of the coordinates, that is, choosing a different set of basis vectors of the lattice of the torus. This group, denoted as $\mathrm{Sl}(2, \mathbb{Z})$, is generated by the following two elements

$$\begin{aligned} \mathcal{T}: &\quad \hat{\tau} \mapsto \hat{\tau}' = \hat{\tau} + 1 \\ \mathcal{S}: &\quad \hat{\tau} \mapsto \hat{\tau}' = -\frac{1}{\hat{\tau}} \end{aligned} \tag{2.1.9}$$

It is easy to find out the action of $\mathcal{T}$:

$$\begin{aligned} X_\omega(\omega + 2\pi) &= GX_\omega(\omega) \\ X_\omega(\omega + 2\pi\hat{\tau}') &= X_\omega(\omega + 2\pi\hat{\tau} + 2\pi) = GX_\omega(\omega + 2\pi\hat{\tau}) \\ &= HGX_\omega(\omega) \end{aligned} \tag{2.1.10}$$

The action of $\mathcal{S}$ is more difficult, since we have to perform a coordinate transformation $\omega' = -\hat{\tau}'\omega = \omega/\hat{\tau}$:

$$\begin{aligned} X_{\omega'}(\omega' + 2\pi) &= X_\omega\left(-\frac{\omega' + 2\pi}{\hat{\tau}'}\right) = X_\omega(\omega + 2\pi\hat{\tau}) = HX_\omega(\omega) \\ &= HX_{\omega'}(\omega') \\ X_{\omega'}(\omega' + 2\pi\hat{\tau}') &= X_\omega\left(-\frac{\omega' + 2\pi\hat{\tau}'}{\hat{\tau}'}\right) = X_\omega(\omega - 2\pi) = G^{-1}X_\omega(\omega) \\ &= G^{-1}X_{\omega'}(\omega') \end{aligned} \tag{2.1.11}$$

Altogether, $\mathcal{T}$ and $\mathcal{S}$ act on the partition functions as

$$\begin{aligned} \mathcal{T}: &\quad Z_{G,H}(\hat{\tau}) \mapsto Z_{G,HG}(\hat{\tau}+1) \\ \mathcal{S}: &\quad Z_{G,H}(\hat{\tau}) \mapsto Z_{H,G^{-1}}(-1/\hat{\tau}) \end{aligned} \tag{2.1.12}$$

This makes immediately clear that the partition function (2.1.8) cannot be modular invariant, since $\mathcal{S}$ maps (2.1.8) into a sum over sectors which are not present in the original theory. To form an invariant partition function, we therefore have to include sectors with nontrivial boundary conditions in the σ direction. In the operator approach, this means that we will have to enlarge the Hilbert space to include these sectors. The modular invariant partition function now reads

$$Z(\hat{\tau}) = \frac{1}{|S|}\sum_{\substack{H\in S\\ G\in S}} Z_{G,H}(\hat{\tau}) \tag{2.1.13}$$

So far, we have been assuming that S is an abelian group, like in toroidal compactifications. But since rotations and translations do not commute, the space group of an orbifold is non-abelian. The solution to this problem, which has been proposed in [33], is based upon the fact that commuting pairs G and H are mapped by $\mathcal{T}$ and $\mathcal{S}$ to other commuting pairs G' and H'. Therefore, a modular invariant partition function can be defined as

$$Z(\hat{\tau}) = \frac{1}{|S|}\sum_{G\in S}\sum_{H\in C(G)} Z_{G,H}(\hat{\tau}) \tag{2.1.14}$$

where $C(H)$ is the centralizer of H in S, that is, the subgroup of elements commuting with H. As we will see below, this is a partition function of S invariant states.

But still, as pointed out in [34] and calculated in detail in [100], this approach does not necessarily lead to a modular invariant theory. This is due to the fact that there are modular transformations which leave G and H invariant and therefore transform $Z_{G,H}$ into itself. If these transformations are plagued by phases, that is, global anomalies, modular invariance is spoiled. For the operator approach, this will be shown in section 2.3. Even more, as considered in [100], there is the possibility of assigning nontrivial phases to the different sectors of the partition function, which have to be consistent with factorization and modular invariance of higher loops (which is guaranteed from one-loop in the usual case studied in this work, see [100, 43]). However, we will not cover these orbifolds, which are called orbifolds with discrete torsion, and the reader is referred to the literature [100, 101].

Finally, we turn to the construction of invariant states. As we have seen above, for each $G \in S$ it is easy to construct a Hilbert space H_G^0 of states invariant under each $D \in C(G)$. If, however, D is not in $C(G)$ it transforms the boundary condition to

$$DX(\omega + 2\pi) = DGD^{-1}\, DX(\omega) \tag{2.1.15}$$

and therefore it maps the state from H_G^0 to $H_{DGD^{-1}}^0$. Hence, an invariant state Ψ^0 must be a superposition of states over all Hilbert spaces in the conjugacy class[1] of some G in S [33, 34]

$$\Psi^0 = \frac{|C(G)|}{|S|} \sum_{G'=DGD^{-1}} D\Psi_G^0 \qquad \text{with} \qquad \Psi_G^0 \in H_G \tag{2.1.16}$$

where Ψ_G^0 is $C(G)$ invariant. Therefore we have

$$\begin{aligned} \mathrm{Tr}_{H_S^0} O &= \sum_{\Psi^0 \in H_S^0} \langle \Psi^0 | O | \Psi^0 \rangle \\ &= \sum_{G \in S} \frac{|C(G)|}{|S|} \sum_{\Psi_G^0 \in H_G^0} \langle \Psi_G^0 | O | \Psi_G^0 \rangle \\ &= \sum_{G \in S} \frac{|C(G)|}{|S|} \frac{1}{|C(G)|} \sum_{\Psi \in H_G} \sum_{H \in C(G)} \langle \Psi | HO | \Psi \rangle \\ &= \frac{1}{|S|} \sum_{G \in S} \sum_{H \in C(G)} \mathrm{Tr}_{H_G} HO \end{aligned} \tag{2.1.17}$$

where H_S^0 denotes the Hilbert space of S invariant states. This shows that (2.1.14) is the partition function of S invariant states.

Implementing the Orbifold Construction

As we will require supersymmetry to be as minimal as possible (for example eight supersymmetries in six dimensions and four in four dimensions, see section

[1] The size of that class is $(|S|)/(|C(G)|)$

16.2 of [76]) it is most natural to write the compact directions in complex notation

$$
\begin{aligned}
Z^i &= X^{2i} + iX^{2i+1} \\
Z^{\bar{i}} &= X^{2i} - iX^{2i+1}
\end{aligned}
\tag{2.1.18}
$$

We let (θ, v) act on Z^i as

$$
(\theta, v)Z^i = e^{2\pi i \phi^i} Z^i + v^i \tag{2.1.19}
$$

The Hilbert space H_G corresponding to the element $G = (\theta, v)$ is defined by the boundary conditions

$$
Z^i(\sigma + 2\pi, \tau) = GZ^i(\sigma, \tau) = e^{2\pi i \phi^i} Z^i(\sigma, \tau) + v^i \tag{2.1.20}
$$

where $Z^i(\sigma, \tau)$ denotes the sum of left and right[2] movers $Z^i(\sigma, \tau) = Z^i_L(\sigma, \tau) + Z^i_R(\sigma, \tau)$. Therefore, the classification of the Hilbert space, except for the projection, carries over from the classification of the space group elements as described above. We note that we have defined the boundary conditions (2.1.20) in a totally left-right symmetric way. Such orbifolds are called symmetric orbifolds. In asymmetric orbifolds, where the action is defined separately for left and right movers, states can no longer be classified as above and, as a consequence, the direct correspondence to geometry is lost [70, 71].

The untwisted sector (of Hilbert space) is defined to correspond to the trivial element of P whereas the twisted sectors are defined to correspond to the other elements[3].

Before we describe the construction of twisted and untwisted sectors, we have to discuss some properties of the bosonic worldsheet fields Z^i_L and Z^i_R (where we will use real notation X^m_L and X^m_R when appropriate).

Since the translations act like (2.1.19) on the coordinates Z^i, we define the following action

$$
\begin{aligned}
(\theta, v)Z^i_L(z) &= e^{2\pi i \phi^i_L} Z^i_L(z) + v^i_L \\
(\theta, v)Z^i_R(\bar{z}) &= e^{2\pi i \phi^i_R} Z^i_R(\bar{z}) + v^i_R
\end{aligned}
\tag{2.1.21}
$$

But since we want to construct a symmetric orbifold, we define

$$
\phi^i = \phi^i_L = \phi^i_R \qquad \text{and} \qquad v^i = v^i_L = v^i_R \tag{2.1.22}
$$

However, we will keep the notation general till the discussion of the level matching condition in section 2.3.

[2] We will be a bit sloppy in our notation: usually a tilde denotes the right movers, only when ambiguities arise we will use subscripts L and R.

[3] As the untwisted sector also contains sectors twisted by translations, this terminology is a bit confusing, but common in the literature.

The mode expansion is in general given by

$$\begin{aligned} X_L^m(z) &= X_{L0}^m - i\frac{\alpha'}{2}p_L^m \ln z + \text{osz} \\ X_R^m(\bar{z}) &= X_{R0}^m - i\frac{\alpha'}{2}p_R^m \ln \bar{z} + \tilde{\text{osz}} \end{aligned} \tag{2.1.23}$$

where the overall momentum is given by $p^m = \hat{n}^m = \frac{1}{2}(p_L^m + p_R^m)$. This implies, as translations are generated by the overall momentum, that (θ, v) acts on Z^i_{LR0} like

$$\begin{aligned} (\theta, v)Z^i_{L0} &= e^{2\pi i\phi^i_L} Z^i_{L0} + v^i_L \\ (\theta, v)Z^i_{R0} &= e^{2\pi i\phi^i_R} Z^i_{R0} + v^i_R \end{aligned} \tag{2.1.24}$$

Next, we discuss the lattice momenta p_{RL} in (2.1.23). The winding can be read off from

$$\begin{aligned} X^m(\sigma + 2\pi, \tau) &= X_L^m(\sigma + 2\pi, \tau) + X_R^m(\sigma + 2\pi, \tau) \\ &= X^m(\sigma, \tau) + 2\pi\frac{\alpha'}{2}(p_R^m - p_L^m) \end{aligned} \tag{2.1.25}$$

and we have

$$\begin{aligned} \hat{w}^m &= \frac{\alpha'}{2}(p_R^m - p_L^m) \\ \hat{n}^m &= \frac{1}{2}(p_L^m + p_R^m) \end{aligned} \tag{2.1.26}$$

It is often convenient to define winding and momentum charge in dimensionless quantities

$$\begin{aligned} w^m &= \left(\frac{\alpha'}{2}\right)^{1/2}(p_R^m - p_L^m) \\ n^m &= \frac{1}{2}\left(\frac{\alpha'}{2}\right)^{1/2}(p_L^m + p_R^m) \end{aligned} \tag{2.1.27}$$

This makes clear that momentum and winding transform like

$$(\theta, v)w = \theta w \qquad (\theta, v)n = \theta n \tag{2.1.28}$$

Finally, the most general mode expansion of the bosons in complex notation is

$$\begin{aligned} Z_L^i(z) &= Z_{L0}^i - i\frac{\alpha'}{2}p_L^i \ln z + i \sum_{\substack{r_i\in\mathbb{Z}+\phi_L^i \\ r_i\neq 0}} \frac{\alpha_{r_i}^i}{r_i z^{r_i}} \\ Z_L^{\bar{i}}(z) &= Z_{L0}^{\bar{i}} - i\frac{\alpha'}{2}\bar{p}_L^i \ln z + i \sum_{\substack{s_i\in\mathbb{Z}-\phi_L^i \\ s_i\neq 0}} \frac{\alpha_{s_i}^{\bar{i}}}{s_i z^{s_i}} \\ Z_R^i(\bar{z}) &= Z_{R0}^i - i\frac{\alpha'}{2}p_R^i \ln \bar{z} + i \sum_{\substack{\tilde{r}_i\in\mathbb{Z}-\phi_R^i \\ \tilde{r}_i\neq 0}} \frac{\tilde{\alpha}_{\tilde{r}_i}^i}{\tilde{r}_i \bar{z}^{\tilde{r}_i}} \\ Z_R^{\bar{i}}(\bar{z}) &= Z_{R0}^{\bar{i}} - i\frac{\alpha'}{2}\bar{p}_R^i \ln \bar{z} + i \sum_{\substack{\tilde{s}_i\in\mathbb{Z}+\phi_R^i \\ \tilde{s}_i\neq 0}} \frac{\tilde{\alpha}_{\tilde{s}_i}^{\bar{i}}}{\tilde{s}_i \bar{z}^{\tilde{s}_i}} \end{aligned} \tag{2.1.29}$$

where Z^i and $Z^{\bar{i}}$ are treated as independent fields. In (2.1.29) the monodromies of the oscillator parts are

$$\begin{aligned} Z_{\rm Losz}^i(ze^{-2\pi i}) &= e^{2\pi i\phi_L^i} Z_{\rm Losz}^i(z) & \qquad Z_{\rm Losz}^{\bar{i}}(ze^{-2\pi i}) &= e^{-2\pi i\phi_L^i} Z_{\rm Losz}^{\bar{i}}(z) \\ Z_{\rm Rosz}^i(\bar{z}e^{+2\pi i}) &= e^{2\pi i\phi_R^i} Z_{\rm Rosz}^i(\bar{z}) & \qquad Z_{\rm Rosz}^{\bar{i}}(\bar{z}e^{+2\pi i}) &= e^{-2\pi i\phi_R^i} Z_{\rm Rosz}^{\bar{i}}(\bar{z}) \end{aligned} \tag{2.1.30}$$

The corresponding boundary conditions in the (σ,τ) frame are identical. The boundary conditions for the zero mode parts are more complicated and will be discussed below (see (2.1.35))

The Virasoro generators are given as[4]

$$\begin{aligned} L_0 &= \frac{\alpha'}{2}\frac{1}{2}p_L^2 + \sum_i \left(\sum_{\substack{r_i\in\mathbb{Z}+\phi_L^i \\ r_i<0}} -r_i N_{r_i}^{\alpha^i} + \sum_{\substack{s_i\in\mathbb{Z}-\phi_L^i \\ s_i<0}} -s_i N_{s_i}^{\alpha^{\bar{i}}} \right) + a_0 \\ \tilde{L}_0 &= \frac{\alpha'}{2}\frac{1}{2}p_R^2 + \sum_i \left(\sum_{\substack{\tilde{r}_i\in\mathbb{Z}-\phi_R^i \\ \tilde{r}_i<0}} -\tilde{r}_i N_{\tilde{r}_i}^{\tilde{\alpha}^{\bar{i}}} + \sum_{\substack{\tilde{s}_i\in\mathbb{Z}+\phi_R^i \\ \tilde{s}_i<0}} -\tilde{s}_i N_{\tilde{s}_i}^{\tilde{\alpha}^{\bar{i}}} \right) + \tilde{a}_0 \end{aligned} \tag{2.1.31}$$

where $p^2 = p^m p^m$ and a_0 and $\tilde{a}_0$ are the vacuum energies which will be calculated in section 2.2.

The oscillators transform under (θ, v) as

$$\begin{aligned} \theta\alpha_r^i &= e^{+2\pi i\phi_L^i}\alpha_r^i & \qquad \theta\alpha_s^{\bar{i}} &= e^{-2\pi i\phi_L^i}\alpha_s^{\bar{i}} \\ \theta\tilde{\alpha}_{\tilde{r}}^i &= e^{+2\pi i\phi_R^i}\tilde{\alpha}_{\tilde{r}}^i & \qquad \theta\tilde{\alpha}_{\tilde{s}}^{\bar{i}} &= e^{-2\pi i\phi_R^i}\tilde{\alpha}_{\tilde{s}}^{\bar{i}} \end{aligned} \tag{2.1.32}$$

[4]We will always give the Virasoro generators in the $(z,\bar{z})$ frame and denote them by L_0 and $\tilde{L}_0$. The corresponding operators in the (σ,τ) frame will be denoted by T_0 and $\tilde{T}_0$.

The Untwisted Sector

The boundary conditions for an element $G = (\mathbb{1}, v)$ are $X(\sigma + 2\pi, \tau) = X(\sigma, \tau) + v$. Hence, we have to set $\phi_L = \phi_R = 0$ in (2.1.29), (2.1.30) and (2.1.31). Furthermore, we have $2\pi\hat{w} = v$, and, since the wave function of the string has to be well defined on the torus,

$$1 = \exp(ip^m v'^m) = \exp(i\hat{n}^m v'^m) = \exp(2\pi i\hat{n}^m \hat{w}'^m) \qquad \text{for all} \qquad v' \in \Gamma \tag{2.1.33}$$

Therefore, we have

$$\hat{w} \in \frac{1}{2\pi}\Gamma \qquad \hat{n} \in 2\pi\Gamma^* \tag{2.1.34}$$

which implies that the lattice momenta (p_L, p_R) lie in an even self-dual lattice.

In order to construct S invariant states, we have to consider two cases:

- No winding, no momentum: $G = (\mathbb{1}, 0)$, $\hat{w} = \hat{n} = 0$

  As this is the trivial element of S, all elements commute with it and the states have to be invariant under all of them. The zero mode of a state (2.1.23) is invariant by itself. We have to form an invariant state by exciting the oscillators in an appropriate way.

- otherwise

  Only translations and a limited number of elements containing rotations (if any) will commute with G. The modes of H_G are the winding and momentum modes of a toroidal compactification, some of which might be projected out by the remaining rotations. S invariant states are superpositions.

  However, as these states will have winding or momentum, by making the volume of the torus big enough (away from enhanced symmetry points), all those states will gain mass and play no role for the massless spectrum. We will therefore ignore them in the following.

The Twisted Sectors

According to (2.1.4) we can always write $v = \tilde{n} + \tilde{w}$ and $X_0 = (1-\theta)^{-1}\tilde{n} + t$. If v is not in I then t can be an arbitrary translation invariant under θ. If $v \in I$,

t is zero. We write the boundary condition (2.1.20) using (2.1.25)

$$\begin{aligned} X(\sigma+2\pi,\tau) &= X(\sigma,\tau) + 2\pi\frac{\alpha'}{2}(p_R - p_L) \\ &= X_0 - i\frac{\alpha'}{2}p_L \ln z - i\frac{\alpha'}{2}p_R \ln \bar{z} + 2\pi\frac{\alpha'}{2}(p_R - p_L) \\ &\stackrel{!}{=} \theta X(\sigma,\tau) + \hat{n} + \hat{w} \qquad (2.1.35) \\ &= \theta X_0 + \tilde{n} + \tilde{w} - i\frac{\alpha'}{2}\theta p_L \ln z - i\frac{\alpha'}{2}\theta p_R \ln \bar{z} \\ &= X_0 + \tilde{w} - i\frac{\alpha'}{2}\theta p_L \ln z - i\frac{\alpha'}{2}\theta p_R \ln \bar{z} \end{aligned}$$

where we have neglected oscillator contributions, which have to be invariant under θ for themselves. Equating the coefficients, we get

$$\tilde{w} = 2\pi\frac{\alpha'}{2}(p_R - p_L) = \hat{w}, \qquad \theta p_L = p_L, \qquad \theta p_R = p_R \qquad (2.1.36)$$

If $v \notin I$ the string can move perpendicular to N, because t is arbitrary in the plane of I. Only in this case momentum is allowed under the constraint that translations $v' \in I$ must leave the wave function invariant:

$$1 = \exp(ip^m v'^m) = \exp(i\hat{n}^m v'^m) \qquad (2.1.37)$$

This together with $p_{LR} \perp N$ restricts $\hat{n}$ to lie in the dual of I with respect to the orthogonal complement of N.

In conclusion, to characterize a twisted sector corresponding to $\theta \in P$ we have to distinguish whether $v \in \Gamma$ is in N or not:

- Given $v \in N$, we have $DX_0 = X_0$ for $X_0 = (1-\theta)^{-1}v$. This means that the center of mass of the string is given by X_0, that is, the string is tied to a fixed point. There is no winding an no momentum. Since translations do not commute with the rotation, states will be superpositions of a given state with all its images under translation. In addition, there might be other elements[5] in the space group which, even up to translations, do not commute with D. The state will be represented by a state in H_G^0 with the fixed point in the fundamental domain of the orbifold. $C(G)$ is given by all elements in S that leave the fixed point of G invariant. Since $D(x_0 + \delta x) = x_0 + \theta\delta x$ invariance under $C(G)$ means that the oscillator excitations of the string have to be invariant under θ. In conclusion, to any fixed point of some $G = (\theta, v)$ in the fundamental domain of the orbifold there is one sector of S invariant states in the twisted sector of θ.

[5] This happens, for instance, in $\mathbb{Z}_N$ orbifolds where N is not a prime number. In this case, there will be unique fixed points of the generator of $\mathbb{Z}_N$, whereas some powers of the generator will have additional fixed points. For example, the standard $\mathbb{Z}_4$ twist $\theta \cong \phi^i = (1/4, -1/4)$ on T^4 has $2 \cdot 2$ fixed points whereas θ^2 has $4 \cdot 4$ fixed points. Of these 16 fixed points, only six are non-equivalent under the space-group, which is easy to find out by direct calculation.

- If $v = \hat{n} + \hat{w} \notin N$, we have $Dx_0 = x_0 + \hat{w}$ from (2.1.5). Again, we get one sector for every plane $x_0 = (1-\theta)^{-1}\hat{n} + t$ in the fundamental domain, but now, in addition, we have winding modes given by all possible $\hat{w}$. Furthermore the string can have momentum in the dual I^* of I with respect to the orthogonal complement of N. However, those sectors, which are often called $N = 2$ sectors in four-dimensional models, will not be present in six-dimensional supersymmetric orbifolds and hence we will not describe them in any further detail.

2.2 Heterotic $E_8 \times E_8$ and Type IIA Orbifolds

Type IIA Orbifolds

We start by describing the worldsheet fields. As before, we have the worldsheet bosons Z_L and Z_R. The sypersymmetric partners are left and right moving fermions Ψ^i and $\tilde{\Psi}^i$ (both in complex notation as given in (2.1.18)). The central charges are $(c, \tilde{c}) = (12, 12)$.

Invariance of the worldsheet supercurrent (which, up to constants, is given as $T_F(z) = \Psi(z)\partial\bar{X}(z) + \bar{\Psi}(z)\partial X(z)$ forces us to define[6]

$$\begin{aligned} (\theta, v)\Psi^i(z) &= e^{2\pi i\phi^i_L}\Psi^i(z) & \qquad (\theta, v)\Psi^{\bar{i}}(z) &= e^{-2\pi i\phi^i_L}\Psi^{\bar{i}}(z) \\ (\theta, v)\tilde{\Psi}^i(\bar{z}) &= e^{2\pi i\phi^i_R}\tilde{\Psi}^i(\bar{z}) & \qquad (\theta, v)\tilde{\Psi}^{\bar{i}}(\bar{z}) &= e^{-2\pi i\phi^i_R}\tilde{\Psi}^{\bar{i}}(\bar{z}) \end{aligned} \tag{2.2.1}$$

Therefore, we have for the oscillators

$$\begin{aligned} (\theta, v)\Psi^i_r &= e^{2\pi i\phi^i_L}\Psi^i_r & \qquad (\theta, v)\Psi^{\bar{i}}_s &= e^{-2\pi i\phi^i_L}\Psi^{\bar{i}}_s \\ (\theta, v)\tilde{\Psi}^i_{\tilde{r}} &= e^{2\pi i\phi^i_R}\tilde{\Psi}^i_{\tilde{r}} & \qquad (\theta, v)\tilde{\Psi}^{\bar{i}}_{\tilde{s}} &= e^{-2\pi i\phi^i_R}\tilde{\Psi}^{\bar{i}}_{\tilde{s}} \end{aligned} \tag{2.2.2}$$

Since now the transformation of the fermionic vacuum is more obscure, we bosonize the complex fermions to real bosons

$$\begin{aligned} \Psi^i(z) &= e^{iH^i(z)} & \qquad \Psi^{\bar{i}}(z) &= e^{-iH^i(z)} \\ \tilde{\Psi}^i(\bar{z}) &= e^{i\tilde{H}^i(\bar{z})} & \qquad \tilde{\Psi}^{\bar{i}}(\bar{z}) &= e^{-i\tilde{H}^i(\bar{z})} \end{aligned} \tag{2.2.3}$$

[6] This point has to be taken with a grain of salt. One might be tempted to allow for a minus sign on the r.h.s. of these equations. However, we are constructing orbifolds in which the discrete symmetries we want to divide out stem from continuous symmetries: translations and rotations, which are part of the Super-Poincaré group of the ten-dimensional superstring theory. Therefore, allowing a minus sign would break supersymmetry.

Bozonization allows us to write down the mode expansion of the H and $\tilde{H}$ fields

$$\begin{aligned} H^i(z) &= H_0^i - iq^i \ln z + i \sum_{\substack{n\in\mathbb{Z}\\ n\neq 0}} \frac{\gamma_n^i}{nz^n} \\ \tilde{H}^i(\bar{z}) &= \tilde{H}_0^i - i\tilde{q}^i \ln \bar{z} + i \sum_{\substack{n\in\mathbb{Z}\\ n\neq 0}} \frac{\tilde{\gamma}_n^i}{n\bar{z}^n} \end{aligned} \tag{2.2.4}$$

and the Virasoro generators

$$\begin{aligned} L_0 &= \frac{1}{2}q^2 + \sum_i \sum_{n=1}^{\infty} nN_n^{\gamma^i} \\ \tilde{L}_0 &= \frac{1}{2}\tilde{q}^2 + \sum_i \sum_{n=1}^{\infty} nN_n^{\tilde{\gamma}^i} \end{aligned} \tag{2.2.5}$$

where $N_n^{\gamma^i}$ is the number of excited states corresponding to the creation operator γ_{-n}^i and $q^2 = q^i q^i$.

The ground states with respect to oscillator excitations are denoted by

$$\begin{aligned} \left|q^i\right\rangle &=: e^{iq^i H^i(0)} : \left|0\right\rangle \qquad q^i \in \Gamma_L \\ \left|\tilde{q}^i\right\rangle &=: e^{i\tilde{q}^i \tilde{H}^i(0)} : \left|0\right\rangle \qquad \tilde{q}^i \in \Gamma_R \end{aligned} \tag{2.2.6}$$

(where a sum over i is understood and we restrict to transversal variables). The lattices Γ_L and Γ_R have to close upon acting by the Ψ and $\tilde{\Psi}$ operators, that is, upon adding integers to the q^i or $\tilde{q}^i$. The monodromies of these states are

$$\begin{aligned} H^i(ze^{-2\pi i}) &= H^i(z) - 2\pi q^i \\ \tilde{H}^i(\bar{z}e^{+2\pi i}) &= \tilde{H}^i(\bar{z}) + 2\pi \tilde{q}^i \end{aligned} \tag{2.2.7}$$

and correspondingly

$$\begin{aligned} \Psi^i(ze^{-2\pi i}) &= e^{iH^i(ze^{-2\pi i})} = \Psi^i(z)e^{-2\pi i q^i} \\ \tilde{\Psi}^i(\bar{z}e^{+2\pi i}) &= e^{i\tilde{H}^i(\bar{z}e^{+2\pi i})} = \tilde{\Psi}^i(\bar{z})e^{+2\pi i \tilde{q}^i} \end{aligned} \tag{2.2.8}$$

and we note that this is strikingly different from (2.2.1). For fields Ψ and $\tilde{\Psi}$ of a given monodromy we can write down mode expansions similar to those of (2.1.29)

$$\begin{aligned} \Psi^i(z) &= \sum_{r_i\in\mathbb{Z}+\nu^i} \frac{\Psi_{r_i}^i}{z^{r_i+1/2}} & \Psi^{\bar{i}}(z) &= \sum_{s_i\in\mathbb{Z}-\nu^i} \frac{\Psi_{s_i}^{\bar{i}}}{z^{s_i+1/2}} \\ \tilde{\Psi}^i(\bar{z}) &= \sum_{\tilde{r}_i\in\mathbb{Z}-\tilde{\nu}^i} \frac{\tilde{\Psi}_{\tilde{r}_i}^i}{\bar{z}^{\tilde{r}_i+1/2}} & \tilde{\Psi}^{\bar{i}}(\bar{z}) &= \sum_{\tilde{s}_i\in\mathbb{Z}+\tilde{\nu}^i} \frac{\tilde{\Psi}_{\tilde{s}_i}^{\bar{i}}}{\bar{z}^{\tilde{s}_i+1/2}} \end{aligned} \tag{2.2.9}$$

Since this has to be compatible to (2.2.8), we have, in a sector of given monodromy,

$$\begin{aligned} -q^i &= r_i + \frac{1}{2} \pmod 1 & +q^i &= s_i + \frac{1}{2} \pmod 1 \\ +\tilde{q}^i &= \tilde{r}_i + \frac{1}{2} \pmod 1 & -\tilde{q}^i &= \tilde{s}_i + \frac{1}{2} \pmod 1 \end{aligned} \tag{2.2.10}$$

which is achieved by

$$\begin{aligned} \nu^i &= \frac{1}{2} - q^i \pmod 1 \\ \tilde{\nu}^i &= \frac{1}{2} - \tilde{q}^i \pmod 1 \end{aligned} \tag{2.2.11}$$

The action of (θ, v) is now, from (2.2.1) and (2.2.3),given as

$$\begin{aligned} (\theta, v)\big|q^i\big\rangle &= e^{2\pi i q^i \phi^i_L}\big|q^i\big\rangle \\ (\theta, v)\big|\tilde{q}^i\big\rangle &= e^{2\pi i \tilde{q}^i \phi^i_R}\big|\tilde{q}^i\big\rangle \end{aligned} \tag{2.2.12}$$

corresponding to

$$\begin{aligned} (\theta, v)H(z) &= 2\pi\phi^i_L + H^i(z) \\ (\theta, v)\tilde{H}(\bar{z}) &= 2\pi\phi^i_R + \tilde{H}^i(\bar{z}) \end{aligned} \tag{2.2.13}$$

As this is just a translation of the bosons, γ^i_n and $\tilde{\gamma}^i_n$ remain invariant. We note that, again, $\big|0\big\rangle$ is invariant under rotations. In addition, since the q^i and $\tilde{q}^i$ can be non-integer, the phases ϕ^i_{LR} must be defined on a interval larger than $(0, 2\pi)$.

The Untwisted Sector of IIA

Type IIA string defines two possible ground states for the worldsheet fermions for left and right movers each. We define the following vacua for R and NS sectors together with their worldsheet fermion numbers[7] F and $\tilde{F}$:

$$\begin{aligned} \big|0\big\rangle_R &= \big|\tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}\big\rangle & F &= 0 \\ \big|0\big\rangle_{NS} &= \big|0, 0, 0, 0\big\rangle & F &= 1 \\ \big|0\big\rangle_{\tilde{R}} &= \big|\tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}\big\rangle & \tilde{F} &= 0 \\ \big|0\big\rangle_{\tilde{NS}} &= \big|0, 0, 0, 0\big\rangle & \tilde{F} &= 1 \end{aligned} \tag{2.2.14}$$

From these vacua the lattices Γ_L and Γ_R are generated.

[7] This assignment of worldsheet fermion numbers stems from the ghosts which have been omitted in our discussion. For details, see section 10.4 of [76]

Type IIA string theory is defined by the GSO-projection[8]

$$(NS+, \tilde{NS}+) \quad (R-, \tilde{NS}+) \quad (NS+, \tilde{R}+) \quad (R-, \tilde{R}+) \tag{2.2.15}$$

where the signs show the values of $(-1)^F$ and $(-1)^{\tilde{F}}$. This restricts the q and $\tilde{q}$ to lie in the lattices $\Gamma^+_{\mathrm{SO}(8)}$ and $\Gamma^-_{\mathrm{SO}(8)}$:

$$\begin{aligned} \Gamma^{\pm}_{\mathrm{SO}(8)} &= \Gamma^R_{\mathrm{SO}(8)} \oplus \left[(\pm\tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}) + \Gamma^R_{\mathrm{SO}(8)}\right] \\ \Gamma^R_{\mathrm{SO}(8)} &= \left\{(n_1, n_2, n_3, n_4) \;\middle|\; n_i \in \mathbb{Z}, \textstyle\sum_i n_i = 0 \pmod 2\right\} \end{aligned} \tag{2.2.16}$$

Since $\Gamma^R_{\mathrm{SO}(8)}$ is the root lattice of SO(8), the little group of massless states in ten dimensions, the R sector contains spinor representations of negative chirality whereas the $\tilde{R}$ sector contains spinor representations of positive chirality. The theory is supersymmetric and contains a negative chirality gravitino in the leftmoving sector and a positive chirality one in the rightmoving sector.

As noted above, (ϕ_L, ϕ_R) must be extended to the interval $-2\pi \le \phi_{LR} \le 2\pi$ since the $(q^i, \tilde{q}^i)$ are now half-integer valued. Furthermore, θ must have a well defined action on the lattices $\Gamma^{\pm}_{\mathrm{SO}(8)}$ and should not project out all spinor representations to preserve some amount of supersymmetry. Let m be a positive integer for which $\theta^m = \mathbb{1}$ (usually we set $m = n$ for $\mathbb{Z}_n$ orbifolds). We write

$$\phi^i_{LR} = \frac{r^i_{LR}}{m} \tag{2.2.17}$$

and demand

$$\sum_i r^i_{LR} = 0 \pmod 2 \tag{2.2.18}$$

which will guarantee that some amount of supersymmetry survives (see section 16.2 of [76]).

The monodromies are

$$\begin{aligned} &\Psi^i(ze^{-2\pi i}) = -\Psi^i(z) \qquad \tilde{\Psi}^i(\bar{z}e^{+2\pi i}) = -\tilde{\Psi}^i(\bar{z}) \qquad \text{R sectors} \\ &\Psi^i(ze^{-2\pi i}) = +\Psi^i(z) \qquad \tilde{\Psi}^i(\bar{z}e^{+2\pi i}) = +\tilde{\Psi}^i(\bar{z}) \qquad \text{NS sectors} \end{aligned} \tag{2.2.19}$$

where the corresponding boundary conditions in the (σ, τ) frame change sign.

A general state (before the projection onto invariant states) can be con-

[8] This theory is denoted as IIA' in [76]. However, since in the heterotic string we will only let the rightmoving sector of IIA survive, we choose positive chirality in the right moving sector.

structed from the following states and creation operators:

$$
\begin{array}{llll}
\text{left:} & |q\rangle \otimes |p_L\rangle & q \in \Gamma^{-}_{\mathrm{SO}(8)} & \\
 & \alpha^i_{r_i} & r_i \in \mathbb{Z} & r_i < 0 \\
 & \alpha^{\bar{i}}_{s_i} & s_i \in \mathbb{Z} & s_i < 0 \\
 & \gamma^i_{-n} & n > 0 & \\
 & & & \\
\text{right:} & |\tilde{q}\rangle \otimes |p_R\rangle & \tilde{q} \in \Gamma^{+}_{\mathrm{SO}(8)} & \\
 & \tilde{\alpha}^i_{\tilde{r}_i} & \tilde{r}_i \in \mathbb{Z} & \tilde{r}_i < 0 \\
 & \tilde{\alpha}^{\bar{i}}_{\tilde{s}_i} & \tilde{s}_i \in \mathbb{Z} & \tilde{s}_i < 0 \\
 & \tilde{\gamma}^i_{-n} & n > 0 &
\end{array}
\tag{2.2.20}
$$

where the components of p_L and p_R are zero in the compact directions. From the fermionic viewpoint, we have the operators (for $n \in \mathbb{Z}, n \geq 0$)

$$
\begin{array}{lllll}
NS: & \Psi^i_{-1/2-n} & \Psi^{\bar{i}}_{-1/2-n} & \tilde{\Psi}^i_{-1/2-n} & \tilde{\Psi}^{\bar{i}}_{-1/2-n} \\
R: & \Psi^i_{-n} & \Psi^{\bar{i}}_{-n} & \tilde{\Psi}^i_{-n} & \tilde{\Psi}^{\bar{i}}_{-n}
\end{array}
\tag{2.2.21}
$$

A state then gets a phase

$$
e^{2\pi i s^{(\hat{\theta},\hat{v})}} = e^{2\pi i (s_L^{(\hat{\theta},\hat{v})} + s_R^{(\hat{\theta},\hat{v})})}
\tag{2.2.22}
$$

upon a transformation $(\hat{\theta}, \hat{v})$. From (2.1.32) and (2.2.12), these phases are given by

$$
\begin{aligned}
s_L^{(\hat{\theta},\hat{v})} &= \sum_i \hat{\phi}^i_L \left(\sum_{\substack{r_i \in \mathbb{Z} \\ r_i < 0}} N^{\alpha^i}_{r_i} - \sum_{\substack{s_i \in \mathbb{Z} \\ s_i < 0}} N^{\alpha^{\bar{i}}}_{s_i} \right) + q^i \hat{\phi}^i_L \\
s_R^{(\hat{\theta},\hat{v})} &= \sum_i \hat{\phi}^i_R \left(\sum_{\substack{\tilde{r}_i \in \mathbb{Z} \\ \tilde{r}_i < 0}} N^{\tilde{\alpha}^i}_{\tilde{r}_i} - \sum_{\substack{\tilde{s}_i \in \mathbb{Z} \\ \tilde{s}_i < 0}} N^{\tilde{\alpha}^{\bar{i}}}_{\tilde{s}_i} \right) + \tilde{q}^i \hat{\phi}^i_R
\end{aligned}
\tag{2.2.23}
$$

From (2.1.31) and (2.2.5) the Virasoro generators transformed to the (σ, τ) frame are (the vacuum energies for ordinary bosons are all zero)

$$\begin{aligned}
T_0 = L_0 - \frac{c}{24} = \frac{\alpha'}{2}\frac{1}{2}p_L^2 + \sum_i \left(\sum_{\substack{r_i \in \mathbb{Z} \\ r_i < 0}} -r_i N_{r_i}^{\alpha^i} + \sum_{\substack{s_i \in \mathbb{Z} \\ s_i < 0}} -s_i N_{s_i}^{\alpha^{\bar{i}}} \right) \\
+ \frac{1}{2}q^2 + \sum_i \sum_{n=1}^{\infty} n N_n^{\gamma^i} - \frac{1}{2} \\
\tilde{T}_0 = \tilde{L}_0 - \frac{\tilde{c}}{24} = \frac{\alpha'}{2}\frac{1}{2}p_R^2 + \sum_i \left(\sum_{\substack{\tilde{r}_i \in \mathbb{Z} \\ \tilde{r}_i < 0}} -\tilde{r}_i N_{\tilde{r}_i}^{\tilde{\alpha}^i} + \sum_{\substack{\tilde{s}_i \in \mathbb{Z} \\ \tilde{s}_i < 0}} -\tilde{s}_i N_{\tilde{s}_i}^{\tilde{\alpha}^{\bar{i}}} \right) \\
+ \frac{1}{2}\tilde{q}^2 + \sum_i \sum_{n=1}^{\infty} n N_n^{\tilde{\gamma}^i} - \frac{1}{2}
\end{aligned} \tag{2.2.24}$$

Since type IIA theory on a torus is modular invariant, we have $T_0 - \tilde{T}_0 = 0$ (mod 1) which yields[9]

$$q^2 - \tilde{q}^2 = 0 \pmod 2 \tag{2.2.25}$$

The projection onto invariant states is given by the constraint

$$s^{(\hat{\theta},\hat{v})} = 0 \pmod 1 \qquad \text{for all} \qquad (\hat{\theta},\hat{v}) \in S \tag{2.2.26}$$

The Twisted Sectors of IIA

As discussed in section 2.1, a twisted sector of $\theta \cong (\phi_L, \phi_R)$ is further specified by a fixed point x_0 in the fundamental domain of the torus. Therefore, we have to set $X_0 = x_0$ in (2.1.23) and ϕ_L and ϕ_R as in (2.1.29) with no momentum and no winding.

Since the OPE of two supercurrents generates the worldsheet energy momentum tensor, which for Poincaré symmetry has to remain periodic with respect to σ, the supercurrent may be periodic or antiperiodic. Therefore, from $T_F(z) = \Psi(z)\partial\bar{X}(z) + \bar{\Psi}(z)\partial X(z)$, the fermions have to have the same monodromy as the bosons up to ± 1. Hence, from the monodromy of Ψ and $\tilde{\Psi}$ in (2.2.8) we get

$$\begin{aligned}
q^i &= p^i - \phi_L^i \qquad & p' \in \Gamma'_L \\
\tilde{q}^i &= \tilde{p}^i + \phi_R^i \qquad & \tilde{p}' \in \Gamma'_R
\end{aligned} \tag{2.2.27}$$

[9] Here we have used the fact that the terms containing the momenta p_{LR} cancel modulo 2: if we leave the non-compact directions non compact we have no winding and therefore $p_L = p_R$. If we further compactify them on a torus, modular invariance of the CFT on that torus will guarantee the cancellation.

for some lattices Γ'_L and Γ'_R. However, since the modular transformation $\mathcal{TST}$ maps (G', H') to $(G'H', H')$ (from (2.1.12)), the σ boundary condition of the untwisted sector G' just maps to a $G'H'$ boundary condition in the twisted sector containing $G'H'$. Therefore, the lattice of the $(p, \tilde{p})$ is exactly the lattice of the untwisted sector[10] and the GSO-projection for $(q, \tilde{q})$ is just the untwisted GSO-projection for $(p, \tilde{p})$. Hence, a general state can be constructed from the following states and creation operators:

$$\begin{array}{llll} \text{left:} & \left|p - \phi_L\right\rangle & p \in \Gamma^-_{\mathrm{SO}(8)} & \\ & \alpha^i_{r_i} & r_i \in \mathbb{Z} + \phi^i_L & r_i \neq 0 \\ & \alpha^{\bar{i}}_{s_i} & s_i \in \mathbb{Z} - \phi^i_L & s_i \neq 0 \\ & \gamma^i_{-n} & n > 0 & \\ \text{right:} & \left|\tilde{p} + \phi_R\right\rangle & \tilde{p} \in \Gamma^+_{\mathrm{SO}(8)} & \\ & \tilde{\alpha}^i_{\tilde{r}_i} & \tilde{r}_i \in \mathbb{Z} - \phi^i_R & \tilde{r}_i \neq 0 \\ & \tilde{\alpha}^{\bar{i}}_{\tilde{s}_i} & \tilde{s}_i \in \mathbb{Z} + \phi^i_R & \tilde{s}_i \neq 0 \\ & \tilde{\gamma}^i_{-n} & n > 0 & \end{array} \qquad (2.2.28)$$

Again, from the fermionic viewpoint, we have the operators (for $n \in \mathbb{Z}$)

$$\begin{array}{lllll} NS: & \Psi^i_{-n+\nu^i} & \Psi^{\bar{i}}_{-n-\nu^i} & \nu^i = \frac{1}{2} + \phi^i_L & (\mathrm{mod}\ 1) \\ \tilde{NS}: & \tilde{\Psi}^i_{-n-\tilde{\nu}^i} & \tilde{\Psi}^{\bar{i}}_{-n+\tilde{\nu}^i} & \tilde{\nu}^i = \frac{1}{2} + \phi^i_R & (\mathrm{mod}\ 1) \\ R: & \Psi^i_{-n+\nu^i} & \Psi^{\bar{i}}_{-n-\nu^i} & \nu^i = \phi^i_L & (\mathrm{mod}\ 1) \\ \tilde{R}: & \tilde{\Psi}^i_{-n-\tilde{\nu}^i} & \tilde{\Psi}^{\bar{i}}_{-n+\tilde{\nu}^i} & \tilde{\nu}^i = \phi^i_R & (\mathrm{mod}\ 1) \end{array} \qquad (2.2.29)$$

where the subscript of Ψ or $\tilde{\Psi}$ is less than or equal to zero.

The transformation phases can be read off from

$$\begin{aligned} s_L^{(\hat{\theta},\hat{v})} &= \sum_i \hat{\phi}^i_L \left(\sum_{\substack{r_i \in \mathbb{Z}+\phi^i_L \\ r_i<0}} N^{\alpha^i}_{r_i} - \sum_{\substack{s_i \in \mathbb{Z}-\phi^i_L \\ s_i<0}} N^{\alpha^{\bar{i}}}_{s_i} \right) + (p^i - \phi^i_L)\hat{\phi}^i_L \\ s_R^{(\hat{\theta},\hat{v})} &= \sum_i \hat{\phi}^i_R \left(\sum_{\substack{\tilde{r}_i \in \mathbb{Z}-\phi^i_R \\ \tilde{r}_i<0}} N^{\tilde{\alpha}^i}_{\tilde{r}_i} - \sum_{\substack{\tilde{s}_i \in \mathbb{Z}+\phi^i_R \\ \tilde{s}_i<0}} N^{\tilde{\alpha}^{\bar{i}}}_{\tilde{s}_i} \right) + (\tilde{p}^i + \phi^i_R)\hat{\phi}^i_R \end{aligned} \qquad (2.2.30)$$

[10] This guarantees (2.2.25) for $(p, \tilde{p})$, a condition that will prove absolutely necessary for modular invariance in section 2.3 (see (2.3.6)).

and we have

$$\begin{aligned}
T_0 = L_0 - \frac{c}{24} = \frac{\alpha'}{2}\frac{1}{2}p_L^2 + \sum_i \left(\sum_{\substack{r_i \in \mathbb{Z}+\phi_L^i \\ r_i<0}} -r_i N_{r_i}^{\alpha^i} + \sum_{\substack{s_i \in \mathbb{Z}-\phi_L^i \\ s_i<0}} -s_i N_{s_i}^{\alpha^{\bar{i}}} \right) \\
+ \frac{1}{2}(p-\phi_L)^2 + \sum_i \sum_{n=1}^{\infty} n N_n^{\gamma^i} + a_0 - \frac{1}{2} \\
\tilde{T}_0 = \tilde{L}_0 - \frac{\tilde{c}}{24} = \frac{\alpha'}{2}\frac{1}{2}p_R^2 + \sum_i \left(\sum_{\substack{\tilde{r}_i \in \mathbb{Z}-\phi_R^i \\ \tilde{r}_i<0}} -\tilde{r}_i N_{\tilde{r}_i}^{\tilde{\alpha}^{\bar{i}}} + \sum_{\substack{\tilde{s}_i \in \mathbb{Z}+\phi_R^i \\ \tilde{s}_i<0}} -\tilde{s}_i N_{\tilde{s}_i}^{\tilde{\alpha}^{i}} \right) \\
+ \frac{1}{2}(\tilde{p}+\phi_R)^2 + \sum_i \sum_{n=1}^{\infty} n N_n^{\tilde{\gamma}^i} + \tilde{a}_0 - \frac{1}{2}
\end{aligned} \tag{2.2.31}$$

It remains to calculate the vacuum energies of the twisted bosons. However, this is an easy problem: worldsheet supersymmetry tells us that the vacuum energy of a complex boson and a complex fermion cancel each other, given both obey the same boundary conditions in the (σ, τ) frame. Thus, given the complex boson obeys $Z(ze^{2\pi i}) = Z(z)e^{2\pi i\phi}$, its vacuum energy is minus the vacuum energy of a complex fermion with $\Psi(ze^{2\pi i}) = \Psi(z)e^{2\pi i(\phi+1/2)}$. Upon bozonization the fermion, because of (2.2.8), will correspond to a real boson of momentum $q = m - \phi + \frac{1}{2}$, $m \in \mathbb{Z}$ with an energy $T_0 = \frac{1}{2}q^2 - \frac{1}{24}$. Obviously, the lowest possible value is given for $-\frac{1}{2} < q < +\frac{1}{2}$ and therefore we set $m = 0$ if $0 \leq \phi \leq 1$. We have

$$T_0^F = \frac{1}{2}\left(\frac{1}{2} - \phi\right)^2 - \frac{1}{24} = \frac{1}{2}\phi(\phi-1) + \frac{1}{12} = -T_0^B \tag{2.2.32}$$

Since a complex boson has $c = 2$ we get

$$a_0 = \frac{1}{2}\phi(1-\phi) \tag{2.2.33}$$

For the whole twisted sector we therefore have

$$\begin{aligned}
a_0 &= \frac{1}{2}\sum_i \phi_{L0}^i(1-\phi_{L0}^i) \\
\tilde{a}_0 &= \frac{1}{2}\sum_i \phi_{R0}^i(1-\phi_{R0}^i)
\end{aligned} \tag{2.2.34}$$

where $0 \leq \phi_{LR0}^i \leq 1$ and $\phi_{LR0}^i = \phi_{LR}^i \pmod 1$.

Heterotic $E_8 \times E_8$ Orbifolds

In heterotic string theory the leftmoving fermions Ψ^i are replaced by sixteen complex fermions Ψ^I, $I = 1, \ldots, 16$ which are equivalent to sixteen leftmoving real bosons H^I. Therefore, we have $(c, \tilde{c}) = (24, 12)$. Upon imposing the appropriate GSO-projection on the H^I (see below) this theory describes $E_8 \times E_8$ gauge fields.

Therefore, as described in section 2.1, we have to equip the space group S with a map to the group of gauge transformations. This is accomplished by amending (θ, v) with a gauge shift[11] β^I which acts on states like

$$(\beta)\big|\text{state}\big\rangle = e^{2\pi i \beta^I H^I}\big|\text{state}\big\rangle \qquad (2.2.35)$$

where the H^I denote the generators of the Cartan subalgebra. In heterotic theory we therefore write (θ, v, β) instead of (θ, v) where it is understood that the (θ, v) is mapped to β by a group homomorphism. In view of (2.2.12) the action of (θ, v, β) will be given by

$$(\theta, v, \beta)\big|q^I\big\rangle = e^{2\pi i q^I \beta^I}\big|q^I\big\rangle \qquad (2.2.36)$$

for gauge degrees of freedom and exactly like the (θ, v) action in case of the other fields. Given an element D of S with $D^m = \mathbb{1}$ for some m, the gauge shift $m\beta$ corresponding to D^m must be a lattice point of the $E_8 \times E_8$ lattice[12].

The Untwisted Sector of Heterotic $E_8 \times E_8$

As in type IIA theory, we have to define vacua for the bosons H^I. In case of the $E_8 \times E_8$ string we divide the H^I in two groups, ranging from 1 to 8 and from 9 to 16. We treat both groups like the eight leftmoving H^i of the type IIA theory. Therefore, we have four ground states as in (2.2.14):

$$\big|0\big\rangle_{R_1} \qquad \big|0\big\rangle_{NS_1} \qquad \big|0\big\rangle_{R_2} \qquad \big|0\big\rangle_{NS_2} \qquad (2.2.37)$$

where subscripts 1 and 2 correspond to the first and second group of H^I respectively and all ground states are assigned $F_1 = F_2 = 0$. The GSO-projection is now given by

$$(\tilde{NS}+) \quad (\tilde{R}+) \qquad \text{and} \qquad (-1)^{F_1} = (-1)^{F_2} = 1 \qquad (2.2.38)$$

[11] We could also have equipped S with a map to automorphisms of the $E_8 \times E_8$ root lattice, but we shall not pursue that approach in this work. As we will see in section 2.5, our construction describes a quite general class of orbifolds.

[12] This is a nontrivial statement as the lattice points of root or weight lattices do correspond to the center of the group, not necessarily to the unit element. But since the center of E_8 is trivial, $m\beta$ can be *any* lattice point.

After applying the projection $F_1 = F_2 = 0$ for the two leftmoving worldsheet fermion numbers the q^I will take values in the lattice $\Gamma_8 \otimes \Gamma_8$ where Γ_8 is the even self-dual root lattice of the group E_8 (see appendix B.1).

If we had chosen the GSO projection as $F_1 + F_2 = 0$, which is the only other consistent choice, the q^I would take values in the lattice Γ_{16}, which is defined like Γ_8 in (B.1.1), except that I runs from 1 to 16. This lattice is self-dual and even aswell and consists of the root lattice of the group Spin(32) together with the weight lattice of the positive chirality spinors of Spin(32). This heterotic string theory[13] is called SO(32) or Spin(32)/$\mathbb{Z}_2$ heterotic theory and will not be described any further as only the $E_8 \times E_8$ theory is directly linked to M-theory.

A general state can be constructed from the following states and creation operators:

$$
\begin{array}{lllll}
\text{left:} & |q\rangle \otimes |p_L\rangle & q \in \Gamma_8 \otimes \Gamma_8 & \\
& \alpha^i_{r_i} & r_i \in \mathbb{Z} & r_i < 0 \\
& \alpha^{\bar{i}}_{s_i} & s_i \in \mathbb{Z} & s_i < 0 \\
& \gamma^I_{-n} & n > 0 & \\
\\
\text{right:} & |\tilde{q}\rangle \otimes |p_R\rangle & \tilde{q} \in \Gamma^+_{\mathrm{SO}(8)} & \\
& \tilde{\alpha}^i_{\tilde{r}_i} & \tilde{r}_i \in \mathbb{Z} & \tilde{r}_i < 0 \\
& \tilde{\alpha}^{\bar{i}}_{\tilde{s}_i} & \tilde{s}_i \in \mathbb{Z} & \tilde{s}_i < 0 \\
& \tilde{\gamma}^i_{-n} & n > 0 &
\end{array}
\tag{2.2.39}
$$

where the components of p_L and p_R are zero in the compact directions. From the fermionic viewpoint, we have the operators (for $n \in \mathbb{Z}, n \geq 0$)

$$
\begin{array}{lllll}
NS: & \Psi^I_{-1/2-n} & \Psi^{\bar{I}}_{-1/2-n} & \tilde{\Psi}^i_{-1/2-n} & \tilde{\Psi}^{\bar{i}}_{-1/2-n} \\
R: & \Psi^I_{-n} & \Psi^{\bar{I}}_{-n} & \tilde{\Psi}^i_{-n} & \tilde{\Psi}^{\bar{i}}_{-n}
\end{array}
\tag{2.2.40}
$$

The transformation phases for $(\hat{\theta}, \hat{v}, \hat{\beta})$ are given by

$$
\begin{aligned}
s_L^{(\hat{\theta},\hat{v},\hat{\beta})} &= \sum_i \hat{\phi}^i_L \left(\sum_{\substack{r_i \in \mathbb{Z} \\ r_i < 0}} N^{\alpha^i}_{r_i} - \sum_{\substack{s_i \in \mathbb{Z} \\ s_i < 0}} N^{\alpha^{\bar{i}}}_{s_i} \right) + q^I \hat{\beta}^I \\
s_R^{(\hat{\theta},\hat{v},\hat{\beta})} &= \sum_i \hat{\phi}^i_R \left(\sum_{\substack{\tilde{r}_i \in \mathbb{Z} \\ \tilde{r}_i < 0}} N^{\tilde{\alpha}^i}_{\tilde{r}_i} - \sum_{\substack{\tilde{s}_i \in \mathbb{Z} \\ \tilde{s}_i < 0}} N^{\tilde{\alpha}^{\bar{i}}}_{\tilde{s}_i} \right) + \tilde{q}^i \hat{\phi}^i_R
\end{aligned}
\tag{2.2.41}
$$

[13] To be precise, gauge bundles of this theory have the structure group Spin(32)/$\mathbb{Z}_2$ which only in the special case of "vector structure" can be embedded into SO(32) (see especially [107]). This is due to the center of Spin(32) which is $\mathbb{Z}_2 \times \mathbb{Z}_2$. SO(32) actually is Spin(32)/$\mathbb{Z}_2'$ for a different generator of $\mathbb{Z}_2'$ as used in the Spin(32)/$\mathbb{Z}_2$ heterotic string theory [53, 54].

The Virasoro generators are

$$\begin{aligned} T_0 = L_0 - \frac{c}{24} &= \frac{\alpha'}{2}\frac{1}{2}p_L^2 + \sum_i \left(\sum_{\substack{r_i \in \mathbb{Z} \\ r_i < 0}} -r_i N_{r_i}^{\alpha^i} + \sum_{\substack{s_i \in \mathbb{Z} \\ s_i < 0}} -s_i N_{s_i}^{\alpha^{\bar{i}}} \right) \\ &\quad + \frac{1}{2}q^I q^I + \sum_{I=1}^{16}\sum_{n=1}^{\infty} n N_n^{\gamma^I} - 1 \\ \tilde{T}_0 = \tilde{L}_0 - \frac{\tilde{c}}{24} &= \frac{\alpha'}{2}\frac{1}{2}p_R^2 + \sum_i \left(\sum_{\substack{\tilde{r}_i \in \mathbb{Z} \\ \tilde{r}_i < 0}} -\tilde{r}_i N_{\tilde{r}_i}^{\tilde{\alpha}^{\bar{i}}} + \sum_{\substack{\tilde{s}_i \in \mathbb{Z} \\ \tilde{s}_i < 0}} -\tilde{s}_i N_{\tilde{s}_i}^{\tilde{\alpha}^{\bar{i}}} \right) \\ &\quad + \frac{1}{2}\tilde{q}^2 + \sum_i \sum_{n=1}^{\infty} n N_n^{\tilde{\gamma}^i} - \frac{1}{2} \end{aligned} \tag{2.2.42}$$

Here, by the same reasoning as in the IIA case, modular invariance of heterotic string theory on the torus gives

$$q^I q^I - \tilde{q}^2 = 1 \pmod 2 \tag{2.2.43}$$

The Twisted Sectors of Heterotic $E_8 \times E_8$

A general state can be constructed from the following states and creation operators:

$$\begin{array}{llll} \text{left:} & \left|p-\beta\right\rangle & p \in \Gamma_8 \otimes \Gamma_8 & \\ & \alpha^i_{r_i} & r_i \in \mathbb{Z} + \phi^i_L & r_i \neq 0 \\ & \alpha^{\bar{i}}_{s_i} & s_i \in \mathbb{Z} - \phi^i_L & s_i \neq 0 \\ & \gamma^I_{-n} & n > 0 & \\ \\ \text{right:} & \left|\tilde{p}+\phi_R\right\rangle & \tilde{p} \in \Gamma^+_{\mathrm{SO}(8)} & \\ & \tilde{\alpha}^i_{\tilde{r}_i} & \tilde{r}_i \in \mathbb{Z} - \phi^i_R & \tilde{r}_i \neq 0 \\ & \tilde{\alpha}^{\bar{i}}_{\tilde{s}_i} & \tilde{s}_i \in \mathbb{Z} + \phi^i_R & \tilde{s}_i \neq 0 \\ & \tilde{\gamma}^i_{-n} & n > 0 & \end{array} \tag{2.2.44}$$

and in the fermionic formulation (for $n \in \mathbb{Z}$)

$$\begin{array}{lllll} NS: & \Psi^I_{-n+\nu^I} & \Psi^{\bar{I}}_{-n-\nu^I} & \nu^I = \frac{1}{2} + \beta^I & \pmod 1 \\ \tilde{NS}: & \tilde{\Psi}^i_{-n-\tilde{\nu}^i} & \tilde{\Psi}^{\bar{i}}_{-n+\tilde{\nu}^i} & \tilde{\nu}^i = \frac{1}{2} + \phi^i_R & \pmod 1 \\ R: & \Psi^I_{-n+\nu^I} & \Psi^{\bar{I}}_{-n-\nu^I} & \nu^I = \beta^I & \pmod 1 \\ \tilde{R}: & \tilde{\Psi}^i_{-n-\tilde{\nu}^i} & \tilde{\Psi}^{\bar{i}}_{-n+\tilde{\nu}^i} & \tilde{\nu}^i = \phi^i_R & \pmod 1 \end{array} \tag{2.2.45}$$

where the subscript of Ψ or $\tilde{\Psi}$ is smaller or equal to zero.

The transformation phases for $(\hat{\theta}, \hat{v}, \hat{\beta})$ are given by

$$
\begin{aligned}
s_L^{(\hat{\theta},\hat{v},\hat{\beta})} &= \sum_i \hat{\phi}_L^i \left(\sum_{\substack{r_i \in \mathbb{Z}+\phi_L^i \\ r_i<0}} N_{r_i}^{\alpha^i} - \sum_{\substack{s_i \in \mathbb{Z}-\phi_L^i \\ s_i<0}} N_{s_i}^{\alpha^{\bar{i}}} \right) + (p^I - \beta^I)\hat{\beta}^I \\
s_R^{(\hat{\theta},\hat{v},\hat{\beta})} &= \sum_i \hat{\phi}_R^i \left(\sum_{\substack{\tilde{r}_i \in \mathbb{Z}-\phi_R^i \\ \tilde{r}_i<0}} N_{\tilde{r}_i}^{\tilde{\alpha}^i} - \sum_{\substack{\tilde{s}_i \in \mathbb{Z}+\phi_R^i \\ \tilde{s}_i<0}} N_{\tilde{s}_i}^{\tilde{\alpha}^{\bar{i}}} \right) + (\tilde{p}^i + \phi_R^i)\hat{\phi}_R^i
\end{aligned}
\tag{2.2.46}
$$

and we have

$$
\begin{aligned}
T_0 = L_0 - \frac{c}{24} &= \frac{\alpha'}{2}\frac{1}{2}p_L^2 + \sum_i \left(\sum_{\substack{r_i \in \mathbb{Z}+\phi_L^i \\ r_i<0}} -r_i N_{r_i}^{\alpha^i} + \sum_{\substack{s_i \in \mathbb{Z}-\phi_L^i \\ s_i<0}} -s_i N_{s_i}^{\alpha^{\bar{i}}} \right) \\
&\quad + \frac{1}{2}(p-\beta)^2 + \sum_{I=1}^{16}\sum_{n=1}^{\infty} n N_n^{\gamma^I} + a_0 - 1 \\
\tilde{T}_0 = \tilde{L}_0 - \frac{\tilde{c}}{24} &= \frac{\alpha'}{2}\frac{1}{2}p_R^2 + \sum_i \left(\sum_{\substack{\tilde{r}_i \in \mathbb{Z}-\phi_R^i \\ \tilde{r}_i<0}} -\tilde{r}_i N_{\tilde{r}_i}^{\tilde{\alpha}^{\bar{i}}} + \sum_{\substack{\tilde{s}_i \in \mathbb{Z}+\phi_R^i \\ \tilde{s}_i<0}} -\tilde{s}_i N_{\tilde{s}_i}^{\tilde{\alpha}^{\bar{i}}} \right) \\
&\quad + \frac{1}{2}(\tilde{p}+\phi_R)^2 + \sum_i \sum_{n=1}^{\infty} n N_n^{\tilde{\gamma}^i} + \tilde{a}_0 - \frac{1}{2}
\end{aligned}
\tag{2.2.47}
$$

where a_0 and $\tilde{a}_0$ are unchanged from type IIA theory (2.2.34).

2.3 Modular Invariance and the Level Matching Condition

As we have discussed in section 2.1, phases occurring for those modular transformations which map sectors to themselves can spoil modular invariance.

Given a transformation $G = (\theta, v) \in S$ with $v \in N$ we have $D^m = \mathbb{1}$ for some integer m. Therefore, from (2.1.10), $\mathcal{T}^m$ transforms boundary conditions into itself: $\mathcal{T}^m(G, H) = (G, HG^m) = (G, H)$. This implies that the partition function of any twisted sector must be invariant under $\hat{\tau} \mapsto \hat{\tau} + m$.

Since the general partition function for the Hilbert space H_G^0 is given by

$$
Z_G^0(\hat{\tau}) = \mathrm{Tr}_{H_G^0}\, q^{L_0 - c/24}\bar{q}^{\tilde{L}_0 - \tilde{c}/24} \qquad q = e^{2\pi i \hat{\tau}} \tag{2.3.1}
$$

(see section 7.2 of [76]), invariance under $\hat{\tau} \mapsto \hat{\tau} + m$ gives the weak form of the level matching condition

$$T_0 - \tilde{T}_0 = 0 \pmod{1/m} \tag{2.3.2}$$

However, as it is argued in [34], since $T_0 - \tilde{T}_0$ generates the worldsheet translations in the σ direction and all states have to be invariant under $\sigma \mapsto \sigma + 2\pi$, we get the level matching condition

$$T_0 - \tilde{T}_0 = 0 \pmod 1 \tag{2.3.3}$$

which includes invariance of the partition function under $\hat{\tau} \mapsto \hat{\tau} + m$. As was shown by an explicit calculation for all modular transformations in [100], invariance of the partition function under $\hat{\tau} \mapsto \hat{\tau} + m$ guarantees modular invariance for symmetric orbifolds, therefore (2.3.3) is the only nontrivial constraint on such orbifolds, to be satisfied for every twisted sector and every fixed point separately.

Type IIA Orbifolds

Writing down the level matching condition gives from (2.2.30) and (2.2.31)

$$\begin{aligned} T_0 - \tilde{T}_0 \pmod 1 &= \\ &= + \sum_i \left(\sum_{\substack{r_i \in \mathbb{Z} + \phi_L^i \\ r_i < 0}} -r_i N_{r_i}^{\alpha^i} + \sum_{\substack{s_i \in \mathbb{Z} - \phi_L^i \\ s_i < 0}} -s_i N_{s_i}^{\alpha^{\bar{i}}} \right) \\ &\quad - \sum_i \left(\sum_{\substack{\tilde{r}_i \in \mathbb{Z} - \phi_R^i \\ \tilde{r}_i < 0}} -\tilde{r}_i N_{\tilde{r}_i}^{\tilde{\alpha}^{\bar{i}}} + \sum_{\substack{\tilde{s}_i \in \mathbb{Z} + \phi_R^i \\ \tilde{s}_i < 0}} -\tilde{s}_i N_{\tilde{s}_i}^{\tilde{\alpha}^{\bar{i}}} \right) \\ &\quad + \frac{1}{2}(p - \phi_L)^2 - \frac{1}{2}(\tilde{p} + \phi_R)^2 + a_0 - \tilde{a}_0 \\ &= -s_L^{(\theta,v)} - s_R^{(\theta,v)} \\ &\quad + \frac{1}{2}(p^2 - \phi_L^2) - \frac{1}{2}(\tilde{p}^2 - \phi_R^2) + a_0 - \tilde{a}_0 \end{aligned} \tag{2.3.4}$$

Since (θ, v) commutes with itself, the state must be invariant under (θ, v) and we impose $0 = s_L^{(\theta,v)} + s_R^{(\theta,v)} \pmod 1$ giving

$$0 = \frac{1}{2}(p^2 - \phi_L^2) - \frac{1}{2}(\tilde{p}^2 - \phi_R^2) + a_0 - \tilde{a}_0 \pmod 1 \tag{2.3.5}$$

But because we are constructing symmetric orbifolds, we have $\phi_L = \phi_R$ and $a_0 = \tilde{a}_0$ and therefore the only remaining constraint is

$$p^2 - \tilde{p}^2 = 0 \pmod 2 \tag{2.3.6}$$

But since the $(p, \tilde{p})$ lattice is identical to that of the untwisted sector, this is nothing but (2.2.25) and the orbifold is modular invariant in any case.

*Heteroti*c $E_8 \times E_8$ *Orbifolds*

By the same calculation as in the type IIA case we get from (2.2.46) and (2.2.47)

$$\begin{aligned} T_0 - \tilde{T}_0 \pmod 1 &= \\ &= -s_L^{(\theta,v,\beta)} - s_R^{(\theta,v,\beta)} \\ &\quad + \frac{1}{2}(p^2 - \beta^2) - \frac{1}{2}(\tilde{p}^2 - \phi_R^2) + a_0 - \tilde{a}_0 - \frac{1}{2} \end{aligned} \tag{2.3.7}$$

As in type IIA, we impose $0 = s_L^{(\theta,v,\beta)} + s_R^{(\theta,v,\beta)} \pmod 1$ and use $\phi_L = \phi_R$ and $a_0 = \tilde{a}_0$ giving

$$\begin{aligned} T_0 - \tilde{T}_0 &= \frac{1}{2}(p^2 - \tilde{p}^2) + \frac{1}{2} + \frac{1}{2}(\phi_R^2 - \beta^2) \pmod 1 \\ &\stackrel{!}{=} 0 \pmod 1 \end{aligned} \tag{2.3.8}$$

Again, we use (2.2.43) from the untwisted sector and get

$$0 = \phi^2 - \beta^2 \pmod 2 \tag{2.3.9}$$

the (strong) level matching condition for heterotic symmetric orbifolds. However, this condition is clearly not invariant under addition of lattice vectors to β, since, for $\beta = s/N$, $s \in \Gamma_8 \otimes \Gamma_8$ and an arbitrary lattice vector $w \in \Gamma_8 \otimes \Gamma_8$

$$(\beta + w)^2 = \beta^2 + w^2 + 2\beta w = \beta^2 + w^2 + \frac{2ws}{N} \tag{2.3.10}$$

Since $\Gamma_8 \otimes \Gamma_8$ is an even self-dual lattice, we can find some lattice vector w with $ws = 1$. Therefore, as a gauge transformation, β is specified by the weak form of the level matching condition

$$0 = \phi^2 - \beta^2 \pmod{2/N} \tag{2.3.11}$$

Given a lattice vector with $ws = 1$, the vector $\beta_s = \beta + nw$ will obey the strong level matching condition if (by (2.3.10))

$$\frac{2n}{N} = \phi^2 - \beta^2 \pmod 2 \tag{2.3.12}$$

is satisfied.

In conclusion, a heterotic orbifold is specified by the data of all (θ, v, β) where, at each fixed point corresponding to (θ, v, β), the weak form of the level matching condition (2.3.11) has to be satisfied. However, the sector of Hilbert space at that fixed point is only well defined when we add some lattice vector $w \in \Gamma_8 \otimes \Gamma_8$ to β such that $\beta_s = \beta + w$ satisfies the strong version of the level matching condition (2.3.8). Since the only place where the addition of w to β matters in our formulas is the transformation phase defined by $s^{(\hat{\theta},\hat{v},\hat{\beta})}$, we learn that w is responsible to render the transformation properties of twisted states consistent.

2.4 Classifying Shift Vectors

As we have seen in the last section, heterotic orbifolds are specified by the data (θ, v, β), where at every fixed point the (weak) level matching condition $0 = \phi^2 - \beta^2 \pmod{2/m}$ has to be satisfied. This makes clear that we will have to show how to classify all possible β satisfying the relation.

The classification of shift vectors for a single E_8 is given in section B.2 of the appendix. Since the gauge group actually is $E_8 \times E_8$, the only symmetry for which we have to care is the exchange of the two group factors.

In six-dimensional orbifolds, the only generator of $\mathbb{Z}_N$ consistent with supersymmetry (2.2.18) is given by

$$\phi^i = \left(\frac{1}{N}, -\frac{1}{N}\right) \tag{2.4.1}$$

Therefore, in the twisted sector θ^k we have

$$(k\phi)^2 = k^2\phi^2 = \frac{2k^2}{N^2} \tag{2.4.2}$$

The general (weak) level matching condition is

$$\frac{2k^2}{N^2} = \frac{s^I s^I}{N^2} \pmod{2/m} \tag{2.4.3}$$

where $m > 1$ is the smallest integer such that $\theta^m = \mathbb{1}$. Therefore we get

$$2k^2 = s_1^I s_1^I + s_2^I s_2^I \pmod{2N^2/m} \tag{2.4.4}$$

Since for the generator of $\mathbb{Z}_N$ m is obviously equal to N, we have

$$2k^2 = s_1^I s_1^I + s_2^I s_2^I \pmod{2N} \tag{2.4.5}$$

To solve this equation, we use the classification of possible E_8 gauge shifts as carried out at the end of appendix B.1. Restricting to orbifolds "without Wilson

$s_1^2 + s_2^2$	$(s_1^I; s_2^I)$	unbroken gauge group
		$\mathbb{Z}_2$
$2 + 0$	$(1, 1, 0^6; 0^8)$	$\mathrm{E}_7 \times \mathrm{SU}(2) \times \mathrm{E}_8$
		$\mathbb{Z}_3$
$4 + 4$	$(2, 0^7; 2, 0^7)$	$\mathrm{SO}(14) \times \mathrm{U}(1) \times \mathrm{SO}(14) \times \mathrm{U}(1)$
$2 + 0$	$(1^2, 0^6; 0^8)$	$\mathrm{E}_7 \times \mathrm{U}(1) \times \mathrm{E}_8$
$2 + 6$	$(1^2, 0^6; 2, 1^2, 0^5)$	$\mathrm{E}_7 \times \mathrm{U}(1) \times \mathrm{E}_6 \times \mathrm{SU}(3)$
$8 + 6$	$(5/2, (1/2)^7; 2, 1^2, 0^5)$	$\mathrm{SU}(9) \times \mathrm{E}_6 \times \mathrm{SU}(3)$

Table 2.1: Six-dimensional $\mathbb{Z}_2$ and $\mathbb{Z}_3$ orbifolds of $\mathrm{E}_8 \times \mathrm{E}_8$ heterotic string theory "without Wilson lines".

lines"[14], that is, without gauge transformations assigned to pure translations, from table B.1, we get $\mathbb{Z}_2$ and $\mathbb{Z}_3$ orbifolds as shown in table 2.1.

2.5 $D = 6$ Orbifolds and Fractional Instanton Numbers

Here we give arguments as already published in [27] and [28].

The level matching condition (2.3.11)

$$\Phi^2 = \beta_1^2 + \beta_2^2 \pmod{2/N} \tag{2.5.1}$$

looks very similar to the anomalous Bianchi identity required for the Green-Schwarz mechanism [50] of heterotic string theory (compare to (3.2.3), see also chapter 12 of [76])

$$\mathrm{d}H = \frac{\alpha'}{4}\left(\operatorname{tr} R^2 - \frac{1}{30}\operatorname{Tr} F_1^2 - \frac{1}{30}\operatorname{Tr} F_2^2\right) \tag{2.5.2}$$

Indeed, since away from the fixed points heterotic string theory on an orbifold should look locally like heterotic string theory on a torus and we did not switch on any B-field backgrounds, equation (2.5.2) should reduce to

$$\operatorname{tr} R^2 = \frac{1}{30}\operatorname{Tr} F_1^2 + \frac{1}{30}\operatorname{Tr} F_2^2 \tag{2.5.3}$$

which even more looks like the level matching condition. In case of the heterotic $\mathrm{Spin}(32)/\mathbb{Z}_2$ string theory both conditions were shown to be equivalent in [43]

[14] As shown at the end of section 3.3, this common terminus is quite confusing, since the gauge shifts associated to fixed points are not related to Wilson lines.

and analyzed in [17, 61] (see also [2]). In this case (2.5.3) is reduced to

$$-\frac{1}{2}\frac{1}{8\pi^2}\int \mathrm{tr}\, R^2 = -\frac{1}{60}\frac{1}{8\pi^2}\int \mathrm{Tr}\, F^2 \pmod 1 \tag{2.5.4}$$

and it is only natural to expect a similar condition for the $E_8 \times E_8$ theory:

$$-\frac{1}{2}\frac{1}{8\pi^2}\int \mathrm{tr}\, R^2 = -\frac{1}{60}\frac{1}{8\pi^2}\int \mathrm{Tr}\, F_1^2 - \frac{1}{60}\frac{1}{8\pi^2}\int \mathrm{Tr}\, F_2^2 \pmod 1 \tag{2.5.5}$$

Level Matching and Fractional Instantons

Indeed, as is shown in appendix A, for an E_8 bundle of a shrunken instanton on a $\mathbb{Z}_N$ orbifold singularity corresponding to (θ, v, β), the fractional part of the instanton number is given by

$$I = -\frac{1}{60}\frac{1}{8\pi^2}\int_U \mathrm{Tr}\, F^2 = \frac{n}{2}\beta^2 \pmod 1 \tag{2.5.6}$$

where U is a small neighborhood surrounding the shrunken instanton. Using this result the heterotic level matching conditions translates into the requirement that locally the sum of the fractional parts of the gauge instanton numbers match the fractional part of the gravitational instanton number (which is $+1/N$ for anti-self-dual curvature, see the computation of (3.1.23)):

$$-\frac{1}{2}\frac{1}{8\pi^2}\int_U \mathrm{tr}\, R^2 = -\frac{1}{60}\frac{1}{8\pi^2}\int_U \mathrm{Tr}\, F_1^2 - \frac{1}{60}\frac{1}{8\pi^2}\int_U \mathrm{Tr}\, F_2^2 \pmod 1 \tag{2.5.7}$$

Even though this might look trivial from a weak coupling perspective, in light of M-theory on $S^1/\mathbb{Z}_2$ (2.5.6) fixes the fractional instanton numbers on each $\mathbb{Z}_2$ fixed point separately. Therefore, the distribution of the integer part on the two $\mathbb{Z}_2$ fixed points is not directly given by (2.5.6) and has to be investigated by different methods [95, 40, 62, 41].

Fractional Instanton Numbers and Orbifold Classification

As discussed in section 2.2 heterotic orbifolds are specified by the space group S together with a group homomorphism of S into the group of gauge transformations. We will show in the following, that this map specifies a flat gauge bundle on the orbifold where the fixed points have been cut out.

Since in six dimensions supersymmetry requires the twist θ to be of the form $\phi^i = (m/N, -m/N)$ (see (2.4.1)), the twist has no fixed points except for the origin. Therefore, we have $N = \Gamma$ and the fixed points are isolated points (their number is given by (2.1.3)). Hence, the fundamental group π_1 of $\mathbb{R}^4 - F$ is zero, where F denotes the set of fixed points. This implies that $\mathbb{R}^4 - F$ is the universal covering space of $(\mathbb{R}^4 - F)/S = O - F$, that is, the orbifold with the fixed points taken out. This further implies (see, for example [22], Chapter III) that π_1 of $O - F$ is isomorphic to S and the map from S to the group of gauge transformations provides a homomorphism of $\pi_1(O - F)$ to the gauge group. This is nothing but the data of a flat gauge bundle[15] on $O - F$ (up to gauge transformations).

In conclusion, we have shown that the orbifolds considered in this work correspond to all possible flat abelian $E_8 \times E_8$ bundles on the orbifold $\mathbb{R}^4/S$ with the fixed points taken out under the only restriction that the sum of the fractional parts of the gauge instanton numbers (computed from the flat bundle data via (2.5.1)) match the fractional part of the gravitational one locally for every fixed point. Since the fractional instanton numbers are computed separately for every E_8, this classification fully applies to M-theory on $S^1/\mathbb{Z}_2$.

2.6 $D = 6$ Orbifold Examples

Preliminaries in Six Dimensions

We begin by discussing some generalities of six-dimensional orbifolds. As the generator of $\mathbb{Z}_N$ is always given by $\phi^i = (1/N, -1/N)$ (see (2.2.18)), its action on spinors in the R sectors is given by the following table

$p^{34} =$ $(\pm\frac{1}{2},$	$\pm\frac{1}{2})$	$p^i\phi^i$	$e^{2\pi i \phi^i p^i}$
$+$	$+$	$\frac{1}{2N}(0)$	α^0
$+$	$-$	$\frac{1}{2N}(2)$	α^1
$-$	$+$	$\frac{1}{2N}(-2)$	$\alpha^{-1} = \alpha^{N-1}$
$-$	$-$	$\frac{1}{2N}(0)$	α^0

(2.6.1)

where the transformation phases are given in powers of

$$\alpha = e^{\frac{2\pi i}{N}} \tag{2.6.2}$$

Spinors of $\mathrm{Spin}(4) = \mathrm{SU}(2)_A \times \mathrm{SU}(2)_B$ will be defined as

$$\begin{array}{llll} (\mathbf{2},\mathbf{1}) & : & (+\frac{1}{2},+\frac{1}{2}) \;\; (-\frac{1}{2},-\frac{1}{2}) & \text{positive chirality} \\ (\mathbf{1},\mathbf{2}) & : & (+\frac{1}{2},-\frac{1}{2}) \;\; (-\frac{1}{2},+\frac{1}{2}) & \text{negative chirality} \end{array} \tag{2.6.3}$$

[15] Since on a flat bundle parallel translation around closed paths (with a fixed starting point) is invariant under continuous deformations of the path, every class in π_1 corresponds to precisely one group element. Since concatenation of two paths corresponds to group multiplication this is a homomorphism from π_1 to the gauge group.

For spinors of SO(8) we then have

$$\begin{array}{rcl} \mathrm{SO}(8) & \to & \mathrm{SU}(2)_1 \times \mathrm{SU}(2)_2 \times \mathrm{SU}(2)_R \times \mathrm{SU}(2)_H \\ \mathbf{8}_+ & \to & (\mathbf{2},\mathbf{1};\mathbf{2},\mathbf{1}) + (\mathbf{1},\mathbf{2};\mathbf{1},\mathbf{2}) \\ \mathbf{8}- & \to & (\mathbf{1},\mathbf{2};\mathbf{2},\mathbf{1}) + (\mathbf{2},\mathbf{1};\mathbf{1},\mathbf{2}) \end{array} \tag{2.6.4}$$

where $\mathrm{SU}(2)_{1,2}$ corresponds to (p^1, p^2) as in (2.6.3) and $\mathrm{SU}(2)_{R,H}$ corresponds to (p^3, p^4). We choose the indices R and H for the latter, since $\mathrm{SU}(R)$ will become the R-symmetry group in $D = 6$ and $\mathrm{SU}(2)_H$ will harbour the holonomy of the orbifold, i.e. the orbifold twist. We let $\mathrm{U}(1)_R \subset \mathrm{SU}(2)_R$ and $\mathrm{U}(1)_H \subset \mathrm{SU}(2)_H$ act like

$$\begin{aligned} (e^{2\pi i \varrho_R})(Z^3, Z^4) &= (e^{2\pi i \varrho_R} Z^3, e^{2\pi i \varrho_R} Z^4) \\ (e^{2\pi i \varrho_H})(Z^3, Z^4) &= (e^{2\pi i \varrho_H} Z^3, e^{-2\pi i \varrho_H} Z^4) \end{aligned} \tag{2.6.5}$$

and

$$\begin{aligned} (e^{2\pi i \varrho_R})(q^3, q^4) &= (q^3 + \varrho_R, q^4 + \varrho_R) \\ (e^{2\pi i \varrho_H})(q^3, q^4) &= (q^3 + \varrho_H, q^4 - \varrho_R) \end{aligned} \tag{2.6.6}$$

Since under $\mathrm{Spin}(4) = \mathrm{SU}(2)_R \times \mathrm{SU}(2)_H$ a vector X^m transforms as $\mathbf{2} \times \mathbf{2}$, we have the following quantum numbers from (2.6.5)

	$\mathrm{U}(1)_R$	$\mathrm{U}(1)_H$
Z^3	$+1$	$+1$
Z^4	$+1$	-1
$Z^{\bar{3}}$	-1	-1
$Z^{\bar{4}}$	-1	$+1$

(2.6.7)

Of course, all this also applies to $\mathrm{U}(1)_1 \subset \mathrm{SU}(2)_1$ and $\mathrm{U}(1)_2 \subset \mathrm{SU}(2)_2$.

By our choice of conventions, spinors in the $(\mathbf{1}, \mathbf{2})$ will not be invariant under the orbifold twist θ (from (2.6.1)). Hence, the holonomy of the orbifold will live in $\mathrm{SU}(2)_H$ and the remaining $\mathrm{SU}(2)_R$ will become the R-symmetry group of the $D = 6$ supersymmetry (see below). Therefore, we write down states with quantum numbers of the little group $\mathrm{SU}(2)_1 \times \mathrm{SU}(2)_2 \times \mathrm{SU}(2)_R$

$$\begin{array}{rcl} \mathrm{SO}(8) & \to & \mathrm{SU}(2)_1 \times \mathrm{SU}(2)_2 \times \mathrm{SU}(2)_R \\ \mathbf{8}_+ & \to & (\mathbf{2},\mathbf{1};\mathbf{2})_0 + (\mathbf{1},\mathbf{2};\mathbf{1})_{+1} + (\mathbf{1},\mathbf{2};\mathbf{1})_{-1} \\ \mathbf{8}- & \to & (\mathbf{1},\mathbf{2};\mathbf{2})_0 + (\mathbf{2},\mathbf{1};\mathbf{1})_{+1} + (\mathbf{2},\mathbf{1};\mathbf{1})_{-1} \end{array} \tag{2.6.8}$$

where a subscript m shows that the state transforms as α^m under the orbifold twist (from (2.6.1)).

Since the gravitinos of type IIA are in the $\mathbf{8}_\pm$ and those of the heterotic theory in the $\mathbf{8}_+$ of SO(8), we see that the only surviving gravitinos of the six-dimensional theory are

$$\begin{array}{ll} (\mathbf{2},\mathbf{1};\mathbf{2})_0 & \\ (\mathbf{1},\mathbf{2};\mathbf{2})_0 & \text{IIA only} \end{array} \tag{2.6.9}$$

Hence, we have $\mathcal{N} = (1,1), D = 6$ SUSY in case of IIA and $\mathcal{N} = (0,1), D = 6$ SUSY in case of the heterotic string[16]. We note that both surviving spinors transform under their own R-symmetries. Therefore, in case of IIA theory, the manifest $\mathrm{SU}(2)_R$ symmetry in (2.6.9) is the diagonal $(\mathrm{SU}(2)_{R1} \times \mathrm{SU}(2)_{R2})_{\mathrm{diag}}$ of both R-symmetries[17].

In these quantum numbers, the supermultiplets of $\mathcal{N} = (1,1), D = 6$ supergravity can be read off from the tables in [96]

$$\begin{array}{rcl} \text{SUGRA} & = & (\mathbf{3},\mathbf{3};\mathbf{1}) + (\mathbf{3},\mathbf{1};\mathbf{1}) + (\mathbf{1},\mathbf{3};\mathbf{1}) \\ & & +(\mathbf{1},\mathbf{1};\mathbf{1}) + (\mathbf{2},\mathbf{2};\mathbf{3}+\mathbf{1}) \\ & & +(\mathbf{3},\mathbf{2};\mathbf{2}) + (\mathbf{2},\mathbf{3};\mathbf{2}) + (\mathbf{1},\mathbf{2};\mathbf{2}) + (\mathbf{2},\mathbf{1};\mathbf{2}) \\ \text{vector} & = & (\mathbf{2},\mathbf{2};\mathbf{1}) + (\mathbf{1},\mathbf{1};\mathbf{3}+\mathbf{1}) \\ & & +(\mathbf{2},\mathbf{1};\mathbf{2}) + (\mathbf{1},\mathbf{2};\mathbf{2}) \end{array} \tag{2.6.10}$$

Those of $\mathcal{N} = (0,1), D = 6$ supergravity are given as

$$\begin{array}{rcl} \text{SUGRA} & = & (\mathbf{3},\mathbf{3};\mathbf{1}) + (\mathbf{3},\mathbf{1};\mathbf{1}) + (\mathbf{3},\mathbf{2};\mathbf{2}) \\ \text{tensor} & = & (\mathbf{1},\mathbf{3};\mathbf{1}) + (\mathbf{1},\mathbf{1};\mathbf{1}) + (\mathbf{1},\mathbf{2};\mathbf{2}) \\ \text{vector} & = & (\mathbf{2},\mathbf{2};\mathbf{1}) + (\mathbf{2},\mathbf{1};\mathbf{2}) \\ \text{half-hyper} & = & (\mathbf{1},\mathbf{1};\mathbf{2}) + (\mathbf{1},\mathbf{2};\mathbf{1}) \\ \text{hyper} & = & 2(\mathbf{1},\mathbf{1};\mathbf{2}) + 2(\mathbf{1},\mathbf{2};\mathbf{1}) \end{array} \tag{2.6.11}$$

where a single half-hypermultiplet is only possible if its bosons are in a real representation where the $\mathrm{SU}(2)_R$ representations have to be taken into account. This is due to CPT symmetry, which guarantees the existence of the CPT-conjugate half-hypermultiplet of opposite charge quantum numbers.

General Abelian Supersymmetric Orbifolds in Six Dimensions

All $\mathbb{Z}_N$ orbifolds of T^4 have been classified in [102]. The only possibilities are $N = 2, 3, 4, 6$. By graphical, methods one can easily derive the following number

[16]This implies a gravitino of positive chirality, which is the common choice in literature on M-theory on $S^1/\mathbb{Z}_2$.

[17]We thank W. Nahm and K. Wendlandt for discussion on that point.

of separate fixed points

	Fixed Points			
N	$\mathbb{Z}_2$	$\mathbb{Z}_3$	$\mathbb{Z}_4$	$\mathbb{Z}_6$
2	16			
3		9		
2	6		4	
2		4		1

(2.6.12)

Care has to be taken not to overcount fixed points [39]. For instance, in the $\mathbb{Z}_4$ model, the $\mathbb{Z}_4$-twist identifies some of the sixteen $\mathbb{Z}_2$ fixed points with each other leaving only six separate $\mathbb{Z}_2$ fixed points.

The IIA Untwisted Sector

We start with the $NS, \tilde{NS}$ sectors, which have vacuum energy $-\frac{1}{2}$ (see (2.2.24)). Since the ground states of the NS sectors have $(-1)^F = -1$, we have to excite with an odd number of fermionic creation operators. The lowest available operators are (see (2.2.20) and (2.2.21))

$$\begin{aligned} \alpha^0 : &\quad \Psi^\mu_{-1/2}, \tilde{\Psi}^\mu_{-1/2}, \alpha^\mu_{-1}, \tilde{\alpha}^\mu_{-1} \\ \alpha^1 : &\quad (\Psi^3_{-1/2}, \Psi^{\bar{4}}_{-1/2}), (\tilde{\Psi}^3_{-1/2}, \tilde{\Psi}^{\bar{4}}_{-1/2}), (\alpha^3_{-1}, \alpha^{\bar{4}}_{-1}), (\tilde{\alpha}^3_{-1}, \tilde{\alpha}^{\bar{4}}_{-1}) \\ \alpha^{-1} : &\quad (\Psi^{\bar{3}}_{-1/2}, \Psi^4_{-1/2}), (\tilde{\Psi}^{\bar{3}}_{-1/2}, \tilde{\Psi}^4_{-1/2}), (\alpha^{\bar{3}}_{-1}, \alpha^4_{-1}), (\tilde{\alpha}^{\bar{3}}_{-1}, \tilde{\alpha}^4_{-1}) \end{aligned} \tag{2.6.13}$$

where the bosonic operators already produce massive states and we have grouped together operators that transform as a **2** of $\mathrm{SU}(2)_R$ (see (2.6.7)). This gives the following invariant states

$$\begin{aligned} \alpha^0\tilde{\alpha}^0 : &\quad \Psi^\mu_{-1/2}\tilde{\Psi}^\nu_{-1/2}\big|0\big\rangle_{NS,\tilde{NS}} \\ \alpha^1\tilde{\alpha}^{-1} : &\quad \big((\Psi^3_{-1/2}, \Psi^{\bar{4}}_{-1/2}), (\tilde{\Psi}^{\bar{3}}_{-1/2}, \tilde{\Psi}^4_{-1/2})\big)\big|0\big\rangle_{NS,\tilde{NS}} \\ \alpha^{-1}\tilde{\alpha}^1 : &\quad \big((\Psi^{\bar{3}}_{-1/2}, \Psi^4_{-1/2}), (\tilde{\Psi}^3_{-1/2}, \tilde{\Psi}^{\bar{4}}_{-1/2})\big)\big|0\big\rangle_{NS,\tilde{NS}} \end{aligned} \tag{2.6.14}$$

and, in the special case $N = 2$

$$\begin{aligned} \alpha^1\tilde{\alpha}^1 : &\quad \big((\Psi^3_{-1/2}, \Psi^{\bar{4}}_{-1/2}), (\tilde{\Psi}^3_{-1/2}, \tilde{\Psi}^{\bar{4}}_{-1/2})\big)\big|0\big\rangle_{NS,\tilde{NS}} \\ \alpha^{-1}\tilde{\alpha}^{-1} : &\quad \big((\Psi^{\bar{3}}_{-1/2}, \Psi^4_{-1/2}), (\tilde{\Psi}^{\bar{3}}_{-1/2}, \tilde{\Psi}^4_{-1/2})\big)\big|0\big\rangle_{NS,\tilde{NS}} \end{aligned} \tag{2.6.15}$$

which in their little group quantum numbers of $\mathrm{SU}(2)_1 \times \mathrm{SU}(2)_2 \times \mathrm{SU}(2)_R$ are

$$\begin{array}{rl} \alpha^0\tilde{\alpha}^0: & (\mathbf{2}\times\mathbf{2},\mathbf{2}\times\mathbf{2};\mathbf{1}) \\ & =(\mathbf{3},\mathbf{3};\mathbf{1})+(\mathbf{3},\mathbf{1};\mathbf{1})+(\mathbf{1},\mathbf{3};\mathbf{1})+(\mathbf{1},\mathbf{1};\mathbf{1}) \\ \alpha^1\tilde{\alpha}^{-1}: & (\mathbf{1},\mathbf{1};\mathbf{2}\times\mathbf{2})=(\mathbf{1},\mathbf{1};\mathbf{3}+\mathbf{1}) \\ \alpha^{-1}\tilde{\alpha}^1: & (\mathbf{1},\mathbf{1};\mathbf{2}\times\mathbf{2})=(\mathbf{1},\mathbf{1};\mathbf{3}+\mathbf{1}) \\ (\alpha^1\tilde{\alpha}^1)_{N=2}: & (\mathbf{1},\mathbf{1};\mathbf{2}\times\mathbf{2})=(\mathbf{1},\mathbf{1};\mathbf{3}+\mathbf{1}) \\ (\alpha^{-1}\tilde{\alpha}^{-1})_{N=2}: & (\mathbf{1},\mathbf{1};\mathbf{2}\times\mathbf{2})=(\mathbf{1},\mathbf{1};\mathbf{3}+\mathbf{1}) \end{array} \tag{2.6.16}$$

In the R sector and the $\tilde{R}$ sector we have from (2.6.8) the degenerate states

$$\begin{array}{rl} R: & (\mathbf{1},\mathbf{2};\mathbf{2})_0,(\mathbf{2},\mathbf{1};\mathbf{1})_{+1},(\mathbf{2},\mathbf{1};\mathbf{1})_{-1} \\ \tilde{R}: & (\mathbf{2},\mathbf{1};\mathbf{2})_0,(\mathbf{1},\mathbf{2};\mathbf{1})_{+1},(\mathbf{1},\mathbf{2};\mathbf{1})_{-1} \end{array} \tag{2.6.17}$$

which can be combined to the following invariant states

$$\begin{array}{rll} \alpha^0\tilde{\alpha}^0: & (\mathbf{1},\mathbf{2};\mathbf{2})_0\otimes(\mathbf{2},\mathbf{1};\mathbf{2})_0 & (\mathbf{2},\mathbf{2};\mathbf{3}+\mathbf{1}) \\ \alpha^1\tilde{\alpha}^{-1}: & (\mathbf{2},\mathbf{1};\mathbf{1})_{-1}\otimes(\mathbf{1},\mathbf{2};\mathbf{1})_{+1} & (\mathbf{2},\mathbf{2};\mathbf{1}) \\ \alpha^{-1}\tilde{\alpha}^1: & (\mathbf{2},\mathbf{1};\mathbf{1})_{+1}\otimes(\mathbf{1},\mathbf{2};\mathbf{1})_{-1} & (\mathbf{2},\mathbf{2};\mathbf{1}) \\ (\alpha^1\tilde{\alpha}^1)_{N=2}: & (\mathbf{2},\mathbf{1};\mathbf{1})_{+1}\otimes(\mathbf{1},\mathbf{2};\mathbf{1})_{+1} & (\mathbf{2},\mathbf{2};\mathbf{1}) \\ (\alpha^{-1}\tilde{\alpha}^{-1})_{N=2}: & (\mathbf{2},\mathbf{1};\mathbf{1})_{-1}\otimes(\mathbf{1},\mathbf{2};\mathbf{1})_{-1} & (\mathbf{2},\mathbf{2};\mathbf{1}) \end{array} \tag{2.6.18}$$

where the right column shows quantum numbers of the little group which identify the states as vectors.

Of course, there are also the states of the NS, $\tilde{R}$ and R, $\tilde{NS}$ sectors, which provide the (space time) fermionic superpartners of the above states. Altogether, from (2.6.10), these states combine into the following multiplets

$$\begin{array}{rl} \alpha^0\tilde{\alpha}^0: & 1\ \text{SUGRA} \\ \alpha^1\tilde{\alpha}^{-1}+\alpha^{-1}\tilde{\alpha}^1: & 2\ \text{vectors} \\ (\alpha^1\tilde{\alpha}^1+\alpha^{-1}\tilde{\alpha}^{-1})_{N=2}: & 2\ \text{vectors} \end{array} \tag{2.6.19}$$

The IIA Twisted Sectors

For the twisted sector θ^k, we have $\phi^i = (k/N, -k/N)$. This implies that, if we compute the $\theta^{N-k} = \theta^{-k}$ twisted sector by using $\phi^i = (-k/N, k/N)$, we get the same results as in the kth twisted sector, up to exchanging the $i = 3$ and $i = 4$ coordinate, which corresponds to conjugating all representations. Therefore we will only treat the case $k \leq N/2$. If $k = N/2$, special things happen and we will mention those cases explicitly.

To compute the worldsheet vacuum energies (2.2.34), we have to normalize the twist vectors ϕ to

$$\phi_0 = \left(\frac{k}{N}, \frac{N-k}{N}\right) \tag{2.6.20}$$

to compute

$$a_0 = \tilde{a}_0 = \frac{1}{2}\sum_i \phi_0^i(1-\phi_0^i) = \frac{k}{N} - \frac{k^2}{N^2} \tag{2.6.21}$$

This gives for the Virasoro generators (2.2.31)

$$\begin{aligned} T_0 &= \ldots + \frac{1}{2}(p-\phi)^2 + \frac{k}{N} - \frac{k^2}{N^2} - \frac{1}{2} \\ \tilde{T}_0 &= \ldots + \frac{1}{2}(\tilde{p}+\phi)^2 + \frac{k}{N} - \frac{k^2}{N^2} - \frac{1}{2} \end{aligned} \tag{2.6.22}$$

From this we find the vacuum states by plugging in the vacuum states (2.2.14) and then reducing the energies by acting with Ψ and $\tilde{\Psi}$ operators to reach $-1/2 \leq q^i \leq +1/2$ and $-1/2 \leq \tilde{q}^i \leq +1/2$

$$\begin{array}{lllll} NS: & \left|0,\ 0, -\frac{k}{N}, +\frac{k}{N}\right\rangle & F=1 & \frac{\alpha'}{4}m^2 = \frac{k}{N} - \frac{1}{2} \\ \tilde{NS}: & \left|0,\ 0, +\frac{k}{N}, -\frac{k}{N}\right\rangle & \tilde{F}=1 & \frac{\alpha'}{4}m^2 = \frac{k}{N} - \frac{1}{2} \\ R: & \left|\frac{1}{2},\ \frac{1}{2},\ \frac{1}{2}-\frac{k}{N},\ \frac{1}{2}+\frac{k}{N}\right\rangle & F=0 & \\ & \left|\frac{1}{2},\ \frac{1}{2},\ \frac{1}{2}-\frac{k}{N}, -\frac{1}{2}+\frac{k}{N}\right\rangle & F=1 & \\ & \left|\frac{1}{2}, -\frac{1}{2},\ \frac{1}{2}-\frac{k}{N}, -\frac{1}{2}+\frac{k}{N}\right\rangle & F=0 & \frac{\alpha'}{4}m^2 = 0 \\ \tilde{R}: & \left|\frac{1}{2},\ \frac{1}{2},\ \frac{1}{2}+\frac{k}{N},\ \frac{1}{2}-\frac{k}{N}\right\rangle & F=0 & \\ & \left|\frac{1}{2},\ \frac{1}{2}, -\frac{1}{2}+\frac{k}{N},\ \frac{1}{2}-\frac{k}{N}\right\rangle & F=1 & \\ & \left|\frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}+\frac{k}{N},\ \frac{1}{2}-\frac{k}{N}\right\rangle & F=0 & \frac{\alpha'}{4}m^2 = 0 \end{array} \tag{2.6.23}$$

where the last state in each sector is the new vacuum and $m^2 = -p_{LR}^2$ is calculated as if the new vacuum state were a physical state with $T_0 = \tilde{T}_0 = 0$. The ν and $\tilde{\nu}$ of (2.2.29) are

$$\begin{array}{llll} NS: & \nu^i = \tilde{\nu}^i & = \frac{1}{2} + \phi^i & = (\frac{1}{2}, \frac{1}{2}, \frac{1}{2}+\frac{k}{N}, \frac{1}{2}-\frac{k}{N}) \pmod 1 \\ R: & \nu^i = \tilde{\nu}^i & = \phi^i & = (0, 0, \frac{k}{N}, -\frac{k}{N}) \pmod 1 \end{array} \tag{2.6.24}$$

and we have the following bosonic operators (from (2.1.29))

$$\begin{array}{llllll} \text{left}: & \alpha_{-1}^{\mu} & \alpha_{-1+k/N}^{3} & \alpha_{-k/N}^{4} & \alpha_{-k/N}^{\bar{3}} & \alpha_{-1+k/N}^{\bar{4}} \\ \text{right}: & \tilde{\alpha}_{-1}^{\mu} & \tilde{\alpha}_{-k/N}^{3} & \tilde{\alpha}_{-1+k/N}^{4} & \tilde{\alpha}_{-1+k/N}^{\bar{3}} & \tilde{\alpha}_{-k/N}^{\bar{4}} \end{array} \tag{2.6.25}$$

and the following fermionic operators (from (2.2.29)) in the respective sectors

$$\begin{array}{llllll} NS: & \Psi^\mu_{-1/2} & \Psi^3_{-1/2+k/N} & \Psi^4_{-1/2-k/N} & \Psi^{\bar 3}_{-1/2-k/N} & \Psi^{\bar 4}_{-1/2+k/N} \\ \tilde{NS}: & \tilde\Psi^\mu_{-1/2} & \tilde\Psi^3_{-1/2-k/N} & \tilde\Psi^4_{-1/2+k/N} & \tilde\Psi^{\bar 3}_{-1/2+k/N} & \tilde\Psi^{\bar 4}_{-1/2-k/N} \\ R: & \Psi^\mu_0 & \Psi^3_{-1+k/N} & \Psi^4_{-k/N} & \Psi^{\bar 3}_{-k/N} & \Psi^{\bar 4}_{-1+k/N} \\ \tilde R: & \tilde\Psi^\mu_0 & \tilde\Psi^3_{-k/N} & \tilde\Psi^4_{-1+k/N} & \tilde\Psi^{\bar 3}_{-1+k/N} & \tilde\Psi^{\bar 4}_{-k/N} \end{array} \tag{2.6.26}$$

Finally, we have in the $NS, \tilde{NS}$ sector[18]

$$(\Psi^3_{-1/2+k/N}, \Psi^{\bar 4}_{-1/2+k/N})(\tilde\Psi^4_{-1/2+k/N}, \tilde\Psi^{\bar 3}_{-1/2+k/N})|0\rangle_{NS,\tilde{NS}} \tag{2.6.27}$$

Explicitly, these four states are given as

$$\begin{pmatrix} |0,0,1-\frac{k}{N},\frac{k}{N}\rangle_{NS} \\ |0,0,-\frac{k}{N},\frac{k}{N}-1\rangle_{NS} \end{pmatrix} \otimes \begin{pmatrix} |0,0,\frac{k}{N},1-\frac{k}{N}\rangle_{\tilde{NS}} \\ |0,0,\frac{k}{N}-1,-\frac{k}{N}\rangle_{\tilde{NS}} \end{pmatrix} \tag{2.6.28}$$

from which we read off

$$s_L^{(\theta,0)} = \frac{k}{N} - 2\frac{k^2}{N^2} \qquad s_R^{(\theta,0)} = -\frac{k}{N} + 2\frac{k^2}{N^2} \tag{2.6.29}$$

In the $R, \tilde R$ sector, where we can only act by Ψ^μ_0 and $\tilde\Psi^\mu_0$,

$$|\pm\tfrac{1}{2},\pm\tfrac{1}{2},\tfrac{1}{2}-\tfrac{k}{N},-\tfrac{1}{2}+\tfrac{k}{N}\rangle_R \otimes |\pm\tfrac{1}{2},\pm\tfrac{1}{2},-\tfrac{1}{2}+\tfrac{k}{N},\tfrac{1}{2}-\tfrac{k}{N}\rangle_{\tilde R} \tag{2.6.30}$$

We have an even number of $-$ signs in the R sector and an odd number of $-$ signs in the $\tilde R$ sector because of the GSO-projection $(-1)^F = -1$ in the leftmoving sector. The (p^1, p^2) charges of these states add up to $(1,0)$, $(0,1)$, $(-1,0)$, $(0,-1)$ which, by (2.6.6) and (2.6.7), correspond to the $\mathrm{U}(1)_1 \times \mathrm{U}(1)_2$ quantum numbers $(1,1)$, $(1,-1)$, $(-1,-1)$, $(-1,1)$ and therefore all states combine into a single vector multiplet.

The IIA $\mathbb{Z}_N$ Fixed Point

Combining our results, a fixed point of a $\mathbb{Z}_N$ twist, which is not at the same time[19] a fixed point of some other $\mathbb{Z}_{N'}$ twist with $N' > N$, has precisely $N-1$ vector multiplets associated to it, one for each twisted sector.

[18]In the case $N = 2k$ one has to be very careful, since the NS vacua are degenerate. It is clear, however, that the only possible excitations are those which change the signs in the $NS, \tilde{NS}$ vacua.

[19]In that case there are additional fields from the twisted sectors of that twist. But of course, then the twisted sectors of the $\mathbb{Z}_N$ twist are also twisted sectors of the $\mathbb{Z}_{N'}$ twist.

The Standard Embedding of Heterotic $E_8 \times E_8$ Orbifolds

As can be seen from the table B.1 the shift vector

$$\beta^I = \frac{1}{N}(1^2, 0^6) \tag{2.6.31}$$

always fulfills the level matching condition $\phi^2 = \beta^2 \pmod 2$. As this β, up to a Weyl reflection (see appendix B.3), is equal to the ϕ vector, this choice of β is called the standard embedding[20]

The Untwisted Sector (Standard Embedding)

The Virasoro generators (2.2.42) read

$$\begin{aligned} T_0 &= \ldots + \frac{1}{2}q^2 - 1 \\ \tilde{T}_0 &= \ldots + \frac{1}{2}\tilde{q}^2 - \frac{1}{2} \end{aligned} \tag{2.6.32}$$

Similar to the type IIA case (2.6.13), in the $\tilde{NS}$ sector we now have the following creation operators

$$\begin{aligned} \alpha^0 : &\quad \tilde{\Psi}^\mu_{-1/2}, \alpha^\mu_{-1}, \tilde{\alpha}^\mu_{-1}, \gamma^I_{-1} \\ \alpha^1 : &\quad (\tilde{\Psi}^3_{-1/2}, \tilde{\Psi}^{\bar{4}}_{-1/2}), (\alpha^3_{-1}, \alpha^{\bar{4}}_{-1}), (\tilde{\alpha}^3_{-1}, \tilde{\alpha}^{\bar{4}}_{-1}) \\ \alpha^{-1} : &\quad (\tilde{\Psi}^{\bar{3}}_{-1/2}, \tilde{\Psi}^4_{-1/2}), (\alpha^{\bar{3}}_{-1}, \alpha^4_{-1}), (\tilde{\alpha}^{\bar{3}}_{-1}, \tilde{\alpha}^4_{-1}) \end{aligned} \tag{2.6.33}$$

From (2.6.32), the only possible states in the leftmoving sector are

$$\begin{aligned} &(\alpha^\mu_{-1}, \alpha^i_{-1}, \alpha^{\bar{i}}_{-1}, \gamma^I_{-1})|0\rangle \\ &|q\rangle \qquad q^2 = 2 \end{aligned} \tag{2.6.34}$$

up to projection onto invariant states. Therefore we have to classify $|q\rangle$ according

[20] This can be understood as embedding $\mathbb{Z}_N \subset \mathrm{SU}(2)$ into $\mathbb{Z}_N \subset \mathrm{SU}(2) \subset \mathrm{E}_8 \times \mathrm{E}_8$ (see appendix B.4), to make contact to smooth K3 and Calabi-Yau compactifications, where embedding the $\mathrm{SU}(n)$ holonomy of the compact manifold M^{2n} into an $\mathrm{SU}(n)$ subgroup of the gauge group is also called standard embedding. However, we note that in case of orbifolds, every shift vector describes an embedding of the $\mathbb{Z}_N$ holonomy into the gauge group.

to their transformation properties under β (see (B.4.7) and (B.4.12)):

$$\begin{array}{lll} & & N=2 \\ \hline & & \mathrm{E}_7 \times \mathrm{SU}(2) \times \mathrm{E}_8 \\ \alpha^0 & \big|q\big\rangle = & (\mathbf{133},\mathbf{1},\mathbf{1}) + (\mathbf{1},\mathbf{3},\mathbf{1}) + (\mathbf{1},\mathbf{1},\mathbf{248}) \\ \alpha^1 & \big|q\big\rangle = & (\mathbf{56},\mathbf{2},\mathbf{1}) \\ \\ & & N>2 \\ \hline & & \mathrm{E}_7 \times \mathrm{E}_8 \times \mathrm{U}(1) \\ \alpha^0 & \big|q\big\rangle = & (\mathbf{133},\mathbf{1})_0 + (\mathbf{1},\mathbf{1})_0 + (\mathbf{1},\mathbf{248})_0 \\ \alpha^1 & \big|q\big\rangle = & (\mathbf{56},\mathbf{1})_{+1} + (\mathbf{1},\mathbf{1})_{-2} \\ \alpha^{-1} & \big|q\big\rangle = & (\mathbf{56},\mathbf{1})_{-1} + (\mathbf{1},\mathbf{1})_{+2} \end{array} \tag{2.6.35}$$

where vectors with $q^2 = 0$ have to be omitted. Hence, we have the states

$$\begin{array}{rl} \alpha^0\tilde{\alpha}^0 : & (\alpha^{\mu}_{-1}, \gamma^{I}_{-1})\tilde{\Psi}^{\nu}_{-1/2}\big|0\big\rangle_{\tilde{NS}} \\ \alpha^1\tilde{\alpha}^{-1} : & ((\alpha^{3}_{-1}, \alpha^{\bar{4}}_{-1}), (\tilde{\Psi}^{\bar{3}}_{-1/2}, \tilde{\Psi}^{4}_{-1/2}))\big|0\big\rangle_{\tilde{NS}} \\ \alpha^{-1}\tilde{\alpha}^{1} : & ((\alpha^{\bar{3}}_{-1}, \alpha^{4}_{-1}), (\tilde{\Psi}^{3}_{-1/2}, \tilde{\Psi}^{\bar{4}}_{-1/2}))\big|0\big\rangle_{\tilde{NS}} \\ (\alpha^1\tilde{\alpha}^1)_{N=2} : & ((\alpha^{3}_{-1}, \alpha^{\bar{4}}_{-1}), (\tilde{\Psi}^{3}_{-1/2}, \tilde{\Psi}^{\bar{4}}_{-1/2}))\big|0\big\rangle_{\tilde{NS}} \\ (\alpha^{-1}\tilde{\alpha}^{-1})_{N=2} : & ((\alpha^{\bar{3}}_{-1}, \alpha^{4}_{-1}), (\tilde{\Psi}^{\bar{3}}_{-1/2}, \tilde{\Psi}^{4}_{-1/2}))\big|0\big\rangle_{\tilde{NS}} \end{array} \tag{2.6.36}$$

which, except for the γ excitation, carry exactly the same quantum numbers as those of the type IIA theory (2.6.16). The other states are

$$\begin{array}{lll} & & N=2 \\ \hline \alpha^0\tilde{\alpha}^0 & : & \tilde{\Psi}^{\mu}_{-1/2}\,\big|q\big\rangle \otimes \big|0\big\rangle_{\tilde{NS}} \\ \alpha^1\tilde{\alpha}^1 & : & ((\tilde{\Psi}^{\bar{3}}_{-1/2}, \tilde{\Psi}^{4}_{-1/2}), (\tilde{\Psi}^{3}_{-1/2}, \tilde{\Psi}^{\bar{4}}_{-1/2}))\,\big|q\big\rangle \otimes \big|0\big\rangle_{\tilde{NS}} \\ \\ & & N>2 \\ \hline \alpha^0\tilde{\alpha}^0 & : & \tilde{\Psi}^{\mu}_{-1/2}\,\big|q\big\rangle \otimes \big|0\big\rangle_{\tilde{NS}} \\ \alpha^1\tilde{\alpha}^{-1} & : & ((\tilde{\Psi}^{\bar{3}}_{-1/2}, \tilde{\Psi}^{4}_{-1/2}))\,\big|q\big\rangle \otimes \big|0\big\rangle_{\tilde{NS}} \\ \alpha^{-1}\tilde{\alpha}^{1} & : & ((\tilde{\Psi}^{3}_{-1/2}, \tilde{\Psi}^{\bar{4}}_{-1/2}))\,\big|q\big\rangle \otimes \big|0\big\rangle_{\tilde{NS}} \end{array} \tag{2.6.37}$$

where the $\big|q\big\rangle$ denote states from (2.6.35) of the respective sector. From the supermultiplets (2.6.11), we have

$$\begin{array}{lll}
 & & N=2 \\
\hline
 & & \mathrm{E}_7\times\mathrm{SU}(2)\times\mathrm{E}_8 \\
\alpha^0\tilde\alpha^0: & \text{SUGRA} & \\
 & \text{tensor} & \\
 & \text{vector} & (\mathbf{133},\mathbf{1},\mathbf{1})+(\mathbf{1},\mathbf{3},\mathbf{1})+(\mathbf{1},\mathbf{1},\mathbf{248}) \\
\alpha^1\tilde\alpha^1: & \text{hyper} & 4(\mathbf{1},\mathbf{1},\mathbf{1}) \\
 & \text{hyper} & (\mathbf{56},\mathbf{2},\mathbf{1}) \\
 & & \\
 & & N>2 \\
\hline
 & & \mathrm{E}_7\times\mathrm{E}_8\times\mathrm{U}(1) \\
\alpha^0\tilde\alpha^0: & \text{SUGRA} & \\
 & \text{tensor} & \\
 & \text{vector} & (\mathbf{133},\mathbf{1})_0+(\mathbf{1},\mathbf{248})_0+(\mathbf{1},\mathbf{1})_0 \\
\alpha^1\tilde\alpha^{-1}+\alpha^{-1}\tilde\alpha^1: & \text{hyper} & 2(\mathbf{1},\mathbf{1})_0 \\
 & \text{hyper} & (\mathbf{1},\mathbf{1})_{-2} \\
 & \text{hyper} & (\mathbf{56},\mathbf{1})_{+1}
\end{array} \tag{2.6.38}$$

We note, that the hypermultiplets consist of two half-hypers each, one from the $\alpha^1\tilde\alpha^{-1}$ sector and one from the $\alpha^{-1}\tilde\alpha^1$ sector. In addition, the neutral hypermultiplets are in a $\mathbf{2}$ of $\mathrm{SU}(2)_R$ as a whole.

The Twisted Sector (Standard Embedding)

Similar to the type IIA case, the gauge shift the the kth twisted sector is given by $\beta = (k/N, k/N, 0^6; 0^8)$. Therefore, we have in the θ^{-k} sector $\beta = (-k/N, -k/N, 0^6; 0^8)$ and again we get the same results as in the kth sector up to conjugating all representations.

As the vacuum energies of the bosons Z carry over from the type IIA case, the Virasoro generators now read

$$\begin{aligned}
T_0 &= \ldots + \frac{1}{2}(p-\beta)^2 + \frac{k}{N} - \frac{k^2}{N^2} - 1 \\
\tilde T_0 &= \ldots + \frac{1}{2}(\tilde p+\phi)^2 + \frac{k}{N} - \frac{k^2}{N^2} - \frac{1}{2}
\end{aligned} \tag{2.6.39}$$

Since the right-moving sectors are identical to those of the type IIA theory, we only have to discuss the left movers

$$\begin{array}{llll} NS_1, NS_2: & \big|(-\tfrac{k}{N})^2, 0^6 \quad ; 0^8\big\rangle & F_1 = F_2 = 0 & \tfrac{\alpha'}{4}m^2 = \tfrac{k}{N} - 1 \\ NS_1, R_2: & \big|(-\tfrac{k}{N})^2, 0^6 \quad ; (\tfrac{1}{2})^8\big\rangle & F_1 = F_2 = 0 & \tfrac{\alpha'}{4}m^2 = \tfrac{k}{N} \\ R_1, NS_2: & \big|(\tfrac{1}{2} - \tfrac{k}{N})^2, (\tfrac{1}{2})^6 \quad ; 0^8\big\rangle & F_1 = F_2 = 0 & \tfrac{\alpha'}{4}m^2 = 0 \\ R_1, R_2: & \big|(\tfrac{1}{2} - \tfrac{k}{N})^2, (\tfrac{1}{2})^6 \quad ; (\tfrac{1}{2})^8\big\rangle & F_1 = F_2 = 0 & \tfrac{\alpha'}{4}m^2 = +1 \end{array} \tag{2.6.40}$$

The ν and $\tilde{\nu}$ of (2.2.29) are

$$\begin{array}{lllll} NS_1: & \nu^I & = \tfrac{1}{2} + \beta^I & = ((\tfrac{1}{2} + \tfrac{k}{N})^2, (\tfrac{1}{2})^6) & (\text{mod } 1) \\ R_1: & \nu^I & = \beta^I & = ((\tfrac{k}{N})^2, 0^6) & (\text{mod } 1) \\ NS_2: & \nu^I & = \tfrac{1}{2} + \beta^I & = ((\tfrac{1}{2})^8) & (\text{mod } 1) \\ R_2: & \nu^I & = \beta^I & = (0^8) & (\text{mod } 1) \end{array} \tag{2.6.41}$$

and we have the following operators in the respective sectors

$$\begin{array}{lllll} & \alpha^3_{-1+k/N} & \alpha^4_{-k/N} & \alpha^{\bar{3}}_{-k/N} & \alpha^{\bar{4}}_{-1+k/N} \\ & \tilde{\alpha}^3_{-k/N} & \tilde{\alpha}^4_{-1+k/N} & \tilde{\alpha}^{\bar{3}}_{-1+k/N} & \tilde{\alpha}^{\bar{4}}_{-k/N} \\ NS_1: & \Psi^{12}_{-1/2+k/N} & \Psi^{\bar{1}\bar{2}}_{-1/2-k/N} & \Psi^{3\ldots8}_{-1/2} & \Psi^{\bar{3}\ldots\bar{8}}_{-1/2} \\ R_1: & \Psi^{12}_{-1+k/N} & \Psi^{\bar{1}\bar{2}}_{-k/N} & \Psi^{3\ldots8}_{0} & \Psi^{\bar{3}\ldots\bar{8}}_{0} \\ NS_2: & \Psi^{9\ldots16}_{-1/2} & \Psi^{\bar{9}\ldots\bar{16}}_{-1/2} & & \\ R_2: & \Psi^{9\ldots16}_{0} & \Psi^{\bar{9}\ldots\bar{16}}_{0} & & \end{array} \tag{2.6.42}$$

Therefore we have the following number of massless states together with their weights in the NS_1, NS_2 sector

$$\begin{array}{lll} 24 & & \Psi^{12}_{-1/2+k/N}(\Psi^{3\ldots8}_{-1/2}, \Psi^{\bar{3}\ldots\bar{8}}_{-1/2})\ \big|0\big\rangle_{NS_1,NS_2} \\ & & (1-\tfrac{k}{N}, -\tfrac{k}{N}, \pm1, 0^5; 0^8) \\ & & (-\tfrac{k}{N}, 1-\tfrac{k}{N}, \pm1, 0^5; 0^8) \\ 2 & & \Psi^1_{-1/2+k/N}\Psi^2_{-1/2+k/N}(\alpha^{\bar{3}}_{-k/N}, \alpha^4_{-k/N})\ \big|0\big\rangle_{NS_1,NS_2} \\ & & ((1-\tfrac{k}{N})^2, 0^6; 0^8) \\ 2 & & (\alpha^3_{-1+k/N}, \alpha^{\bar{4}}_{-1+k/N})\ \big|0\big\rangle_{NS_1,NS_2} \\ & & ((-\tfrac{k}{N})^2, 0^6; 0^8) \\ 2 & N=2 & \Psi^1_0\Psi^2_0(\alpha^3_{-1+k/N}, \alpha^{\bar{4}}_{-1+k/N})\ \big|0\big\rangle_{NS_1,NS_2} \\ & & ((1-\tfrac{k}{N})^2, 0^6; 0^8) \\ N & k=1 & (\alpha^{\bar{3}4}_{-1/N}\ldots\alpha^{\bar{3}4}_{-1/N})\ \big|0\big\rangle_{NS_1,NS_2} \\ & & ((-\tfrac{k}{N})^2, 0^6; 0^8) \end{array} \tag{2.6.43}$$

where the ±1 is permuted over the $q^{3\ldots8}$. To derive these states, we explicitly have to check for invariance under the twist. However, this is easy since the oscillators have to transform as $1/N$. The states special for $k = 1$ have to contain precisely $N - 1$ creation operators.

Finally, in the R_1, NS_2 sector there are $2^6/2 = 32$ massless states

$$\big|(\tfrac{1}{2} - \tfrac{k}{N})^2, (\pm\tfrac{1}{2})^6; 0^8\big\rangle \qquad \text{even number of '}-\text{'} \tag{2.6.44}$$

All states containing α excitations have weights proportional to the U(1) charge vector (B.4.11) and therefore are singlets under $\mathrm{E}_7 \times \mathrm{E}_8$. Their respective U(1) charges are given by $q'' = 2 - 2k/N$ and $q'' = -2k/N$ (since $q'' = q^1 + q^2 = -2k/N$, from (B.4.8)).

The highest weight of the 56 states without α excitations is

$$(1 - \tfrac{k}{N}, -\tfrac{k}{N}, 1, 0^5; 0^8) \tag{2.6.45}$$

All 56 states without α excitations have the U(1) charge

$$q'' = 1 - \frac{2k}{N} \tag{2.6.46}$$

corresponding to the charge vector (see (B.4.11))

$$q^I = (\tfrac{1}{2} - \tfrac{k}{N}, \tfrac{1}{2} - \tfrac{k}{N}, 0^6; 0^8) \tag{2.6.47}$$

Subtracting that vector from the highest weight gives

$$(\tfrac{1}{2}, -\tfrac{1}{2}, 1, 0^5; 0^8) \tag{2.6.48}$$

which is the highest weight of the **56** of E_7. Since we have only 56 remaining states, there are no other representations.

Altogether, after combining with the rightmoving states (2.6.27), the particle content of a twisted sector is

$$\begin{array}{rll} & \text{half-hyper} & (\mathbf{56}, \mathbf{1})_{1-2k/N} \\ & \text{half-hyper} & 2(\mathbf{1}, \mathbf{1})_{2-2k/N} \\ & \text{half-hyper} & 2(\mathbf{1}, \mathbf{1})_{-2k/N} \\ k = 1 & \text{half-hyper} & N(\mathbf{1}, \mathbf{1})_{-2/N} \\ N = 2 & \text{half-hyper} & 2(\mathbf{1}, \mathbf{1})_{+1} \end{array} \tag{2.6.49}$$

The $\mathbb{Z}_N$ Orbifold Fixed Point of Heterotic $E_8 \times E_8$ String Theory in the Standard Embedding

For N odd, we have the following particle content (without U(1) quantum numbers)

$$\begin{array}{lcl} \text{hyper} & (N-1)/2 & (\mathbf{56}, \mathbf{1}) \\ \text{hyper} & 3N - 2 & (\mathbf{1}, \mathbf{1}) \end{array} \tag{2.6.50}$$

We note that all half hypers of twisted sector k, by their U(1) quantum numbers, have to combine with their CPT conjugates in twisted sector $N - k$ to full hypermultiplets.

For N even, we have the following particle content (without U(1) quantum numbers)

$$\begin{array}{rcl} \text{hyper} & (N-2)/2 & (\mathbf{56}, \mathbf{1}) \\ \text{half-hyper} & 1 & (\mathbf{56}, \mathbf{1}) \\ \text{hyper} & 3N-2 & (\mathbf{1}, \mathbf{1}) \end{array} \tag{2.6.51}$$

Again, all half hypers of twisted sector k have to combine with their conjugates in sector $N - k$, including the half-hypers of the $2k = N$ sector which have to combine among themselves. The only exception is the single $(\mathbf{56}, \mathbf{1})$ half-hypermultiplet in that sector which is neutral under U(1) and is its own conjugate.

Finally, in the very special $N = 2$ case the unbroken group is $E_7 \times SU(2) \times E_8$ and the two generic half-hypermultiplets of U(1) charge +1 combine with the other two generic half-hypermultiplet of opposite charge into two full hypermultiplets in the $\mathbf{2}$ of SU(2). However, *the same* happens for the 2 ($= N$) special half-hypermultiplets appearing only for $k = 1$ and the other two half-hypermultiplets appearing only for $2k = N$. The particle content of the twisted sector of one fixed point then is

$$\begin{array}{rcl} \text{hyper} & 2 & (\mathbf{1}, \mathbf{2}, \mathbf{1}) \\ \text{half-hyper} & 1 & (\mathbf{56}, \mathbf{1}, \mathbf{1}) \end{array} \tag{2.6.52}$$

To sum up, disregarding U(1) or SU(2) quantum numbers, the particle content of a $\mathbb{Z}_N$ orbifold singularity of heterotic $E_8 \times E_8$ string theory in the standard embedding is given by

$$\begin{array}{rcl} \text{half-hyper} & N-1 & (\mathbf{56}, \mathbf{1}) \\ \text{hyper} & 3N-2 & (\mathbf{1}, \mathbf{1}) \end{array} \tag{2.6.53}$$

The $\mathbb{Z}_3, \beta = \frac{1}{3}(5/2, (1/2)^7; 2, 1^2, 0^5)$ *Orbifold*

This shift vector only fulfills the weak level matching condition $\phi^2 = \beta^2 \pmod{2/3}$ and we choose $w = (0^8; 0, 1, 0, -1, 0^5)$ as a root vector with $w \cdot s = w \cdot 3\beta = 1$. Then, from (2.3.12), we have

$$\frac{2n}{3} = \phi^2 - \beta^2 = \frac{2}{9} - \frac{14}{9} = -\frac{4}{3} \pmod{2} \tag{2.6.54}$$

and therefore $n=1$ and $\beta_s=\beta+w$. At the end of the calculation, we will verify that this is the correct choice. From table 2.1 we see that the unbroken gauge group is

$$\mathrm{SU}(9)\times \mathrm{E}_6\times \mathrm{SU}(3) \tag{2.6.55}$$

which in general allows for complex representation. This implies that all charged hypermultiplets will have to combine from half-hypermultiplets of the θ^1 and θ^{-1} sectors.

The Untwisted Sector

As (2.6.32) and (2.6.33) remain unchanged from the standard embedding, we only need to write down the decomposition of the adjoint representations (B.4.20) and (B.4.26)

$$\begin{array}{lll} & & \mathrm{SU}(9)\times \mathrm{E}_6\times \mathrm{SU}(3) \\ \alpha^0 & |q\rangle = & (\mathbf{80},\mathbf{1},\mathbf{1})+(\mathbf{1},\mathbf{78},\mathbf{1})+(\mathbf{1},\mathbf{1},\mathbf{8}) \\ \alpha^1 & |q\rangle = & (\mathbf{84},\mathbf{1},\mathbf{1})+(\mathbf{1},\mathbf{27},\mathbf{3}) \\ \alpha^{-1} & |q\rangle = & (\overline{\mathbf{84}},\mathbf{1},\mathbf{1})+(\mathbf{1},\overline{\mathbf{27}},\bar{\mathbf{3}}) \end{array} \tag{2.6.56}$$

Therefore, from (2.6.34), (2.6.36) and (2.6.37) for $N=3$, we have the states

		$\mathrm{SU}(9)\times \mathrm{E}_6\times \mathrm{SU}(3)$
$\alpha^0\tilde{\alpha}^0$:	SUGRA	
	tensor	
	vector	$(\mathbf{80},\mathbf{1},\mathbf{1})+(\mathbf{1},\mathbf{78},\mathbf{1})+(\mathbf{1},\mathbf{1},\mathbf{8})$
$\alpha^1\tilde{\alpha}^{-1}+\alpha^{-1}\tilde{\alpha}^1$:	hyper	$2(\mathbf{1},\mathbf{1},\mathbf{1})$
	hyper	$(\mathbf{84},\mathbf{1},\mathbf{1})+(\mathbf{1},\mathbf{27},\mathbf{3})$

(2.6.57)

where the charged hypermultiplets now contain the conjugate representations found in (2.6.56) in their conjugate halves.

The Twisted Sector

Since there are only two twisted sectors, conjugate to each other, we only have to treat the case $\beta=\frac{1}{3}(5/2,(1/2)^7;2,1^2,0^5)$.

The Virasoro generators are from (2.6.39) for $N=3$

$$\begin{aligned} T_0 &= \ldots+\frac{1}{2}(p-\beta)^2-\frac{14}{18} \\ \tilde{T}_0 &= \ldots+\frac{1}{2}(\tilde{p}+\phi)^2-\frac{5}{18} \end{aligned} \tag{2.6.58}$$

The vacua are

$$
\begin{array}{llll}
NS_1, NS_2: & \left|-\frac{5}{6}, (-\frac{1}{6})^7 \;; -\frac{2}{3}, (-\frac{1}{3})^2, 0^5\right\rangle & & \\
 & \left|+\frac{1}{6}, (-\frac{1}{6})^7 \;; +\frac{1}{3}, (-\frac{1}{3})^2, 0^5\right\rangle & F_1 = F_2 = 1 & \frac{\alpha'}{4}m^2 = -\frac{1}{2} \\
NS_1, R_2: & \left|-\frac{5}{6}, (-\frac{1}{6})^7 \;; -\frac{1}{6}, (+\frac{1}{6})^2, (\frac{1}{2})^5\right\rangle & & \\
 & \left|+\frac{1}{6}, (-\frac{1}{6})^7 \;; -\frac{1}{6}, (+\frac{1}{6})^2, (\frac{1}{2})^5\right\rangle & F_1 = 1, F_2 = 0 & \frac{\alpha'}{4}m^2 = 0 \\
R_1, NS_2: & \left|-\frac{1}{3}, (+\frac{1}{3})^7 \;; -\frac{2}{3}, (-\frac{1}{3})^2, 0^5\right\rangle & & \\
 & \left|-\frac{1}{3}, (+\frac{1}{3})^7 \;; +\frac{1}{3}, (-\frac{1}{3})^2, 0^5\right\rangle & F_1 = 0, F_2 = 1 & \frac{\alpha'}{4}m^2 = -\frac{1}{6} \\
R_1, R_2: & \left|-\frac{1}{3}, (+\frac{1}{3})^7 \;; -\frac{1}{6}, (+\frac{1}{6})^2, (\frac{1}{2})^5\right\rangle & F_1 = F_2 = 0 & \frac{\alpha'}{4}m^2 = +\frac{1}{3}
\end{array}
\tag{2.6.59}
$$

The ν and $\tilde{\nu}$ of (2.2.29) are

$$
\begin{array}{llll}
NS_1: & \nu^I = \frac{1}{2} + \beta^I & = (\frac{1}{3}, (\frac{2}{3})^7) & (\text{mod } 1) \\
R_1: & \nu^I = \beta^I & = (\frac{5}{6}, (\frac{1}{6})^7) & (\text{mod } 1) \\
NS_2: & \nu^I = \frac{1}{2} + \beta^I & = (\frac{1}{6}, (\frac{5}{6})^2, (\frac{1}{2})^5) & (\text{mod } 1) \\
R_2: & \nu^I = \beta^I & = (\frac{2}{3}, (\frac{1}{3})^2, 0^5) & (\text{mod } 1)
\end{array}
\tag{2.6.60}
$$

and we have the following operators in the respective sectors

$$
\begin{array}{lllllll}
 & \alpha^3_{-2/3} & \alpha^4_{-1/3} & \alpha^{\bar{3}}_{-1/3} & \alpha^{\bar{4}}_{-2/3} & & \\
 & \tilde{\alpha}^3_{-1/3} & \tilde{\alpha}^4_{-2/3} & \tilde{\alpha}^{\bar{3}}_{-2/3} & \tilde{\alpha}^{\bar{4}}_{-1/3} & & \\
NS_1: & \Psi^1_{-2/3} & \Psi^{\bar{1}}_{-1/3} & \Psi^{2\ldots 8}_{-1/3} & \Psi^{\bar{2}\ldots\bar{8}}_{-2/3} & & \\
R_1: & \Psi^1_{-1/6} & \Psi^{\bar{1}}_{-5/6} & \Psi^{2\ldots 8}_{-5/6} & \Psi^{\bar{2}\ldots\bar{8}}_{-1/6} & & \\
NS_2: & \Psi^9_{-5/6} & \Psi^{\bar{9}}_{-1/6} & \Psi^{10,11}_{-1/6} & \Psi^{\bar{10},\bar{11}}_{-5/6} & \Psi^{12\ldots 16}_{-1/2} & \Psi^{\bar{12}\ldots\bar{16}}_{-1/2} \\
R_2: & \Psi^9_{-1/3} & \Psi^{\bar{9}}_{-2/3} & \Psi^{10,11}_{-2/3} & \Psi^{\bar{10},\bar{11}}_{-1/3} & \Psi^{12\ldots 16}_{0} & \Psi^{\bar{12}\ldots\bar{16}}_{0}
\end{array}
\tag{2.6.61}
$$

Therefore, we have the following massless states

$$
\begin{array}{ll}
NS_1, NS_2: & (\Psi^{\bar{1}}_{-1/3}, \Psi^{2\ldots 8}_{-1/3})(\Psi^{\bar{9}}_{-1/6}, \Psi^{10,11}_{-1/6})\,\left|0\right\rangle \\
R_1, NS_2: & (\Psi^{\bar{9}}_{-1/6}, \Psi^{10,11}_{-1/6})\,\left|0\right\rangle
\end{array}
\tag{2.6.62}
$$

with the highest weights

$$
\begin{array}{ll}
NS_1, NS_2: & (+\frac{1}{6}, +\frac{5}{6}, (-\frac{1}{6})^6; +\frac{1}{3}, +\frac{2}{3}, -\frac{1}{3}, 0^5) \\
R_1, NS_2: & (-\frac{1}{3}, (+\frac{1}{3})^7; +\frac{1}{3}, +\frac{2}{3}, -\frac{1}{3}, 0^5)
\end{array}
\tag{2.6.63}
$$

which comprise a $(\mathbf{9}, \mathbf{1}, \mathbf{3})$ (see (B.4.25) and (B.4.19)).

Finally, we come back to transformation properties of these states and the issue of weak and strong level matching. Either from (2.6.63) or from (B.4.25) and (B.4.19) we get

$$
e^{2\pi i s_L^{(\theta,0,\beta_s)}} = \alpha^{1+\frac{1}{3}} = \alpha^{\frac{4}{3}} = e^{2\pi i \frac{4}{9}}
\tag{2.6.64}
$$

which clearly does not cancel the phase of the rightmoving states (2.6.29)

$$e^{2\pi i s_R^{(\theta,0)}} = e^{2\pi i(-\frac{1}{3}+2\frac{1}{9})} = e^{-2\pi i\frac{1}{9}} \tag{2.6.65}$$

However, calculating s_L using $\beta_s = \beta + w$ instead of β, we have

$$e^{2\pi i s_L^{(\theta,0,\beta)}} = e^{2\pi i(\frac{4}{9}+\frac{6}{9})} \stackrel{!}{=} e^{2\pi i\frac{1}{9}} \tag{2.6.66}$$

Chapter 3

Kaluza-Klein-Monopoles

In this chapter we will discuss the multiple Kaluza-Klein-monopole solution of [55] and [92] which is a generalized Taub-NUT space [38, 46] (see generally [37]). For our conventions and details on the various properties of the solution the reader is referred to appendix C.

3.1 The Kaluza-Klein-Monopole Solution

The metric of N KK-monopoles is given as (see [78] or [88])

$$\mathrm{d}s^2 = U^{-1}(\mathrm{d}x^4 + \vec{\omega}\cdot\mathrm{d}\vec{r})^2 + U\mathrm{d}\vec{r}^{\,2} \tag{3.1.1}$$

where $x^4 = x^4 + 2\pi R$ is a compact direction and $\vec{x} = \vec{r}$ is treated as three-dimensional euclidian space with the standard scalar product. We use the orientation $\mathrm{d}x^1 \wedge \mathrm{d}x^2 \wedge \mathrm{d}x^3 \wedge \mathrm{d}x^4$. U and $\vec{\omega}$ are defined as

$$U = 1 + \sum_{I=1}^{N} U_I \qquad \vec{\omega} = \sum_{I=1}^{N} \vec{\omega}_I \tag{3.1.2}$$

with

$$U_I = \frac{R/2}{|\vec{r} - \vec{r}_I|} \qquad \vec{\nabla}\times\vec{\omega}_I = \vec{\nabla}U_I \tag{3.1.3}$$

where we demand the $\vec{\omega}_I$ to be divergence free: $\vec{\nabla}\cdot\vec{\omega} = 0$. It is clear that $\vec{\omega}_I$ is not globally well defined and a Dirac string has to emerge from every $\vec{r}_I$. As shown in appendix C.1, the last equation in (3.1.3) actually comprises an anti-self-duality condition on the curvature tensor. This implies the existence of a hyperkähler structure, derived in appendix C.1.

This space will be denoted as K_N. Its metric, except for the factors of U and U^{-1}, is the same metric as it is used in Kaluza-Klein theories on S^1 with the Kaluza-Klein U(1) potential $\vec{\omega}$. By (3.1.2) and (3.1.3), the solutions are defined to have magnetical charge[1] N with respect to $\vec{\omega}$ and can therefore be regarded as as special kind of magnetic monopole for $|\vec{r}| \to \infty$.

As we will see below, as long as the $\vec{r}_I$ are all distinct, the singularities at $\vec{r} = \vec{r}_I$ are all coordinate singularities which disappear by a proper choice of coordinates. When we take the limit where all U_I become very large simultaneously, we get

$$U = \sum_{I=1}^{N} U_I \tag{3.1.4}$$

and the metric degenerates to the multi-center Eguchi-Hanson gravitational instanton [37] (denoted by X_N) which is known to exhibit a $\mathbb{Z}_m$ orbifold singularity as m of the $\vec{r}_I$ approach each other.

Most important for string theory (see, for example, [88]), the KK solutions support N normalizable self-dual harmonic two forms Ω_I, which locally can be given as follows (see [78])

$$\Omega_I = \mathrm{d}\xi_I \qquad \xi_I = U^{-1}U_I(\mathrm{d}x^4 + \vec{\omega}\cdot\mathrm{d}\vec{r}) - \vec{\omega}_I\cdot\vec{r} \tag{3.1.5}$$

Using the vielbein $e^{\underline{a}}$ as defined in appendix C.1, we globally have

$$\Omega_I = \partial_i(U^{-1}U_I)\left(e^{\underline{i}}\wedge e^{\underline{4}} - \frac{1}{2}\epsilon_{ijk}e^{\underline{j}}\wedge e^{\underline{k}}\right) \tag{3.1.6}$$

The Single KK-Monopole Solution

To unfold the behavior of the solution at $\vec{r} = \vec{r}_I$ we study the $N = 1$ solution K_1 and then apply our calculation to the general solution.

First, we give an explicit realization of $\vec{\omega}$ in the $N = 1$ case [55]. We use spherical coordinates

$$\begin{aligned} x^1 &= r\sin\vartheta\cos\varphi \\ x^2 &= r\sin\vartheta\sin\varphi \\ x^2 &= r\cos\vartheta \end{aligned} \tag{3.1.7}$$

set $\vec{r}_1 = 0$ and write

$$\mathrm{d}s^2 = U^{-1}(\mathrm{d}x^4 + R\,\tfrac{1}{2}(\cos\vartheta - 1)\,\mathrm{d}\varphi)^2 + U\big(\mathrm{d}r^2 + r^2(\mathrm{d}\vartheta^2 + \sin^2\vartheta\mathrm{d}\varphi^2)\big) \tag{3.1.8}$$

with

$$U = 1 + \frac{R}{2r} \tag{3.1.9}$$

[1] It is clear that the magnetical charge cannot be zero around every $\vec{r}_I$, otherwise the $\vec{\omega}_I$ were globally well defined.

That this solution is a realization of the general metric (3.1.1) is an easy exercise in spherical coordinates. In the $r/R \to 0$ limit, in the coordinates $(r_1, \varphi_1, r_2, \varphi_2)$ with

$$\begin{aligned} \varrho^2 &= r_1^2 + r_2^2 = 2Rr \\ r_1 &= \varrho \cos \vartheta/2 \qquad \varphi_1 = \frac{x^4}{R} \\ r_2 &= \varrho \sin \vartheta/2 \qquad \varphi_2 = \varphi - \frac{x^4}{R} \end{aligned} \tag{3.1.10}$$

the metric (3.1.8) reduces to

$$\mathrm{d}s^2 = \mathrm{d}r_1^2 + \mathrm{d}r_2^2 + r_1^2 \mathrm{d}\varphi_1^2 + r_2^2 \mathrm{d}\varphi_2^2 \tag{3.1.11}$$

which is the euclidian metric on $\mathbb{C}^2$ with

$$z^1 = r_1 e^{i\varphi_1} \qquad z^2 = r_2 e^{i\varphi_2} \tag{3.1.12}$$

Since the coordinate transformation (3.1.10) is certainly not singular for $r > 0$ and therefore well defined on the solution (3.1.8), we see that the single Kaluza-Klein-monopole has the topology of $\mathbb{R}^4$. This implies, of course, that the "compact coordinate" x^4 does not parametrize a compact direction, since any loop around x^4 can be contracted to a point (since the fundamental group of $\mathbb{R}^4$ is trivial).

Having a closer look on x^4, we rewrite the "translation" $x^4 \mapsto x^4 + \phi R$ to the coordinates (3.1.10)

$$\begin{aligned} \varphi_1 &= \tfrac{x^4}{R} & &\mapsto \varphi_1 + \phi \\ \varphi_2 &= \varphi - \tfrac{x^4}{R} & &\mapsto \varphi_2 - \phi \end{aligned} \tag{3.1.13}$$

Therefore, $x^4 \mapsto x^4 + \phi R$ acts on (z^1, z^2) as

$$(z^1, z^2) \mapsto (\exp(i\phi) z^1, \exp(-i\phi) z^2) \tag{3.1.14}$$

which already is very reminiscent of (2.4.1), the orbifold twist used in chapter 2.

N KK-Monopoles on Top of Each Other

We can easily generalize (3.1.8) to the case for N KK-monopoles K_N with $\vec{r}_I = 0$, since we only have to replace $\vec{\omega}$ by $N\vec{\omega}$

$$\mathrm{d}s^2 = U^{-1}(\mathrm{d}x^4 + NR\,\tfrac{1}{2}(\cos\vartheta - 1)\,\mathrm{d}\varphi)^2 + U\big(\mathrm{d}r^2 + r^2(\mathrm{d}\vartheta^2 + \sin^2\vartheta \mathrm{d}\varphi^2)\big) \tag{3.1.15}$$

and modify U to

$$U = 1 + \frac{NR}{2r} \tag{3.1.16}$$

These formulas are exactly the same as those of the $N = 1$ case with R replaced by NR. However, the difference between the two cases lies in the fact that the x^4 coordinate is still identified under $x^4 \mapsto x^4 + 2\pi R$. Therefore, the new coordinates $(r_1, \varphi_1, r_2, \varphi_2)$ now look like (see (3.1.10))

$$\begin{aligned} &\varrho^2 = r_1^2 + r_2^2 = 2NRr \\ &r_1 = \varrho \cos\vartheta/2 \qquad \varphi_1 = \frac{x^4}{NR} \\ &r_2 = \varrho \sin\vartheta/2 \qquad \varphi_2 = \varphi - \frac{x^4}{NR} \end{aligned} \tag{3.1.17}$$

but the "translation" $x^4 \mapsto x^4 + \phi R$ now acts like

$$\begin{array}{lll} \varphi_1 & = \frac{x^4}{NR} & \mapsto \varphi_1 + \phi/N \\ \varphi_2 & = \varphi - \frac{x^4}{NR} & \mapsto \varphi_2 - \phi/N \end{array} \tag{3.1.18}$$

and the action on (z^1, z^2) is

$$(z^1, z^2) \mapsto (\exp(i\phi/N)z^1, \exp(-i\phi/N)z^2) \tag{3.1.19}$$

which is identical to the orbifold twist of an $\mathbb{Z}_N$ orbifold fixed point (2.4.1). That this singularity can not be resolved by a coordinate transformation will be shown below, as a byproduct of the calculation of the gravitational instanton number.

The Gravitational Instanton Number

It is straightforward but tedious to calculate the first Pontrjagin number of the single KK-monopole solution

$$P_1(K_1) = \int_{K_1} p_1 = -\frac{1}{2}\left(\frac{1}{2\pi}\right)^2 \int_{K_1} \mathrm{tr}\, R^2 = -2 \tag{3.1.20}$$

Since P_1 is even, K_1 admits a spin connection. However, as in section 2.5, to make contact to Yang-Mills instantons, we give that result in terms of the gravitational instanton number $I_L = P_1/2$

$$I_L(K_1) = -\frac{1}{2}\frac{1}{8\pi^2}\int_{K_1} \mathrm{tr}\, R^2 = -1 \tag{3.1.21}$$

Using this result, we can now easily calculate the gravitational instanton number of N KK-monopoles by the following argument: we start by N monopoles whose cores at $\vec{r} = \vec{r}_I$ are all far apart from each other in units of R. Then each monopole, in a region of scale R around it, sees a small perturbation of the metric due to the presence of the other monopoles, which will go to zero by some negative power of the distance to the other monopoles (from (3.1.3)). Since $P_1 = 2$ for a single monopole, the integral $\int p_1$ limited to a fixed region around the monopole will approach 2 in the $R \to 0$ limit. Therefore, in this limit, the whole integral approaches $2N$. But since P_1 is a topological invariant and the integrand of (3.1.20) contains only U and four of its derivatives (see (C.1.10) and (C.1.11) in the appendix) and therefore goes with r^{-6} as $r \to \infty$, the integral is invariant under deformations of R and $\vec{r}_I$. We therefore have

$$I_L(K_N) = -N \tag{3.1.22}$$

But, we can go further: as we have seen, at the $\vec{r}_I = 0$ point the metric is given by (3.1.15) and the integral $\int p_1$ over this space after taking out the singularity is $1/N$ (since the x^4 direction is only $1/N$ times as big as for K_1). Therefore, the singularity must account for a gravitational instanton number of $N - 1/N$. This calculation stays valid as we move the $\vec{r}_I$ apart on a scale l which is much smaller that that of R: the sum $\sum_I U_I$ can be expanded in multipole moments for $r >> l$: the leading term is $NR/2r$ and the dipole is by definition kept zero since it corresponds to a displacement of the center of mass of the $\vec{r}_I$. Therefore, all terms except the leading term fall at least as $(r/l)^{-2}$. Hence, for $r >> l$, the space, up to coordinate transformations, approaches (3.1.15) and the integral over K_N, where the region of scale l around the origin is taken out, approaches $1/N$. But, since the solution in the $l << r << R$ region approaches a flat metric, the integral over the region of scale l is a topological invariant as long as the $\vec{r}_I$ stay at scale l or smaller. Since the scale l region is nothing but the Eguchi-Hanson multi-center gravitational instanton, we have

$$I_L(X_N) = -N + \frac{1}{N} \tag{3.1.23}$$

This implies, of course, that the curvature goes to infinity as we let the $\vec{r}_I$ approach each other. Therefore, the orbifold singularity cannot be removed by a coordinate transformation.

3.2 Supersymmetry in Low Energy Heterotic String Theory

To embed the Kaluza-Klein-monopole solution into heterotic string theory and to make contact to supersymmetric orbifolds, we have to study the conditions of unbroken supersymmetry in the low energy effective action of heterotic string theory in ten dimensions. The approach we follow was developed in [97] and used in the famous series of publications [98, 23, 24]. Since we heavily use knowledge of low energy effective actions of string theory, the reader is referred to [76], especially to chapter 12.

The general outline of the procedure is as follows: we are looking for states which preserve some amount of supersymmetry and therefore are uncharged with respect to some supercharges Q. For a classical background, this corresponds to invariance of all fields, bosonic and fermionic, under supersymmetry transformations induced by Q. However, since some supersymmetry survives, which includes some subgroup of the original Poincaré symmetry, all fermionic fields have to vanish, since fields in spinor representations always transform under rotations. Because supersymmetry transforms bosons into fermions and vice versa, supersymmetry transformations of bosons automatically vanish in backgrounds without fermions. The only remaining condition of unbroken supersymmetry is then the vanishing of the supersymmetry transformations of the fermionic fields.

We start by giving the low energy effective action of the bosonic degrees of freedom of heterotic string theory in the string frame to lowest order in α', but including the Lorentz Chern-Simons term required by the Green-Schwarz mechanism [50] (in the conventions of [76])

$$S = \frac{1}{2\kappa_{10}^2} \int \mathrm{d}^{10}x \sqrt{-G_H} e^{-2\Phi} \Big[R_H + 4\partial_M \Phi \partial^M \Phi \\ - \frac{1}{2}\frac{1}{6} H_{MNP} H^{MNP} - \frac{\alpha'}{8} \operatorname{tr} F_{MN} F^{MN} \Big] \tag{3.2.1}$$

with[2]

$$\begin{aligned} H &= \mathrm{d}B + \frac{\alpha'}{4}(\omega_{3L} - \omega_{3Y}) \\ \omega_{3L} &= \operatorname{tr}(\omega^H R^H - \tfrac{1}{3}(\omega^H)^3) \\ \omega_{3Y} &= \operatorname{tr}(AF - \tfrac{1}{3}A^3) \end{aligned} \tag{3.2.2}$$

The last equations imply

$$\begin{aligned} \mathrm{d}H &= \frac{\alpha'}{4}(\omega_{4L} - \omega_{4Y}) \\ \omega_{4L} &= \operatorname{tr}(R^H)^2 \\ \omega_{4Y} &= \operatorname{tr} F^2 \end{aligned} \tag{3.2.3}$$

The Yang-Mills coupling constant g_{10} is

$$\frac{1}{2\kappa_{10}^2}\frac{\alpha'}{8} = \frac{1}{8}\frac{1}{g_{10}^2} \tag{3.2.4}$$

This action can be computed in a straightforward way from the action given in [54] and is a generalization of the $\mathcal{N} = (1,0)$, $D = 10$ supergravity Yang-Mills theory derived in [16, 25]. Here ω^H denotes one-form of the Lorentz connection,

[2]For convenience, the trace symbol used for gauge fields is defined as $\operatorname{tr} = 1/30 \operatorname{Tr}$.

R^H its two-form curvature and a superscript or subscript H signals the use of the string frame metric G^H_{MN}.

To discuss supersymmetry, we need to give the supersymmetry transformations of the fermions left out from (3.2.1). These can be derived from those given in the original literature [16, 25], where the action is given in the Einstein frame. We therefore convert the action to the Einstein frame

$$S = \frac{1}{2\kappa_{10}^2} \int \mathrm{d}^{10}x \sqrt{-G_E} \Big[R_E - \frac{1}{2} \partial_M \Phi \partial^M \Phi - \frac{1}{2} \frac{1}{6} e^{-\Phi} H_{MNP} H^{MNP} - \frac{\alpha'}{8} e^{-\Phi/2} \operatorname{tr} F_{MN} F^{MN} \Big] \quad (3.2.5)$$

where the metric of the Einstein frame G^E_{MN} is related to that of the string frame

$$G^E = e^{-\Phi/2} G^H \quad (3.2.6)$$

(see, for example, [76], section 3.7). The supersymmetry transformations of the dilatino λ, the gravitino Ψ_M and the gaugino χ are given as[3]

$$\begin{aligned} \delta_{\epsilon^E} \lambda^E &= -\frac{1}{2\sqrt{2}} e^{\Phi/4} \left(-\not{D}\Phi + \frac{\sqrt{2}}{24\kappa_{10}} \Gamma^{MNP} H_{MNP} \right) \epsilon^E \\ \delta_{\epsilon^E} \Psi^E_M &= \mathrm{D}_M \epsilon^E + \frac{1}{8} \frac{\sqrt{2}}{24\kappa_{10}} ({\Gamma_M}^{NPQ} - 9\, \delta^N_M \Gamma^{PQ}) H_{NPQ} \epsilon^E \\ \delta_{\epsilon^E} \chi^E &= -\frac{1}{4g_{10}} e^{\Phi/4} \Gamma^{MN} F_{MN} \epsilon^E \end{aligned} \quad (3.2.7)$$

where we again have neglected fermions contributions on the rhs.. To convert to the string frame and to simplify the formulas, we use the definitions [97]

$$\begin{aligned} \epsilon &= e^{\Phi/8} \epsilon^E \\ \lambda &= e^{-\Phi/8} \lambda^E \\ \chi &= e^{-\Phi/8} \chi^E \\ \Psi_M &= e^{-\Phi/8} \left(\Psi^E_M - \frac{1}{2\sqrt{2}} \Gamma^E_M \lambda^E \right) \end{aligned} \quad (3.2.8)$$

Here we have explicitly indicated the dependence of the Γ-matrices on the Einstein or string vielbein. After some algebra we arrive at

$$\begin{aligned} \delta_\epsilon \lambda &= \frac{1}{2\sqrt{2}} \left(-\not{D}\Phi + \frac{1}{2\sqrt{2}\, 6\kappa_{10}} \Gamma^{MNP} H_{MNP} \right) \epsilon \\ \delta_\epsilon \tilde{\Psi}_M &= \mathrm{D}_M \epsilon - \frac{1}{2\sqrt{2}\, 6\kappa_{10}} \frac{3}{2} \Gamma^{PQ} H_{MPQ} \epsilon \\ \delta_\epsilon \chi &= -\frac{1}{4g_{10}} \Gamma^{MN} F_{MN} \epsilon \end{aligned} \quad (3.2.9)$$

[3] Since the condition of unbroken supersymmetry corresponds to the vanishing of these supersymmetry transformation, the normalization of the fermions is not of importance.

Six Flat Directions

We now focus on solutions which preserve at least eight supersymmetries, which for six non-compact directions correspond to $\mathcal{N} = (1,0)$ or $\mathcal{N} = (0,1)$, $D = 6$ supersymmetry. As in $D = 6$ orbifolds in section 2.6, we split indices $M = 0, \ldots, 9$ into $\mu = 0, \ldots, 5$ and $a = 6, \ldots, 9$. We will only consider excited background fields[4] that have tensor indices ranging at most from 6 to 9. We use the epsilon tensors with $\epsilon_{\underline{0}\ldots\underline{5}} = +1, \epsilon_{\underline{6}\ldots\underline{9}} = +1$ where underlined indices denote vielbein indices.

First, we have to discuss the decomposition of the supersymmetry parameter ϵ. In ten dimensions we take ϵ to be of positive chirality: $\omega_{1,9}\epsilon = +\epsilon$ where $\omega^{\mathbb{C}}_{1,9} = \Gamma^{\underline{0}} \cdots \Gamma^{\underline{9}}$ is the chirality operator for the mostly plus convention of the metric. We split the Clifford algebra[5] of the ten-dimensional gamma matrices as $\mathrm{Cl}_{1,9} = \bar{\bar{\mathrm{Cl}}}_{1,5} \otimes \bar{\mathrm{Cl}}_{0,4}$ (see [66]). Then the gamma matrices decompose as

$$\begin{aligned} \Gamma^{\underline{\mu}} &= \bar{\bar{\Gamma}}^{\underline{\mu}} \otimes \mathbb{1} & \qquad \omega^{\mathbb{C}}_{1,5} &= -\bar{\bar{\Gamma}}^{\underline{0}} \cdots \bar{\bar{\Gamma}}^{\underline{5}} \\ \Gamma^{\underline{a}} &= -\omega^{\mathbb{C}}_{1,5} \otimes \bar{\Gamma}^{\underline{a}} & \qquad \omega^{\mathbb{C}}_{0,4} &= -\bar{\Gamma}^{\underline{6}} \cdots \bar{\Gamma}^{\underline{9}} \end{aligned} \tag{3.2.10}$$

where $\omega^{\mathbb{C}}_{1,5}$ and $\omega^{\mathbb{C}}_{0,4}$ are the chirality operators of the respective Clifford algebras.

Then we can split the spinor ϵ into its positive and negative chirality parts with respect to the four-dimensional chirality $\epsilon = \epsilon_1 + \epsilon_1$, $\omega^{\mathbb{C}}_{0,4}\epsilon_\pm = \pm\epsilon_\pm$. This implies, after some algebra

$$\begin{aligned} \epsilon_{abcd}\Gamma^{abcd}\epsilon_\pm &= \mp 24\,\epsilon_\pm \\ \epsilon_{\nu_0\ldots\nu_5}\Gamma^{\nu_0\ldots\nu_5}\epsilon_\pm &= \mp 720\,\epsilon_\pm \\ \Gamma^{ab}\epsilon_\pm &= \pm\,\tfrac{1}{2}\,\epsilon^{abcd}\Gamma_{cd}\epsilon_\pm \end{aligned} \tag{3.2.11}$$

This decomposition allows us to rewrite the supersymmetry conditions $0 = \delta_\epsilon\lambda = \delta_\epsilon\tilde{\Psi}_M = \delta_\epsilon\chi$. We set $\epsilon = \epsilon_\pm$ and fix the sign from now on. From (3.2.9), $0 = \delta_\epsilon\lambda$ now reads[6]

$$\partial_d\Phi = \pm\frac{1}{2\sqrt{2}\,6\kappa_{10}}\epsilon^{abc}{}_d H_{abc} \qquad \text{or} \qquad H_{abc} = \pm 2\sqrt{2}\,6\kappa_{10}\epsilon_{abc}{}^d\partial_d\Phi \tag{3.2.12}$$

[4]This is clear in case of six non-compact directions. In case of a toroidal compactification there might be Wilson lines or B-field backgrounds, which however do not affect our arguments given below.

[5]We use the mathematical definition of Clifford algebras (see [66]). The gamma matrices of the algebra $\mathrm{Cl}_{r,s}$ fulfill

$$\Gamma^a\Gamma^b + \Gamma^b\Gamma^a = -2G^{ab}\,\mathbb{1}$$

where G^{ab} has r positive and s negative eigenvalues.

[6]We have used that $\Gamma^a v_a\epsilon = 0$ implies $v_a = 0$. Since the metric G_{ab} is positive definite, $0 = \Gamma^a v_a \Gamma^b v_b \epsilon = G^{ab} v_a v_b \epsilon$ implies $v_a = 0$.

After decomposing F_{ab} into its self-dual and anti-self-dual parts $F_{ab} = F^+_{ab} + F^-_{ab}$ with $F^\pm_{ab} = \pm\frac{1}{2}\epsilon_{ab}{}^{cd}F^\pm_{cd}$, we get

$$\delta_\epsilon\chi = -\frac{1}{4g_{10}}\left(F^+_{ab}\Gamma^{ab}\epsilon_+ + F^-_{ab}\Gamma^{ab}\epsilon_-\right) \tag{3.2.13}$$

which implies, by $\epsilon = \epsilon_\pm$,

$$F^\pm_{ab} = 0 \tag{3.2.14}$$

Finally, plugging (3.2.12) into the equation $0 = \delta_\epsilon\tilde{\Psi}_M$ gives

$$0 = \mathrm{D}_a\epsilon_\pm - \frac{1}{2}\Gamma_a{}^d\partial_d\Phi\,\epsilon_\pm \qquad \text{and} \qquad 0 = \mathrm{D}_\mu\epsilon_\pm \tag{3.2.15}$$

Setting $G^H_{ab} = e^{-\Phi}G^0_{ab}$, the left equation is equivalent to

$$0 = \mathrm{D}^0_a\epsilon_\pm \tag{3.2.16}$$

where D^0_a is the covariant derivative with respect to the connection of the metric G^0_{ab}. As we have already seen in the beginning of section 2.6, chiral spinors of $\mathrm{Spin}(4) = \mathrm{SU}(2)_1 \times \mathrm{SU}(2)_2$ transform as $(\mathbf{2},\mathbf{1})$ or $(\mathbf{1},\mathbf{2})$. In general, covariant constancy of a spinor means that the result of parallel transporting the spinor around any closed loop over the manifold (including not transporting it at all) will be independent of the chosen loop. This is the same as the holonomy of the manifold acting trivially on the spinor. Therefore, if there is a covariant constant spinor, say, in the $(\mathbf{2},\mathbf{1})$, the holonomy, and with it the curvature two-form, must be confined to $\mathrm{SU}(2)_2$. Since the curvature two-form takes values in a two-tensor of $\mathrm{SO}(4)$, which transforms as $(\mathbf{3},\mathbf{1})+(\mathbf{1},\mathbf{3})$, it must be self-dual or anti-self-dual in order to fit into a single $\mathrm{SU}(2)$. Therefore, if $\epsilon = \epsilon_+$ has positive chirality, the curvature must be anti-self-dual and, if $\epsilon = \epsilon_-$ has negative chirality, self-dual. We note that the curvature tensor, from (3.2.14), shares this property with the curvature tensor of the gauge connection. Therefore, we have

$$F^\pm_{ab} = 0 \qquad R^{0\pm}_{ab} = 0 \tag{3.2.17}$$

where R^0 has to be evaluated using the metric G^0.

Finally, (3.2.12) implies that the equation $\mathrm{d}H = \frac{\alpha'}{4}(\omega_{4L} - \omega_{4Y})$ becomes

$$\Box\Phi = \pm\frac{1}{2\sqrt{2}\kappa_{10}}\frac{\alpha'}{4}\frac{1}{4}\left(\mathrm{tr}\,R^H_{ab}R^H_{cd} - \mathrm{tr}\,F_{ab}F_{cd}\right)\epsilon^{abcd} \tag{3.2.18}$$

where $\Box = \nabla^e\nabla_e$.

From these equations, there are basically two possibilities to find supersymmetric solutions of heterotic string theory. The first approach, which is natural from the viewpoint of string theory as such, is to find solutions perturbatively in α'. In this approach, (3.2.18) reduces to $\Box\Phi = 0$ and the other equations are

$F^{\pm}_{ab} = R^{H\pm}_{ab} = 0$. Of course, at first order in α', (3.2.18) will have to be fulfilled and now reads

$$\Box\Phi = -\frac{1}{2\sqrt{2}\kappa_{10}}\frac{\alpha'}{4}\frac{1}{2}\left(\mathrm{tr}\, R^H_{ab}R^{H\,ab} - \mathrm{tr}\, F_{ab}F^{ab}\right) \tag{3.2.19}$$

Therefore, curvature of the metric drives the dilaton to negative values and the string to weak coupling whereas gauge curvature drives the dilaton to positive values and the string to strong coupling. Since (3.2.19), by (3.2.3), implies charge of the three-form field strength H_{abc} at first order, this order in α' is the lowest order at which one can expect to find the proper quantum numbers of the solution. Extending a solution of zeroth order to include (3.2.3) is, even though a bit tedious, possible and can be extended to solutions of M-theory on $S^1/\mathbb{Z}$ (see [106] and references therein). We note that the α'-corrections given in (3.2.2), as noted at the beginning of section 3.2, provide only those corrections needed for anomaly cancellation, that is, topological consistency of the solution. Therefore, finding the solutions including all first order α'-corrections is an extremely difficult task.

The other possibility to solve the supersymmetry conditions and to overcome the problems of higher order α'-corrections is to consider effects of first order in α' from the beginning and hold the rhs. of (3.2.18) zero by embedding the spin connection into the gauge connection[7] [23]. These solutions correspond to enhanced $\mathcal{N} = (4,4)$ worldsheet supersymmetry and, because of the enhanced symmetry, one expects the solution to provide an exact solution of heterotic string theory (see chapter 3 of [23]). However, it is clear, that those solutions provide only a very limited subset of all states in the spectrum which preserve eight supersymmetries.

3.3 KK-Monopoles in Heterotic String Theory

In light of the conditions for supersymmetric solutions given in the last section, Kaluza-Klein-monopoles provide a good candidate solution for heterotic string theory: they have an (anti-) self-dual spin connection and approach $\mathbb{R}^3 \times S^1$ Kaluza-Klein backgrounds at $r \to \infty$. To be solutions, they therefore should appear with their correct quantum numbers in all toroidal compactification of heterotic string theory from three to eight non-compact space-like directions. Hence, to identify the solution, one has to analyze toroidal compactifications of the heterotic string.

[7] To be precise, at first order in α' one has to embed the generalized connection ${\omega_a}^{bc} + {H_a}^{bc}$ into the gauge connection; see [23].

The Charge Lattice of Toroidal Compactifications

The full spectrum and moduli space of toroidal heterotic compactifications has been computed in the breakthrough publications of Narain [69] and Narain, Sarmadi and Witten [72] (see chapters 8 and 11 of [76]). Here, we shall use the formalism of [67] as used in [84, 85].

In a compactification of the heterotic string on the d-dimensional torus T^d we have to treat the worldsheet bosons (Z_R^m, Z_L^m, H^I) on an equal footing [69] (we use $m = 9-d+1, \ldots, 9$ and $\mu = 0, \ldots, 9-d$): the lattice momenta (k_R^m, k_L^m, k^I), converted to dimensionless quantities as in (2.1.27), have to span an even self-dual lattice Λ of the signature $(d, 16+d)$. All such lattices are equivalent to each other and therefore to a basis lattice Λ_0 by $\mathrm{O}(d, 16+d)$ rotations and the moduli space of the heterotic compactification is given as

$$\mathrm{O}(\mathbb{Z}, d, 16+d) \backslash \mathrm{O}(d, 16+d) / \mathrm{O}(d) \times \mathrm{O}(16+d) \tag{3.3.1}$$

where $\mathrm{O}(\mathbb{Z}, d, 16+d)$ denotes the subset of $\mathrm{O}(d, 16+d)$ rotations that leave the lattice Λ_0 invariant.

We choose Λ_0 in a canonical way for $\mathrm{E}_8 \times \mathrm{E}_8$ heterotic string theory[8]: we let the lattice momenta k^I live on $\Gamma_8 \times \Gamma_8$ and constrain the coordinates X^m to lie on the torus lattice Γ_{T^d} defined by

$$X^m = X^m + 2\pi \left(\frac{\alpha'}{2}\right)^{1/2} \tag{3.3.2}$$

By the same argument as in the untwisted sector of orbifolds, we have for winding and momentum (see (2.1.34) and (2.1.26))

$$\hat{w} \in \frac{1}{2\pi}\Gamma_{T^d} \qquad \hat{n} \in 2\pi\Gamma_{T^d}^* \tag{3.3.3}$$

and in dimensionless quantities

$$w^m = \left(\frac{\alpha'}{2}\right)^{-1/2} \hat{w}^m \in \mathbb{Z}^d \qquad n_m = \left(\frac{\alpha'}{2}\right)^{+1/2} \hat{n}_m \in \mathbb{Z}^d \tag{3.3.4}$$

The dimensionless lattice momenta are then

$$\begin{aligned} k_R^m &= n^m + w^m/2 & \qquad n_m &= \frac{1}{2}(k_{Rm} + k_{Lm}) \\ k_L^m &= n^m - w^m/2 & \qquad w^m &= k_R^m - k_L^m \end{aligned} \tag{3.3.5}$$

[8] By the above construction, heterotic $\mathrm{E}_8 \times \mathrm{E}_8$ string theory is clearly equivalent to $\mathrm{Spin}(32)/\mathbb{Z}_2$ heterotic string theory upon compactification on T^d.

We let the lattice Λ_0 be given by the vectors

$$k_0 = \begin{pmatrix} n^0_{\underline{m}} \\ w_0^{\underline{m}} \\ k_0^I \end{pmatrix} = \begin{pmatrix} n_{\underline{m}} \\ w^{\underline{m}} \\ q^I \end{pmatrix} \qquad n_{\underline{m}}, w^{\underline{m}} \in \mathbb{Z},\ (q^I) \in \Gamma_8 \times \Gamma_8 \tag{3.3.6}$$

where we have used that, on our standard torus, vielbein indices $\underline{m}$ are identical to curved indices m. On this lattice, the $(d, d+16)$ metric L reads

$$L = \begin{pmatrix} & \mathrm{I}_d & \\ \mathrm{I}_d & & \\ & & -\mathrm{I}_{16} \end{pmatrix} \qquad \text{with} \qquad L^2 = \mathrm{I} \tag{3.3.7}$$

and we have, from (3.3.5),

$$k_0^T L k_0 = 2 n_m w^m - q^I q^I = k_R^2 - k_L^2 - k^I k^I \tag{3.3.8}$$

which shows that the signature of L is $(d, d+16)$ indeed.

On a general torus, we simply have to define vielbeine $e_m{}^{\underline{m}}$ and $\hat{e}^m{}_{\underline{m}}$ with $e_p{}^{\underline{m}} \hat{e}^q{}_{\underline{m}} = \delta_p^q$ where the torus metric is given as $G_{mn} = e_m{}^{\underline{p}} e_n{}^{\underline{q}} \delta_{\underline{pq}}$. In this case, the lattice momenta are

$$\begin{pmatrix} n_m \\ w^m \\ k^I \end{pmatrix} = E\underline{k} = E \begin{pmatrix} n_{\underline{m}} \\ w^{\underline{m}} \\ q^I \end{pmatrix} \qquad E = \begin{pmatrix} e & & \\ & \hat{e} & \\ & & \mathrm{I}_{16} \end{pmatrix} \tag{3.3.9}$$

The key point in this formalism is that the matrix E satisfies $E^T L E = L$ and therefore is an element of $\mathrm{SO}(d, d+16)$ with respect to the metric L. Defining

$$\hat{E} = \begin{pmatrix} \hat{e} & & \\ & e & \\ & & \mathrm{I}_{16} \end{pmatrix} \tag{3.3.10}$$

we have the relations

$$E^T L E = L \qquad \hat{E} E^T = \mathrm{I} \qquad \hat{E}^T L \hat{E} = L \tag{3.3.11}$$

and

$$E E^T = \begin{pmatrix} G & & \\ & G^{-1} & \\ & & \mathrm{I}_{16} \end{pmatrix} \qquad \hat{E} \hat{E}^T = \begin{pmatrix} G^{-1} & & \\ & G & \\ & & \mathrm{I}_{16} \end{pmatrix} \tag{3.3.12}$$

As shown further in [69, 72], the general lattice Λ is given as $\Lambda = ED\Lambda_0$ and the lattice momenta are therefore given as $k = (n, w, k) = EDk_0$ where

$$D = \begin{pmatrix} \mathrm{I}_d & \frac{1}{2}(B+C) & A \\ & \mathrm{I}_d & \\ & A & \mathrm{I}_{16} \end{pmatrix} \qquad C_{\underline{mn}} = A^I_{\underline{m}} A^I_{\underline{n}} \tag{3.3.13}$$

satisfies $D^T L D = L$ and therefore also is an element of $\mathrm{SO}(d, d+16)$. The $B_{\underline{mn}}$ define a constant B-field background and the $A^I_{\underline{m}}$ provide a Wilson line background. For the flat momenta $\underline{k} = Dk_0$ converted back to (k_R, k_L, k) via (3.3.5), we have the perhaps more familiar formulas (see, for example, section 11.6 of [76])

$$\begin{aligned} k_{R\underline{m}} &= n_{\underline{m}} + (+\delta_{\underline{mn}} + B_{\underline{mn}})\frac{w^{\underline{n}}}{2} + q^I A^I_{\underline{m}} + A^I_{\underline{m}} A^I_{\underline{n}} \frac{w^{\underline{n}}}{2} \\ k_{L\underline{m}} &= n_{\underline{m}} + (-\delta_{\underline{mn}} + B_{\underline{mn}})\frac{w^{\underline{n}}}{2} + q^I A^I_{\underline{m}} + A^I_{\underline{m}} A^I_{\underline{n}} \frac{w^{\underline{n}}}{2} \\ k^I &= q^I + w^{\underline{m}} A^I_{\underline{m}} \end{aligned} \tag{3.3.14}$$

As we have already seen in section 2.6, metric G_{MN}, B-field B_{MN} and the generators of the Cartan subalgebra A^I_M correspond to the states

$$\alpha^M_{-1} \tilde{\Psi}^N_{-1/2} \big|0\big\rangle_{\tilde{N}S} \qquad \gamma^I_{-1} \tilde{\Psi}^N_{-1/2} \big|0\big\rangle_{\tilde{N}S} \tag{3.3.15}$$

To reveal the properties of the left states, since we have to deal with fermionic vertex operators, we have to discuss the vertex operators in their different pictures (for all this, see chapter 12 of [76]). Happily, we can treat these precisely as in the purely bosonic case (see section 12.3 of [76]), where the corresponding states are

$$\alpha^N_{-1} \tilde{\alpha}^N_{-1} \big|0\big\rangle \tag{3.3.16}$$

This implies (see section 8.3 of [76]), that $B_\mu{}^m$ measures the winding charge given by w^m and $G_\mu{}^m$ measures the Kaluza-Klein or (compact) momentum charge given by n_m.

Finally, since all charges so far were electric charges, we have to discuss the possible magnetic charges $\tilde{k}$ appearing in the theory. These are constrained by the requirement of Dirac-Schwinger-Zwanziger charge quantization in four dimensions [32, 80, 81, 110, 111] or its higher dimensional analog [73, 99]

$$k^T \tilde{k} \in \mathbb{Z} \tag{3.3.17}$$

Since $k = \hat{E}Dk_0$ is related to k_0 by a $\mathrm{SO}(d, d+16)$ rotation, this requirement is easy to solve by the ansatz $\tilde{k} = LEDL\tilde{k}_0$ which implies $\tilde{k} = \tilde{k}_0$ in case of trivial background fields. Then we have $k^T \tilde{k} = k_0^T (ED)^T L (ED) \tilde{k}_0 = k_0 LL\tilde{k}_0$.

Now, since Λ_0 is an self-dual lattice with respect to the metric L, $L\tilde{k}_0$ must be an element of Λ_0. Therefore $\tilde{k}_0$ is an element of $L\Lambda_0$ and reads

$$\tilde{k}_0 = \begin{pmatrix} \tilde{n}_0^{\underline{m}} \\ \tilde{w}^0_{\underline{m}} \\ \tilde{k}_0^I \end{pmatrix} = \begin{pmatrix} \tilde{n}^{\underline{m}} \\ \tilde{w}_{\underline{m}} \\ \tilde{q}^I \end{pmatrix} \qquad \tilde{n}_{\underline{m}}, \tilde{w}_{\underline{m}} \in \mathbb{Z}, \; (\tilde{q}^I) \in \Gamma_8 \times \Gamma_8 \tag{3.3.18}$$

We have chosen our notation such that the quantization condition (3.3.17) is given as

$$\begin{aligned} k^T \tilde{k} &= n_m \tilde{n}^m + w^m \tilde{w}_m + k^I \tilde{k}^I \\ &= n^0_m \tilde{n}^m_0 + w^m_0 \tilde{w}^0_m + k^I_0 \tilde{k}^I_0 \\ &\in \mathbb{Z} \end{aligned} \tag{3.3.19}$$

Therefore, $\tilde{n}^m$ is the magnetic charge with respect to $G_\mu{}^m$, $\tilde{w}^m$ is the magnetic charge with respect to $B_\mu{}^m$ and $\tilde{k}^I$ is the magnetic charge with respect to A^I_μ.

BPS States in Toroidal Compactifications

As we have seen in section 3.2, the conditions for partly unbroken supersymmetry, which are called BPS conditions if the solution can be interpreted as a particle or extended object of a supersymmetric theory, can be very complicated in the low energy effective theory. In toroidal compactifications of the heterotic string, however, the conditions have a very simple form. Since their derivation requires a lot more technology which will not be important for our arguments, we shall not repeat the derivation here. Instead, the reader is referred to [76], section 11.6. The condition is simply

$$\frac{\alpha'}{4} G_{mn} p^m_R p^n_R = \frac{\alpha'}{4} m^2 \tag{3.3.20}$$

which states that the square of rightmoving lattice momentum of the compact directions is given by the mass squared of the state. The Virasoro generators in the (σ, τ)-frame are given (see (2.2.42))

$$\begin{aligned} T_0 &= \frac{\alpha'}{4} G_{\mu\nu} p^m_L p^n_L + \frac{1}{2} G_{mn} k^m_L k^n_L + \frac{1}{2} k^I k^I + N - 1 \\ \tilde{T}_0 &= \frac{\alpha'}{4} G_{\mu\nu} p^m_R p^n_R + \frac{1}{2} G_{mn} k^m_R k^n_R + \frac{1}{2} \tilde{N} - \frac{1}{2} \end{aligned} \tag{3.3.21}$$

where N counts all integer valued leftmoving oscillator excitations and $\tilde{N}$ contains all rightmoving oscillator excitations together with the lattice excitations of the rightmoving fermions. Upon imposing (3.3.20), together with the physical state condition $T_0 = \tilde{T}_0 = 0$, we have $\tilde{N} = 1$. Since the μ directions are indeed non-compact, we have $G_{\mu\nu} p_R^\mu p_R^\nu = G_{\mu\nu} p_L^\mu p_L^\nu$ and arrive at (see (3.3.8))

$$\begin{aligned} 2N - 2 &= G_{mn} k_R^m k_R^n - G_{mn} k_L^m k_L^n - k^I k^I \\ &= k^T L k \\ &= 2 n_m^0 w_0^m - q^I q^I \end{aligned} \tag{3.3.22}$$

Therefore, the BPS condition for the electrically charged, particle like states is (since $N \geq 0$)

$$k^T L k = 2 n_m^0 w_0^m - q^I q^I \geq -2 \tag{3.3.23}$$

Finally, we turn to magnetically charged states dual to those of (3.3.23). As the electrically charged BPS states were particle like, their charges are integrals over two-form field strength F_2. Therefore, in D-dimensional flat space, the magnetically dual states have charges with respect to the $(D-2)$-form dual field strength $(*F_2)_{D-2}$, whose potential is a $(D-3)$-form. Therefore, the magnetically charged states of charge $\tilde{k}$ are extended objects of $D-4$ space-like dimensions.

This implies that the case of $D = 4$ is very special, since particles may carry both, magnetic and electric charges. In addition, the supersymmetry algebra of $\mathcal{N} = 4$, $D = 4$ supersymmetry allows for an $\mathrm{Sl}(2, \mathbb{Z})$ symmetry that treats magnetical and electrical charges on an equal footing and especially contains an element that exchanges k and $\tilde{k}$ quantum numbers (see [84] and references therein). In fact, it is conjectured that this symmetry is an exact symmetry of $\mathcal{N} = 4$, $D = 4$ Super-Yang-Mills theory and heterotic compactifications on T^6. Here we need only the proven fact that the BPS condition can be written in a manifestly $\mathrm{Sl}(2, \mathbb{Z})$ invariant way and therefore has precisely the same form as (3.3.23):

$$\tilde{k}^T L \tilde{k} = 2 \tilde{n}_0^m \tilde{w}_m^0 - \tilde{q}^I \tilde{q}^I \geq -2 \tag{3.3.24}$$

Even more, since this condition is a statement on supersymmetry alone[9], and supersymmetry is preserved in toroidal compactifications, (3.3.24) is valid for all $D \geq 4$.

Of course, for $D > 4$, there might in general be other electrically or magnetically charged BPS states which are neither particles nor $D-4$ dimensional extended objects. But since we will not need these states, we do not consider them here in any further detail.

Identifying the Kaluza-Klein-Monopole

As stated in the introduction to this section, to identify the Kaluza-Klein-monopole solution in heterotic string theory, we have to compute its correct

[9] This is a very powerful argument, which was used as a key argument for the existence of M-theory in [104].

quantum numbers and show that these are consistent with those of toroidal compactifications. For KK-monopoles with no Wilson lines switched on, the identification which we shall present has been carried out in [86]. We start by a compactification on $S^1 = T^1$ to $D = 9$ and identify the $6 \dots 9$ directions with the $1 \dots 4$ directions of the K_1 solution (3.1.1) or (3.1.8). Therefore, the solution has $8 - 3 = 5$ non-compact space-like directions, which is precisely right to carry magnetic charge of the kind studied above.

By construction, this solution carries one unit of magnetical Kaluza-Klein charge $\tilde{n} = 1$. To zeroth order in α', this would be the only magnetic charge of the solution, but, as argued in section 3.2, we have to look at first order in α' to make contact to string theory. At this order, by (3.2.3), we have

$$\mathrm{d}H = \frac{\alpha'}{4} \operatorname{tr}(R^H)^2 \tag{3.3.25}$$

which, by a detailed straightforward calculation, implies a magnetic charge of -1 unit with respect to $B_{\mu 9}$, since the single KK-monopole solution carries one unit of gravitational instanton number [86].

Plugging the quantum numbers $\tilde{w} = -1, \tilde{n} = 1$ into the BPS condition (3.3.24), we have

$$-2 = 2\tilde{N} - 2 = 2\tilde{n}^m \tilde{w}_m \qquad \tilde{N} = 0 \tag{3.3.26}$$

and we expect this to be the exact result, by the standard argument of nonrenormalization of the BPS condition [109] and the fact that all terms of topological relevance have been included in the calculation. Furthermore, since the calculation involved only four coordinates, in compactifications on higher dimensional tori T^d the calculation goes through as long as we do not switch on background fields which mix the individual S^1 factors of $T^d = (S^1)^d$.

Despite these results, it should be clear that simply showing that these states fit into the spectrum does not necessarily mean that they are actually present. However, there are indeed very powerful arguments as to why such states have to be present in the spectrum:

- Enhanced gauge symmetries at critical radii

  As a consequence of Narain compactification [69] on a single S^1 as presented above, at special points in the moduli space a U(1) symmetry can be enhanced to a full SU(2) gauge symmetry. This can already be achieved by varying the size of the S^1. The W^+ and W^- gauge bosons which need to become massless at these points are provided by BPS winding and momentum modes[10] with the quantum numbers $n = -w = 1$ and $n = -w = -1$ (see, for example, [76], section 8.3). But since an SU(2) Yang-Mills theory

[10] That these modes have to be BPS is a trivial consequence of the fact that massless multiplets for more than eight supersymmetries only have a number of degrees of freedom that is the square root of the number of degrees of freedom of a massive multiplet.

higgsed to U(1), especially with high supersymmetry, contains BPS magnetic monopoles which precisely have the quantum numbers $\tilde{n} = -\tilde{w} = \pm 1$, such states have to be present and are therefore identified with Kaluza-Klein-monopoles [88]. Since the explicit supergravity solution is only valid for R much larger than the string scale $\sqrt{\alpha'}$, whereas the description of the Higgs effect in terms of a Yang-Mills theory is only valid as long as the radius of the S^1 is of order of the string scale, this identification provides a very powerful tool, which allows for the calculation of the moduli space of heterotic string theory on orbifold singularities without gauge background [88].

- Heterotic Type II Duality

  This duality, which by now has become widely accepted and is supported by great amount of spectacular successes, relates heterotic string theory compactified on T^4 to type II string theory on K3 (see every reference on nonperturbative string theory, such as [76] or [89]). Especially electrically charged states of one theory are identified with magnetically charged states of the other theory and vice versa.

- $\mathrm{Sl}(2,\mathbb{Z})$ S-duality of $\mathcal{N} = 4$, $D = 4$ Super-Yang-Mills and heterotic string theory on T^6

  In this duality, which goes back to the old duality conjecture of Montonen and Olive [68, 48] is now expected to be an exact duality of the above theories. The action of $\mathrm{Sl}(2,\mathbb{Z})$ acts on electric and magnetic charges combined into one vector $(k, \tilde{k})$. It therefore predicts the existence of whole orbits of dyonic states, that is, states charged both electrically and magnetically. As has been shown in [86], KK-monopoles provide the basic solution in the construction of many of these states.

Of the above arguments, in particular $\mathrm{Sl}(2,\mathbb{Z})$ duality provides a powerful test of the identification of the KK-monopole with states in toroidal compactifications: if $\mathrm{Sl}(2,\mathbb{Z})$ symmetry is expected to hold exactly, states in the same orbit should share the same degeneracy of the ground state. As we have seen above in (3.3.26), $\tilde{N} = 0$ and therefore the degeneracy of the KK-monopole solution should correspond to an elementary string state with $N = 0$, that is, no leftmoving oscillators excited. As the rightmoving sector transforms as $\mathbf{8} + \mathbf{8}_+$ of the little group SO(8), in such states, the degeneracy has to be 16.

To calculate the degeneracy of the KK-monopole solution, we have to use the fact that the solution ist constructed to preserve eight supersymmetries (order by order, of course). That this is possible is a mild assumption since the solution approaches a flat background and there are no global topological consistency conditions. Indeed, as the solution is smooth and has a nonvanishing gravitational instanton number, its holonomy is SU(2) and not just a subgroup of

SU(2). Therefore, it breaks precisely eight supersymmetries. But, as explained in section 2.5 of [98], this implies the existence of eight fermionic zero modes on the solution, whose quantization gives a $2^4 = 16$-fold degenerate state (for the KK-monopole this was computed first in [58]).

Turning on Wilson Lines

Despite of the success of the arguments given above, there remains a little conundrum: by the formulas (3.3.14) or (3.3.13) of Narain compactification, by turning on a Wilson line in the 9 direction, we get for a state of quantum numbers $\tilde{n}_0^9 = 1$, $\tilde{w}_9^0 = -1$

$$\begin{aligned} \tilde{n}^9 &= \tilde{n}_0^9 = 1 \\ \tilde{w}_9 &= \tilde{w}_9^0 + \tfrac{1}{2} A_9^I A_9^I \tilde{n}_0^9 = -1 + \tfrac{1}{2} A_9^I A_9^I \\ \tilde{k}^I &= \tilde{n}^9 A_9^I = A_9^I \end{aligned} \tag{3.3.27}$$

(this can be computed easily as if the charges were electric: only n and w have to be exchanged). As $\tilde{w}_9$ is the magnetic charge of $B_{\mu 9}$, which is determined by $\mathrm{d}H = \frac{\alpha'}{4} \operatorname{tr} R^2$ (3.2.3), a Wilson line has to induce an abelian anti-self-dual instanton on the background of the K_1 solution which provides the source for the H-field. In addition, this solution has now magnetical gauge charge proportional to the Wilson line. As shown in appendix C.1, (C.1.24) and (C.1.25), this is provided by the following gauge field

$$A^I = a^I \, U^{1/2} \, e^{\underline{4}} \tag{3.3.28}$$

where $e^{\underline{m}}$ is the vielbein on K_N as given in (C.1.6) in the appendix and a^I is the Wilson background at $r \to \infty$. From this we easily get the gauge field strength (see (C.1.24))

$$F^I = \mathrm{d}A^I = U^{-2} \left(-\partial_k U e^{\underline{k}} \wedge e^{\underline{4}} + \frac{1}{2} \epsilon_{mnk} \partial_k U e^{\underline{m}} \wedge e^{\underline{n}} \right) a^I \tag{3.3.29}$$

and the magnetic gauge charge

$$\int_{S^2 \to \infty} F^I = -\frac{1}{2} (2\pi R) a^I \tag{3.3.30}$$

and the integral

$$\int F^I \wedge F^I = -(2\pi R)^2 a^I a^I \tag{3.3.31}$$

Up to irrelevant constants, this confirms the quantum numbers of (3.3.27).

To analyze this background in the core region of the K_1 region, we transform to the coordinates of (3.1.10) and take the $r \to 0$ limit, which results in

$$U^{-1/2} e^{\underline{4}} = R^{-1}(r_1^2 \mathrm{d}\phi_1 - r_2^2 \mathrm{d}\phi_2) \qquad (3.3.32)$$

Therefore, we have a gauge background whose potential goes linearly with the distance to the origin. As this calculation goes through for the K_N case, we learn that we have a non-constant gauge background around the orbifold singularity. However, as we have seen in chapter 2, away from the orbifold singularity heterotic string theory on orbifolds behaves exactly like a string theory on a flat space. Therefore, despite other claims in the recent literature [49], even though the gauge shift corresponding to a fixed point provides exactly the data corresponding to a Wilson line, an orbifold singularity with a gauge shift is totally different from a orbifold singularity at the core of a KK-monopole in a Wilson line background. This is also clear from the fact that the Eguchi-Hanson space X_N supports anti-self-dual abelian instantons after blowing up the orbifold singularity whereas on K3 the cohomology classes of the S^2 after blowing up are not anti-self-dual (see the end of section 2.1 in [17]).

3.4 KK-Monopoles with Non-Abelian Instantons

As we have have seen in section 2.5, the level matching condition of heterotic orbifolds is deeply related to fractional instanton numbers of non-abelian instantons sitting on orbifold singularities (That these are non-abelian is clear from the analysis of the last section.). To analyze that relationship in greater detail, we now study non-abelian instantons on (multiple) KK-monopole backgrounds.

Our starting point is the standard embedding, where the SU(2) anti-self-dual spin connection is embedded into an SU(2) subgroup[11] of $E_8 \times E_8$. From the quantum numbers of the "bare" KK-monopole $\tilde{n} = +1, \tilde{w} = -1$ and the fact that the density of the gauge instanton number enters the anomalous Bianchi identity (3.2.3) with opposite sign compared to that of the gravitational instanton number, we expect that the magnetic winding number -1 is canceled by the instanton number of the gauge instanton [86]. As it turns out, for heterotic strings on T^6, these states provide precisely the dyonic states predicted by S-duality in four dimensions [86]. The BPS condition (3.3.24) is

$$0 = 2\tilde{N} - 2 = 2\tilde{n}^m \tilde{w}_m \qquad \tilde{N} = 1 \qquad (3.4.1)$$

Since now $\tilde{N} = 1$, $\mathrm{Sl}(2,\mathbb{Z})$ duality, as discussed in the last section, implies that the degeneracy of this state should be exactly the same as that of an elementary

[11] At the classical level this is equivalent to the same construction for the $\mathrm{Spin}(32)/\mathbb{Z}_2$ heterotic string theory.

string state with $N = 1$. Since there are 24 leftmoving oscillators available, the degeneracy is $16 \cdot 24$.

In fact, explaining this degeneracy is an *extremely difficult problem* which was addressed first in the calculation of the degeneracy of H-monopoles [15, 44]. Later, as it became clear how small instantons of the heterotic $\mathrm{Spin}(32)/\mathbb{Z}_2$ theory are related to the 5-brane solution of heterotic $\mathrm{Spin}(32)/\mathbb{Z}_2$ theory [105], arguments were given that the desired degeneracy arises from singularities in the moduli space of the $\mathrm{Spin}(32)/\mathbb{Z}_2$ heterotic 5-brane [105, 90, 77]. In [77], the problem was also addressed in the context of heterotic type II duality.

All this provides mounting evidence that the degeneracy of an instanton sitting on a KK-monopole indeed is enhanced by a factor of 24 compared to the bare KK-monopole. However, as we shall see below, $\mathrm{E}_8 \times \mathrm{E}_8$ heterotic string theory behaves quite different from $\mathrm{Spin}(32)/\mathbb{Z}_2$ theory, since deforming the instanton away from the standard embedding explicitly allows for $\mathrm{E}_8 \times \mathrm{E}_8$ instantons to become pointlike. At such points in the moduli space, $\mathrm{E}_8 \times \mathrm{E}_8$ is expected to undergo a phase transition where the theory at the transition point is a theory of tensionless strings [83] and an infinite number of states becomes massless.

The t'Hooft Ansatz

We start by the t'Hooft ansatz (see, for example [37], section 9) on the K_N background

$$A_a{}^{\underline{i}} = -s_{ab}{}^{\underline{i}} G^{bc} \partial_c \ln g \tag{3.4.2}$$

where $A^{\underline{i}}$ is considered as gauge potential in the adjoint of SU(2) with field strength

$$F^{\underline{i}} = \mathrm{d}A^{\underline{i}} + \frac{1}{2}\epsilon^{ijk} A^{\underline{j}} \wedge A^{\underline{j}} \tag{3.4.3}$$

and the $s^{\underline{i}}$ comprise the (self-dual) hyperkähler structure[12] of K_N (see (C.1.19))

$$s^{\underline{i}} = e^{\underline{i}} \wedge e^{\underline{4}} + \frac{1}{2}\epsilon^{ijk} e^{\underline{j}} \wedge e^{\underline{k}} \tag{3.4.4}$$

To discuss the t'Hooft ansatz in curved space, we begin by flat $\mathbb{R}^4$ equipped with the vielbein $e_a{}^{\underline{b}} = \delta_a^b$ where we do not have to distinguish flat and curved indices. Then, upon imposing anti-self-duality $F^i = -{*}F^i$, after a straightforward by lengthy calculation, the ansatz reduces to

$$g^{-1}\Box g = 0 \tag{3.4.5}$$

Anti-self-duality, by the Bianchi identity $\mathrm{D}F = 0$, implies the fulfillment of the equations of motion $\mathrm{D}{*}F = 0$.

[12]In the context of the t'Hooft ansatz these are often called t'Hooft tensor.

In curved space, we have to replace all derivatives by covariant derivatives and get additional terms from the covariant derivative of the hyperkähler structure. After a moderate, but tricky calculation using the closedness of the $s^{\underline{i}}$ (see appendix C.1) and some standard vielbein calculus, one again arrives at

$$g^{-1}\Box g = 0 \tag{3.4.6}$$

where now $\Box$ is the covariant Laplace operator. On the K_N background, $\Box g$ is given as (see (C.1.14))

$$\Box g = U^{-1}\big(\Delta g - 2\vec{\omega}\cdot\vec{\partial}g' + (U^2+\omega^2)g''\big) \tag{3.4.7}$$

where a prime denotes differentiation with respect to x^4 and $\Delta = \partial_i\partial_i$. Using this operator, it seems quite impossible to solve (3.4.6) in general, except for the case where g is independent of x^4 and the ansatz reduces to

$$g^{-1}U^{-1}\Delta g = 0 \tag{3.4.8}$$

Happily, we are already equipped with a solution of this equations, simply by setting $g = U$.

From (3.4.2), the explicit form of the gauge potential is

$$\begin{aligned} A_4{}^{\underline{i}} &= U^{-2}\partial_i U \\ A_j{}^{\underline{i}} &= U^{-2}\omega_j\partial_i U - U^{-1}\epsilon_{ijk}\partial_k U \end{aligned} \tag{3.4.9}$$

From the spin connection of the K_N solution (C.1.9), we have

$$\begin{aligned} \omega_{4\underline{4i}} &= -\tfrac{1}{2}U^{-2}\partial_i U &&= -\tfrac{1}{2}A_4{}^{\underline{i}} \\ \omega_{j\underline{4i}} &= -\tfrac{1}{2}U^{-2}\omega_j\partial_i U + \tfrac{1}{2}U^{-1}\epsilon_{ijk}\partial_k U &&= -\tfrac{1}{2}A_j{}^{\underline{i}} \end{aligned} \tag{3.4.10}$$

Which identifies the gauge instanton $g = U$ as the standard embedding. Therefore, from (3.1.2) and (3.1.3), a solution of the t'Hooft ansatz (3.4.6) on the K_N background is given by

$$g = 1 + \sum_{I=1}^{N}\frac{R/2}{|\vec{r}-\vec{r}_I|} \tag{3.4.11}$$

This is a very interesting result, since this ansatz is known to produce divergent solutions in field theory (see [64, 45], especially for a discussion of solutions of this kind in string theory). Therefore, it seems that this g is related to the details of the K_N background and it is not clear what the moduli space of the instanton is.

However, as we have seen in section 3.1, in the $N = 1$ case we have the possibility to make a coordinate transformation that resolves the apparent singularity at $\vec{r} = 0$. From (3.1.10) we have for the euclidian distance ϱ from the origin $\varrho^2 = r_1^2 + r_2^2 = 2Rr$. Therefore, in that coordinates, g is given as

$$g = 1 + \frac{R^2}{\varrho^2} \tag{3.4.12}$$

which is just the ansatz for an SU(2) instanton in flat space of scale R. This shows that there is absolutely no problem in scaling the instanton to an arbitrary scale (at least classically), since any singularity at the origin is resolved precisely as in the flat space case. Now, this argument turned the other way round, we can make the scale R of the KK-monopole solution arbitrary big, where, now even in a quantum theory, an instanton sitting at the origin with a fixed scale λ is well approximated by the classical solution. As the scale R gets very big, the instanton is effectively point like (but still macroscopic with respect to the string scale) and therefore some part of its moduli space should be identical to the background space K_N. This should even continue to hold as we increase the scale size of the instanton to values where the one in

$$g = 1 + \frac{\lambda^2}{\varrho^2} \tag{3.4.13}$$

can be neglected, since in that case the scale size drops out of the calculation and the moduli space, by the Kronheimer-Nakajima theorem[13] of [65], is known to be identical to X_1 (even to X_N in case of K_N) as long as the instanton position is confined to a region of scale less than R within the origin, or, in other words, as long as the instanton and the KK-monopole sit on top of each other. This is also compatible with the assumption that the moduli space of an instanton whose center is very far from the origin of the KK-monopole in units of R and whose scale size is smaller than that distance locally is $\mathbb{R}^3 \times S^1$. In that case g should approximate

$$g(\vec{r}, x^4) = 1 + \frac{\frac{\tau^2}{2Rd} \sinh \frac{d}{R}}{\cosh \frac{d}{R} - \cos \frac{x^4 - x^4_0}{R}} \tag{3.4.14}$$

where $(\vec{r}_0, x^4_0)$ is the position of the instanton and $d = |\vec{r} - \vec{r}_0|$ is the distance to the instanton core. This formula can be obtained from summing over an infinite number of instantons located at $x^4 = x^4_0 + 2\pi R\mathbb{Z}$ (see [45]).

In conclusion, we propose that the eight parameters of the t'Hooft ansatz give precisely the classical one instanton moduli space on K_1.

Small $E_8 \times E_8$ Instantons

As we have seen above, classically, there is no problem in scaling an instanton down to arbitrary small size. In string theory, however, special things might happen. In case of the Spin(32)/$\mathbb{Z}_2$ heterotic string theory, this can be analyzed purely by arguments of $\mathcal{N} = (0,1)$, $D = 6$ supersymmetry and the effect of

[13] It would be interesting to extend the Kronheimer-Nakajima construction of instantons of X_N to K_N.

higgsing on the moduli space. This argument suggests that a nonperturbative SU(2) enhanced gauge symmetry appears in the zero scale size limit [105]. In somewhat sloppy words, the instanton shrinks to zero size and turns into a Spin(32)/$\mathbb{Z}_2$ 5-brane that carries the SU(2) gauge theory on its worldvolume. In $E_8 \times E_8$ string theory from the M-theory on $S^1/\mathbb{Z}_2$ perspective, however, the 5-brane that appears as the instanton shrinks is sitting at the E_8 "end-of-the-world" 9-brane that produced the instanton. Since M-theory membranes which stretch from the M-5-brane to the 9-branes have zero length, they have zero tension from the ten-dimensional viewpoint where they therefore appear as tensionless strings with an infinite number of massless modes [83].

By the arguments given above, this should be possible for an instanton sitting on a KK-monopole. Hence, as the degeneracy of this state depends on the precise form of the moduli space and is especially sensitive to singularities in that space, the expected enhancement by a factor of 24 is an even more intriguing feature of heterotic $E_8 \times E_8$ string theory.

Wilson Lines

Again, as already in section 3.3 for the bare KK-monopole, we might switch on Wilson lines. As long as we embed the instanton in such a way into $E_8 \times E_8$ that the Wilson line commutes with the SU(2) of the instanton, all computations go though as before. However, the charges in toroidal compactifications do not at all depend on whether this is the case or not, and therefore we have to explain what happens if we turn on Wilson lines that do not commute with the instanton. As explained in [105], a non-abelian instanton in flat space has to turn to zero size when a Wilson line is switched on. As it is clear from (3.3.32), a Wilson line at $r \to \infty$ on a K_N background provides an abelian background on the whole of the space and therefore the instanton on K_N has to turn to zero size when a Wilson line with which it does not commute is switched on.

3.5 The Standard Embedding of Heterotic Orbifolds

We now will make contact to the standard embedding of heterotic orbifolds as presented in chapter 2 and especially in the examples of section 2.6. We begin by presenting standard arguments on the dynamics of KK-monopoles in string theory and M-theory, such as can be found in [88] or [87].

Type IIA Theory and M-Theory

We start by looking at a $\mathbb{Z}_N$ fixed point of type IIA orbifolds. As shown in section 2.6, such a fixed point supports $N-1$ massless vector multiplets of $\mathcal{N}=(1,1)$, $D=6$ supersymmetry. As noted in section 3.1, the K_N solution supports precisely N *normalizable* anti-self-dual harmonic two forms Ω_I. As these forms are harmonic, they immediately provide us with N solutions of the field equations of the B-field B_{MN} and the RR three-form C_{MNP} by the ansatz

$$B_{mn} = \sum_I \Omega_{Imn} b^I \qquad C_{\mu mn} = \sum_I \Omega_{Imn} c^I_\mu \tag{3.5.1}$$

(see, for example, [52], chapter 14). Together with the $3N$ parameters $\vec{r}_I$, this is precisely the bosonic content of N vector multiplets which therefore comprise a $\mathcal{N}=(1,1)$ supersymmetric $(\mathrm{U}(1))^N$ gauge theory in $D=6$. Since all modes are normalizable on K_N, when we bring all the $\vec{r}_I$ together to a distance of scale R, the modes will be localized at a scale R.

However, as we bring the $\vec{r}_I$ close together at a scale l much below the scale of R, at this scale around the center of the $\vec{r}_I$ the solution will look like the Eguchi-Hanson multi-center gravitational instanton X_N, and we have to distinguish modes which are normalizable with respect to X_N (at the scale l) from those which are only normalizable at the scale R. In terms of an low energy effective description at scales above R, the modes of scale l can be treated by an effective quantum action whereas the modes of scale R can be treated as classical background fields, since the interaction between the two should roughly be controlled by the ratio l/R.

As discussed in appendix C.2, (C.2.6) and (C.2.9), modes that fall faster than r^{-1} are normalizable on K_N whereas modes that fall faster than $r^{-3/4}$ are normalizable on X_N (and K_N).

As we have seen in the calculation of the gravitational instanton number in section 3.1, the function U can be expanded in multipole moments the leading term of which is given by $R/2r$ where $r=|\vec{r}-\vec{r}_{CM}|$ is the distance to the center of mass coordinate $\vec{r}_{CM}$. The other terms fall with $(r/l)^{-2}$ or faster. Therefore, the $N-1$ linearly independent modes that correspond to a variation of the relative coordinates $\vec{r}_I-\vec{r}_J$ without variation of $\vec{r}_{CM}$ fall with $(r/l)^{-2}$ and are normalizable on X_N whereas the displacement of the center off mass coordinate $\vec{r}_{CM}$ goes with $(r/l)^{-1}$ and corresponds to the dipole moment which is therefore normalizable only on K_N.

Therefore, the modes normalizable on X_N will be given by the relative coordinates $\vec{r}_I-\vec{r}_J$. There are precisely $N-1$ linearly independent such modes.

This essentially carries over to the two-forms Ω_I. Since Ω_I is given by (see (3.1.6))

$$\Omega_I = \partial_i(U^{-1}U_I)\left(e^{\underline{i}}\wedge e^{\underline{4}} - \frac{1}{2}\epsilon_{ijk}e^{\underline{j}}\wedge e^{\underline{k}}\right) \tag{3.5.2}$$

the difference $\Omega_I - \Omega_J$ is controlled by $U_I - U_J$ where the R/r term cancels. Since the remaining terms are falling like $(r/l)^{-1}$ or faster, $\Omega_I - \Omega_J$ falls like $(r/l)^{-2}$ or faster and the $N-1$ independent forms of $\Omega_I - \Omega_J$ are normalizable on X_N whereas $\sum_I \Omega_I$ is normalizable only on K_N and happens to be given by

$$\sum_I \Omega_I = -\mathrm{d}(U^{-1/2} e^{\underline{4}}) \tag{3.5.3}$$

In conclusion, the modes normalizable on X_N comprise a $(\mathrm{U}(1))^{N-1}$ supersymmetric gauge theory. At distances of scale R or greater, the system is described by a $(\mathrm{U}(1))^N$ supersymmetric gauge theory where one U(1) factor can be attributed to the center of mass motion of the whole solution.

As type II heterotic duality predicts the apparence of an SU(N) enhanced gauge symmetry at a $\mathbb{Z}_N$ orbifold singularity, precisely when all the vacuum expectation values of the bosonic scalars vanish (see, for example [104]), we expect the same to happen for N coincident KK-monopoles, where the enhanced gauge group is $U(N)$. Since type IIA theory is expected to be equivalent to M-theory on S^1, M-theory on a $\mathbb{Z}_N$ orbifold singularity also leads to an enhanced gauge symmetry of SU(N) [104], and the same argument applies to N coincident KK-monopoles in M-theory [88]. Since $\mathrm{U}(N) = \mathrm{SU}(N) \times \mathrm{U}(1)$ and the U(1) is attributed to the center of mass motion, the U(1) is expected to remain free in the infrared. This is completely analogous to the case U(N) gauge theories on D-branes as both cases are dual to each other when type IIA string theory is dual to M-theory on S^1 [87]. Of course, this can be applied to type IIB string theory as well [88].

Heterotic String Theory

Now turning to the standard embedding of heterotic $\mathrm{E}_8 \times \mathrm{E}_8$ string theory, we have shown in section 2.6 that a $\mathbb{Z}_N$ orbifold fixed point supports the following half-hypermultiplets (see (2.6.53))

$$(N-1)(\mathbf{56}, \mathbf{1}) + (6N-4)(\mathbf{1}, \mathbf{1}) \tag{3.5.4}$$

where the U(1) quantum numbers are as given in (2.6.49).

For N KK-monopoles with the standard embedding, we again have the modes of the $\vec{r}_I$ together with those of the B-field, giving N hypermultiplets. In addition to these, as argued in section 3.4, we have $4N$ bosonic degrees of freedom corresponding to the centers of the N instantons. As a single instanton as three global SU(2) rotations[14] associated to it, we expect one more hypermultiplet per

[14] From the t'Hooft ansatz, only the global SU(2) (or even E_8) rotations of all instantons at the same time are manifest. Nevertheless, to compute the multiplet structure of the moduli, one can treat the N instantons as independent, which can be seen from the explicit construction in [35].

instanton, which, as its fourth bosonic component, must contain the scale factor. Finally, since the instantons are in an SU(2) subgroup of E_8 and the adjoint of E_8 decomposes as (see (B.4.7))

$$\begin{array}{rcl} E_8 & \rightarrow & E_7 \times SU(2) \\ \mathbf{248} & \rightarrow & (\mathbf{133},\mathbf{1}) + (\mathbf{1},\mathbf{3}) + (\mathbf{56},\mathbf{2}) \end{array} \tag{3.5.5}$$

an embedding of SU(2) into E_8 is specified by $2 \cdot 56$ parameters. Hence, we expect N half-hypermultiplets in the **56** of one E_8 controlling the embedding of the N instantons into E_8. Summing up, we have accounted for N half-hypermultiplets in the $(\mathbf{56},\mathbf{1})$ and $3N$ neutral hypermultiplets.

Now, by the same arguments as in the type IIA case, we expect the following multiplets which are normalizable only on K_N but not on X_N: one hypermultiplet containing the center of mass coordinates of the KK-monopoles together with the overall B-field mode and one hypermultiplet containing the overall scale factor of all instantons. That that scale factor λ has to be non-normalizable on X_N is clear, because by taking the $r \to 0$ limit the one in $g = 1 + \lambda \ldots$ can be neglected and therefore, since g enters the formulas in a logarithm, the overall scale drops out. The other three components of the hypermultiplet are the three global overall SU(2) rotations of the solution, which certainly are normalizable only on K_N. This applies as well to one half-hypermultiplet in the $(\mathbf{56},\mathbf{1})$, since these modes are responsible for the embedding of SU(2) into E_8.

Finally, there remains one hypermultiplet corresponding the center of mass coordinates of all instantons. Since displacing all the instantons induces a dipole moment in g and since, in addition, g enters the field by gradient of $\ln g$, the modes are falling like $(r/l)^{-2}$ and therefore are normalizable.

To verify these results by another method, we compute the number of modes using Dirac indices for manifolds with boundary. As explained in section 4.1 in detail, by the high amount of eight supersymmetries in six dimensions, index theory allows for the direct computation of the multiplet content. The computation of index formulas for X_N can be found in appendix C.3.

Setting $I_{YM} = N - 1/N$ (and working with self-dual KK-monopoles, for simplicity), by (C.3.33) and (C.3.34), we arrive at the following indices together with the number of half-hypermultiplets and their representation in $E_7 \times E_8$

Index		No.	Irrep.
$\mathrm{ind}\,\not{D}_{1/2}$	$= 0$	0	
$\mathrm{ind}\,\not{D}_{3/2}$	$= 2N - 2$	$2N - 2$	$(\mathbf{1},\mathbf{1})$
$\mathrm{ind}\,\not{D}_{\mathbf{2}}$	$= I_{YM} - 1 + \frac{1}{N} = N - 1$	$N - 1$	$(\mathbf{56},\mathbf{1})$
$\mathrm{ind}\,\not{D}_{\mathbf{3}}$	$= 4I_{YM} - 2 + \frac{4}{N} = 4N - 2$	$4N - 2$	$(\mathbf{1},\mathbf{1})$

(3.5.6)

(since Dirac indices count zero modes in their complex dimensionality, they count half-hypermultiplets directly).

All in all, we have found $N-1$ normalizable half-hypermultiplets in the $(\mathbf{56},\mathbf{1})$ and $3N-2$ normalizable neutral hypermultiplets. $N-1$ of the neutral hypermultiplets belong to supergravity modes which already were present in the type IIA case. Therefore, we have exactly reproduced the particle content of the $\mathbb{Z}_N$ orbifold singularity in the standard embedding. This suggests that the physics of the scale l region is indeed described by heterotic string theory on a $\mathbb{Z}_N$ orbifold singularity in the standard embedding. Since string theory in general can be applied to non-compact orbifolds such as $\mathbb{C}^2/\mathbb{Z}_N$ (see [35]), we expect that the above correspondence indeed works for general N.

Since away from the scale l region the gauge fields exactly look like a single SU(2) instanton on top of a single KK-monopole, it is clear that the gauge symmetry is broken to E_7 in the infrared. It can not be enhanced as in the type IIA case since massless U(1) vectors, which could serve as generators of the Cartan subalgebra, are not present. However, to every half-hypermultiplet in the $(\mathbf{56},\mathbf{1})$ there is a hypermultiplet containing three modes of global SU(2) rotations together with a relative scaling mode. By the presence of the scaling mode, there is the possibility to shrink an instanton down to zero size to produce a small $E_8 \times E_8$ instanton which can lead to non-trivial physics in the infrared (see [83]).

For length scales below l, the gauge symmetry appears unbroken and symmetry enhancement might be possible. In fact, as argued in the next chapter based on [62, 49], some SU(m) subgroup of the gauge symmetries in the orbifold limit has to be ascribed to enhanced symmetries as in the type II case.

Chapter 4

Low Energy Effective Description of $D = 6$ Orbifolds

Having focused on individual orbifold fixed points in the last chapter, we now turn to the relation of type IIA and heterotic orbifolds as discussed in chapter 2 to smooth compactifications of the respective theories on K3.

4.1 K3 Compactification of the $E_8 \times E_8$ Heterotic String Theory

Since compactification of superstring theories on Calabi-Yau manifolds such as K3 is a standard topic of introductions on superstring theory, we will not attempt to review the details of such constructions here (see [76] chapters 17 and 19, or [52], chapters 12 to 16; for explicit details on K3, the reader is referred to [4]). Instead, we will use index theory as in [51] and give the relevant features of K3 as appropriate.

The Standard Embedding

In the standard embedding of $\mathcal{N} = (0, 1)$, $D = 6$ heterotic compactifications the SU(2) anti-self-dual spin connection of K3 is embedded into the gauge group $E_8 \times E_8$ simply by choosing an SU(2) subgroup and setting equal the gauge connection to the spin connection. Hence, the anomalous Bianchi identity (3.2.3) is $\mathrm{d}H = 0$ and anomaly cancellation by the Green-Schwarz mechanism can be achieved.

We will use the same conventions as in section 2.6 and in section 3.2. Since we choose the spin connection is anti-self dual, it will live in the $(\mathbf{1}, \mathbf{3})$ of $\mathrm{SU}(2)_R \times \mathrm{SU}(2)_H$ and, as in the case of the orbifolds, $\mathrm{SU}(2)_H$ will host the holonomy of K3. This implies, as already explained in section 3.2 on low energy effective field theory, that covariantly constant spinors have to live in the $(\mathbf{2}, \mathbf{1})$ representation. This, however, allows us to compute the $D = 6$ massless field content purely from index theory, since the index of Dirac operators is the number of positive chirality zero modes minus the number of negative chirality ones. As supersymmetry guarantees that hypermultiplet modes are in the $(\mathbf{1}, \mathbf{2})$ of $\mathrm{SU}(2)_1 \times \mathrm{SU}(2)_2$, by (2.6.4), the spinor transforms as $(\mathbf{1}, \mathbf{2})$ of $\mathrm{SU}(2)_R \times \mathrm{SU}(2)_H$ and the Dirac index on K3 gives minus the number of zero modes. For vector multiplets, by the same argument, the Dirac index on K3 gives directly the number of zero modes.

From the field equations in ten-dimensions

$$\not{D}\Psi = \Gamma^M \mathrm{D}_M \Psi = 0 \tag{4.1.1}$$

which, by (3.2.10), decompose as

$$0 = \Gamma^\mu \partial_\mu \Psi + \Gamma^a \mathrm{D}_a \Psi = \bar{\bar{\Gamma}}^\mu \partial_\mu \otimes \mathbb{1}\, \Psi - \omega^{\mathbb{C}}_{1,5} \otimes \bar{\Gamma}^a \mathrm{D}_a \Psi \tag{4.1.2}$$

it is clear that massless modes in $D = 6$ correspond to covariantly constant spinors and hence zero modes on K3 (a very pedagogical introduction into those matters can be found in chapter 14 of [52]).

We start by the fields charged under the gauge group $E_8 \times E_8$, which decomposes as in the orbifold case (2.6.35) for $N = 2$

$$\begin{aligned} E_8 \times E_8 &\to E_7 \times \mathrm{SU}(2) \times E_8 \\ (\mathbf{248}, \mathbf{1}) + (\mathbf{1}, \mathbf{248}) &\to (\mathbf{133}, \mathbf{1}, \mathbf{1}) + (\mathbf{1}, \mathbf{3}, \mathbf{1}) + (\mathbf{1}, \mathbf{1}, \mathbf{248}) \\ &\quad + (\mathbf{56}, \mathbf{2}, \mathbf{1}) \end{aligned} \tag{4.1.3}$$

Since the spin connection is embedded in the gauge $\mathrm{SU}(2)$, $D = 6$ vector multiplets must be invariant under $\mathrm{SU}(2)_H$ and the $D = 6$ gauge group is E_7 times E_8. The remaining charged fields must originate from the A^I_a and their superpartners and have to appear in hypermultiplets. For the $\mathbf{2}$ of $\mathrm{SU}(2)$, the Dirac index is (very useful tables of index polynomials can be found in [3])

$$\begin{aligned} \mathrm{ind}\, \not{D}_{\mathbf{2}} &= \int_{\mathrm{K3}} -\frac{1}{8\pi^2} \mathrm{tr}_{\mathbf{2}}\, F^2 + \frac{\mathrm{Dim}\, \mathbf{2}}{12 \cdot 16\pi^2} \mathrm{tr}\, R^2 \\ &= I_{YM} - \frac{\mathrm{Dim}\, \mathbf{2}}{12} I_L \end{aligned} \tag{4.1.4}$$

where I_{YM} is the instanton number and I_L is the gravitational instanton number as used in (3.1.21). Since the gravitational instanton number of $K3$ is -24, by the embedding of the spin connection into the gauge connection, the gauge instanton number must be -24 as well and we arrive at

$$\mathrm{ind}\, \not{D} = -24 + 4 = -20 \tag{4.1.5}$$

Therefore, we have 20 (complex) zero modes of the Dirac operator in the $\mathbf{2}$ of $\mathrm{SU}(2)$ on K3. Since a $D = 6$ hypermultiplet contains two complex spinors, we arrive at 10 hypermultiplets in the $(\mathbf{56}, \mathbf{1})$.

For the **3** of SU(2) we have

$$\begin{aligned}\operatorname{ind} \not{D}_{\mathbf{2}} &= \int_{\mathrm{K3}} -\frac{1}{8\pi^2} \operatorname{tr}_{\mathbf{3}} F^2 + \frac{\operatorname{Dim} \mathbf{3}}{12 \cdot 16\pi^2} \operatorname{tr} R^2 \\ &= 4 I_{YM} - \frac{\operatorname{Dim} \mathbf{3}}{12} I_L = -4 \cdot 24 + 6 = -90\end{aligned} \tag{4.1.6}$$

where we have used that $\operatorname{tr}_{\mathbf{3}} = 4 \operatorname{tr}_{\mathbf{2}}$ and therefore arrive at 45 neutral hypermultiplets.

For spinors neutral under SU(2), the index is

$$\operatorname{ind} \not{D} = \int_{\mathrm{K3}} \frac{1}{12 \cdot 16\pi^2} \operatorname{tr} R^2 = -\frac{1}{12} I_L = +2 \tag{4.1.7}$$

and therefore there are no other hypermultiplets from ten-dimensional gauge fields. Applied to vector multiplets, every gauge generator in ten dimensions neutral under SU(2) survives and the gauge group is $E_7 \times E_8$ as expected.

There is, however, the possibility of additional hypermultiplets arising from the modes Ψ_a of the ten-dimensional gravitino together with the modes of the ten-dimensional spinor from the supergravity multiplet. Since both fields arise from the representation $\mathbf{8_v} \otimes \mathbf{8}_+$ of SO(8), the relevant Dirac index is that of the Rarita-Schwinger operator (see [3])

$$\operatorname{ind} \not{D}_{3/2} = \int_{\mathrm{K3}} \frac{1}{16\pi^2} \frac{d-24}{12} \operatorname{tr} R^2 = \frac{20}{12} I_L = -40 \tag{4.1.8}$$

where we used $d = 4$ for K3. Therefore, we have 20 additional neutral hypermultiplets. Since these stem from the ten-dimensional supergravity multiplet, the bosons in the hypermultiplets should correspond to the K3 modes of the bosonic fields in the supergravity multiplet, that is, the metric and the B-field. And in fact, string theory on K3 has 80 moduli, 38 of which are complex structure moduli and the rest of which are called Kähler moduli (see [76], section 19.8).

It remains to discuss the modes of the ten-dimensional supergravity multiplet which do not result in hypermultiplets. This is not very illuminating (see [51]) and we will simply decompose the B-field into its six-dimensional self-dual and anti-self-dual parts which, by (2.6.11), makes clear that the ten dimensional supergravity multiplet decomposes into a six-dimensional supergravity multiplet and one tensor multiplet.

Summing up, the standard embedding of heterotic $E_8 \times E_8$ theory on K3 yields the following massless spectrum in $\mathcal{N} = (0, 1)$, $D = 6$ supermultiplets

$$\begin{array}{rl} \text{SUGRA} & 1 \\ \text{tensor} & 1 \\ \text{vector} & (\mathbf{133}, \mathbf{1}, \mathbf{1}) + (\mathbf{1}, \mathbf{1}, \mathbf{248}) \\ \text{hyper} & 65(\mathbf{1}, \mathbf{1}) + 10(\mathbf{56}, \mathbf{1}) \end{array} \tag{4.1.9}$$

$SU(2) \times SU(2)$ Nonstandard Embedding

Since the anomalous Bianchi identity for the ansatz $H = 0$ reduces to

$$\operatorname{tr} R^2 = \frac{1}{30}\operatorname{Tr} F_1^2 + \frac{1}{30}\operatorname{Tr} F_2^2 \tag{4.1.10}$$

(see (2.5.3)), we can try and embed n_1 anti-instantons of SU(2) in the first E_8 and $n_2 = -(24 - n_1)$ anti-instantons of SU(2) in the second E_8 to solve this equation. Hence the unbroken gauge group is $\mathrm{E}_7 \times \mathrm{E}_7$ and all index calculations go through as before except that we will now have to calculate (4.1.4) and (4.1.6) for the two E_8 groups separately. This gives

$$\begin{aligned} \operatorname{ind} \not{D}_{\mathbf{2}}^{(1)} &= -n_1 + 4 \qquad & \operatorname{ind} \not{D}_{\mathbf{3}}^{(1)} &= -4n_1 + 6 \\ \operatorname{ind} \not{D}_{\mathbf{2}}^{(2)} &= -n_2 + 4 \qquad & \operatorname{ind} \not{D}_{\mathbf{3}}^{(2)} &= -4n_2 + 6 \end{aligned} \tag{4.1.11}$$

resulting in $n_1 - 4$ half-hypermultiplets in the $(\mathbf{56}, \mathbf{1})$, $n_2 - 4$ half-hypermultiplets in the $(\mathbf{1}, \mathbf{56})$ and $2n_1 + 2n_2 - 6 = 42$ neutral hypermultiplets, all stemming from the gauge degrees of freedom. The massless spectrum is

$$\begin{array}{rl} \text{SUGRA} & 1 \\ \text{tensor} & 1 \\ \text{vector} & (\mathbf{133}, \mathbf{1}, \mathbf{1}) + (\mathbf{1}, \mathbf{1}, \mathbf{248}) \\ \text{hyper} & 62(\mathbf{1}, \mathbf{1}) \\ \text{half-hyper} & (n_1 - 4)(\mathbf{56}, \mathbf{1}) + (n_2 - 4)(\mathbf{1}, \mathbf{56}) \end{array} \tag{4.1.12}$$

As can be seen from the above calculation, the gauge instanton numbers are constrained to satisfy $n_1, n_2 \geq 4$. This is a quite general feature of nonstandard embeddings, since from the above compactification one can obtain other compactifications with the same gauge instanton numbers simply by higgsing some part of the remaining gauge symmetry [83]. As further noted in [83], there are singularities in the moduli space of nonstandard embeddings corresponding to small[1] $\mathrm{E}_8 \times \mathrm{E}_8$ instantons turning to heterotic 5-branes which allow for phase transitions between compactifications with different (n_1, n_2). At the phase transition points, tensionless strings appear.

4.2 Blow Ups and Heterotic Type IIA Duality

[1] As shown in section 3.4, "small" instantons are instantons where the scale size of the instanton goes to zero, which is not the same as such an instanton being point-like, which is possible at orbifold singularities.

Type IIA Compactification on K3

As we have seen in the last section, in heterotic compactifications on K3 there always appear 20 hypermultiplets which are related to deformations of the metric and B-field backgrounds. Since $\mathcal{N} = (1,1)$, $D = 10$ supersymmetry contains $\mathcal{N} = (0,1)$ supersymmetry as a subsector, the analysis goes through in type IIA compactifications on K3, where the only possible matter multiplet in $D = 6$ is the vector multiplet (see (2.6.10)). In $\mathcal{N} = (0,1)$ notation this indeed splits into a vector and a hypermultiplet. Therefore, there should be 20 vector multiplets in $D = 6$, the vector of which must arise from the three-form RR-potential C_{MNP} in ten dimensions, since this potential appears in the $\mathcal{N} = (1,1)$ supergravity multiplet and hence must arise as a superpartner of the $\mathcal{N} = (0,1)$ part of the supergravity multiplet in $D = 10$ (for the detailed multiplet structure, see [76], chapter 12 and appendix B). The same argument applies to the supergravity multiplet in $D = 6$ and we arrive at the following massless particle content of type IIA compactifications on K3:

$$1 \text{ SUGRA}, \; 20 \text{ vector} \tag{4.2.1}$$

Of course, we would have arrived at the same spectrum by the usual methods of Calabi-Yau or K3 compactification (see [76], section 19.8).

However, because of the high amount of 16 surviving supersymmetries, we can say more about the moduli space of type IIA compactifications on K3. Firstly, from conformal field theory arguments (see [4] and references therein), the 16 space-time supersymmetries correspond to $\mathcal{N} = (4,4)$ worldsheet supersymmetry and the moduli space should be approximated by

$$\mathrm{O}(4, 16+4) \Big/ \mathrm{O}(4) \times \mathrm{O}(16+4) \tag{4.2.2}$$

As explained in section 3.3 of [4], this is compatible with the low energy effective description of type IIA string theory on K3, where a

$$\mathrm{O}(3, 16+3) \Big/ \mathrm{O}(3) \times \mathrm{O}(16+3) \tag{4.2.3}$$

subspace of the above space is attributed to the geometric symmetries of the K3 surface and the remaining coordinates parametrize B-field backgrounds and the overall volume of K3. This space-time perspective, however, reveals that the above description breaks down at special points in the moduli space of the K3 surface where singularities appear and the curvature diverges.

As already explained in section 3.3, this problem can be addressed by heterotic type II duality, which states that the moduli space of type IIA theory on $\mathrm{K3} \times T^2$ is equivalent to the moduli space of heterotic compactifications on T^6. This implies, when we go to a point in the moduli space of the heterotic compactification where it becomes a product corresponding to the product $T^6 = T^4 \times T^2$, that we can decompactify the T^2 by sending its volume to infinity and arrive at the moduli space of heterotic compactifications on T^4 (see (3.3.1))

$$\mathrm{O}(\mathbb{Z},4,16+4) \Big\backslash \mathrm{O}(4,16+4) \Big/ \mathrm{O}(4)\times\mathrm{O}(16+4) \tag{4.2.4}$$

Since the decompactification, by supersymmetry, works for the type IIA side equally well, this moduli space should be equal to the moduli space of type IIA theory on K3, including points where singularities of K3 appear. The $\mathrm{O}(\mathbb{Z},4,16+4)$ subgroup that was not present in (4.2.2) corresponds to mirror symmetry of type IIA string theory on K3 which we will not consider in this work (see section 3.4 of [4]). Of course, this subgroup leaves invariant the BPS condition (3.3.23). In fact, it is not difficult to verify this condition on the type IIA side, using the results on supersymmetric cycles of [18].

From this identification (see section 2.6 of [4]), the points in the moduli space where enhanced gauge symmetries arise on the heterotic side correspond to those points on the type II side where orbifold singularities of K3 arise. Therefore, as already discussed in section 3.5, this implies that type IIA and M-theory on a $\mathbb{Z}_N$ orbifold singularity show an enhanced $\mathrm{SU}(N)$ gauge symmetry at these points [104].

Blowups

The connection to orbifolds as discussed in chapter 2 lies in the fact that one can go to points in the moduli space of K3 at which only isolated orbifold singularities appear and the curvature outside the orbifold singularities tends to zero. At this point, the K3 surface is identical to the global geometrical orbifolds as discussed in section 2.1. In fact, this was one of the starting points for the orbifold construction in string theory [33]. Reversing the procedure, one can "blow up" the orbifold singularities in a completely local way (for details see section 2.6 of [4]), that is, individually for every orbifold point at an arbitrarily small scale. Roughly, this can be imagined as cutting out a little cone around the orbifold singularity and replacing it by a Eguchi-Hanson gravitational instanton (see section 3.1). Therefore, the results of section 3.1 fully apply as long as we have a separation of scales R and l of KK-monopole and Eguchi-Hanson gravitational instanton respectively. Especially, blowing up corresponds to giving the moduli $\vec{r}_I - \vec{r}_J$ a vacuum expectation value.

For type IIA theory, this implies a conundrum, since type IIA orbifolds as constructed in section 2.6 do not show enhanced $\mathrm{SU}(N)$ gauge symmetries. This is resolved (see section 4.3 of [4]) by noting that unbroken $\mathrm{SU}(N)$ symmetry requires vanishing of the vacuum expectation values of all scalars in the vector multiplets which are in the adjoint of $\mathrm{SU}(N)$. Therefore, perturbative orbifolds must correspond to points in the moduli space where some vacuum expectation values are nonzero. Since those moduli corresponding to the blowup modes must be zero at the orbifold point, the only remaining possibility is a nonvanishing B-field background on the shrunken gravitational instanton.

As we have seen in section 3.5, heterotic orbifolds in the standard embedding can be described in a similar way. Around a $\mathbb{Z}_N$ orbifold singularity the orbifold string theory is described as heterotic string theory on X_N which in turn is the description of the interior of heterotic string theory on N KK-monopoles on top of N SU(2) instantons. In contrast to the type IIA case, the spectrum directly corresponds to that of the smooth K3 compactification (4.1.9).

4.3 Higgsing Heterotic Models at the Orbifold Point

The Standard Embedding for $N = 2$

As constructed in section 2.6, the massless spectrum is

		$E_7 \times SU(2) \times E_8$	
untwisted	SUGRA		
	tensor		
	vector	$(\mathbf{133},\mathbf{1},\mathbf{1})+(\mathbf{1},\mathbf{3},\mathbf{1})+(\mathbf{1},\mathbf{1},\mathbf{248})$	(4.3.1)
	hyper	$4(\mathbf{1},\mathbf{1},\mathbf{1})+(\mathbf{56},\mathbf{2},\mathbf{1})$	
16 fixed points	hyper	$16\cdot 2(\mathbf{1},\mathbf{2},\mathbf{1})$	
	half-hyper	$16\cdot(\mathbf{56},\mathbf{1},\mathbf{1})$	

The obvious way to make contact to the standard embedding of smooth K3 compactifications is to break SU(2) by giving a vacuum expectation value (vev) to two different[2] twisted fields in the $(\mathbf{1},\mathbf{2},\mathbf{1})$. In this case, the vector multiplets in the adjoint $\mathbf{3}$ of SU(2) can only get mass by swallowing three hypermultiplets neutral under $E_7 \times E_8$. One might be tempted to expect that these modes are among the four neutral untwisted hypermultiplets, but this cannot be since these are geometric moduli of K3 and hence have to remain massless[3].

[2]That there have to be at least two vevs is a well known feature of eight unbroken supercharges: to hold the triplet of D-terms zero, contributions from different multiplets have to cancel out. See [82], section 3.

[3]This is clear from direct analysis of space time fields G_{ab} and B_{ab} or simply from noting that these modes remain present in the type IIA case even after blowing up some orbifold singularities.

Therefore, the only possibility is that the three vectors swallow modes from the twisted sector. After all, this is not very unusual, since the same mechanism is at work in the well known example of the standard embedding in four-dimensional heterotic orbifolds, as explained in [76], section 16.4. This implies, that massless modes of the smooth standard embedding can no longer be ascribed to individual fixed points, since to reach a smooth K3, all orbifold singularities have to be blown up, and hence all twisted fields have to be treated at the same footing.

We note, however, that the geometrical modes associated to fixed points remain localized, since their vev directly corresponds to the scale of the blowup. Because there are 20 K3 hypermultiplets four of which are already present in the untwisted sector, we expect that at least one E_7 singlet hypermultiplet remains massless at every fixed point. Therefore, even if only a single orbifold singularity is blown up, the swallowed modes have to come from different fixed points and hence should be a linear combination of all available hypermultiplets.

The Standard Embedding for $N > 2$

For $N > 2$ the massless spectrum is given as

		$E_7 \times E_8 \times U(1)$	
untwisted	SUGRA		
	tensor		
	vector	$(\mathbf{133},\mathbf{1})_0 + (\mathbf{1},\mathbf{248})_0$	(4.3.2)
	hyper	$2(\mathbf{1},\mathbf{1})_0 + (\mathbf{1},\mathbf{1})_{-2} + (\mathbf{56},\mathbf{1})_{+1}$	
$\mathbb{Z}_N$ fixed points	hyper	$(3N-2)(\mathbf{1},\mathbf{1})_{\dots}$	
	half-hyper	$(\mathbf{56},\mathbf{1})_{\dots}$	

Here, to break U(1), again we would have to give at least two vevs to $E_7 \times E_8$ singlets charged under U(1) (see [82], section 2). As above, the two neutral singlets in the untwisted sector belong to geometric modes of K3 and have to remain massless. However, since $\mathcal{N} = (0,1)$, $D = 6$ supersymmetry is inherently chiral, a U(1) gauge symmetry with charged hypermultiplets is in general anomalous and a variation of the mechanism of [31] has to resolve the problem (for details, see [35], section 7). As a result, the U(1) symmetry will be higgsed and by the same argument as in the $N = 2$ case, the massive mode should be a linear combination of all U(1) charged singlet hypermultiplets.

The $\mathbb{Z}_3, \beta = \frac{1}{3}(5/2,(1/2)^7;2,1^2,0^5)$ Orbifold

We now turn to nonstandard embeddings and explicitly look at the $\mathbb{Z}_3$,

$\beta = \frac{1}{3}(5/2, (1/2)^7; 2, 1^2, 0^5)$ orbifold of the heterotic $E_8 \times E_8$ string theory as constructed in section 2.6.

In the previous literature [2], orbifold models were related to smooth K3 compactifications by higgsing the spectrum (4.1.12) of the smooth nonstandard K3 compactification and finding a "similar" orbifold model. However, as we will see below, higgsing of the orbifold to meet the spectrum of the smooth compactification is much more involved as in the case of the standard embedding. The main reason for this is that all fields in this orbifold transform in complex representations of the gauge group, whereas in the other examples the representations were pseudo-real.

As shown in section 2.6, the massless spectrum of the orbifold is given as follows

		$SU(9) \times E_6 \times SU(3)$	
untwisted	SUGRA		
	tensor		(4.3.3)
	vector	$(\mathbf{80}, \mathbf{1}, \mathbf{1}) + (\mathbf{1}, \mathbf{78}, \mathbf{1}) + (\mathbf{1}, \mathbf{1}, \mathbf{8})$	
	hyper	$2(\mathbf{1}, \mathbf{1}, \mathbf{1}) + (\mathbf{84}, \mathbf{1}, \mathbf{1}) + (\mathbf{1}, \mathbf{27}, \mathbf{3})$	
9 fixed points	hyper	$9(\mathbf{9}, \mathbf{1}, \mathbf{3})$	

We begin by trying to give a vacuum expectation value (vev) to the bulk hypermultiplet in the $(\mathbf{84}, \mathbf{1}, \mathbf{1})$. Since SU(9) decomposes as

$$\begin{array}{rcl} SU(9) & \rightarrow & SU(6) \times SU(3) \times U(1) \\ \mathbf{9} & \rightarrow & (\mathbf{6}, \mathbf{1})_{+3} + (\mathbf{1}, \mathbf{3})_{-6} \\ \mathbf{80} & \rightarrow & (\mathbf{35}, \mathbf{1})_0 + (\mathbf{1}, \mathbf{8})_0 + (\mathbf{1}, \mathbf{1})_0 + (\mathbf{6}, \bar{\mathbf{3}})_{+9} + (\bar{\mathbf{6}}, \mathbf{3})_{-9} \\ \mathbf{84} & \rightarrow & (\mathbf{20}, \mathbf{1})_{+9} + (\mathbf{15}, \mathbf{3})_{+9} + (\mathbf{6}, \bar{\mathbf{3}})_0 + (\mathbf{1}, \mathbf{1})_{-9} \end{array} \tag{4.3.4}$$

a vev in the **84** breaks SU(9) to SU(6) × SU(3) (this is clear from the charged singlet in the decomposition of the **84**). However, since we have $\mathcal{N} = (0, 1)$, $D = 6$ supersymmetry, the Higgs effect requires two massless full hypermultiplets in the $(\mathbf{6}, \bar{\mathbf{3}})_{+9}$+ $(\bar{\mathbf{6}}, \mathbf{3})_{-9}$ be eaten up by the vector multiplets in the same representations. Since these are not present, higgsing by a vev in the **84** is not possible.

Next we try to give a vev to a hypermultiplet in the $(\mathbf{9}, \mathbf{1}, \mathbf{3})$. The relevant decomposition of SU(9) and SU(3) are

$$
\begin{array}{rcl}
\mathrm{SU}(9) & \rightarrow & \mathrm{SU}(8)\times\mathrm{U}(1) \\
\mathbf{9} & \rightarrow & \mathbf{1}_8+\mathbf{8}_{-1} \\
\mathbf{80} & \rightarrow & \mathbf{63}_0+\mathbf{1}_0+\mathbf{8}_{-9}+\bar{\mathbf{8}}_{+9} \\
\mathbf{84} & \rightarrow & \mathbf{56}_{+3}+\mathbf{28}_{+6} \\
\\
\mathrm{SU}(3) & \rightarrow & \mathrm{SU}(2)\times\mathrm{U}(1) \\
\mathbf{3} & \rightarrow & \mathbf{1}_2+\mathbf{2}_{-1} \\
\mathbf{8} & \rightarrow & \mathbf{3}_0+\mathbf{1}_0+\mathbf{2}_{-3}+\mathbf{2}_{+3}
\end{array}
\tag{4.3.5}
$$

Hence, a vev in the $(\mathbf{9},\mathbf{1},\mathbf{3})$ breaks SU(9)×E_6×SU(3) to SU(8)×E_6×SU(2)×U(1) (the U(1) is a linear combinations of the two U(1) appearing in (4.3.5)). However, again there are no full hypermultiplets in the $\mathbf{8}$ and $\bar{\mathbf{8}}$ of SU(8) or in the $\mathbf{2}$ and $\mathbf{2}$ of SU(2), as needed for the Higgs effect. Therefore, giving a vev to fields at a single fixed point is not possible and the moduli space corresponding to a single fixed point is trivial.

Finally, we could try to give a vev to a hypermultiplet in the $(\mathbf{1},\mathbf{27},\mathbf{3})$. However, the same problem as above arises and, furthermore, all representations of E_7 are invariant under the gauge shift and there is no need to break E_7 to make contact to smooth K3 compactifications.

In view of all this, there remains only one solution: we have to give vevs to *all* hypermultiplets sitting at fixed points to break SU(9) and SU(3) completely. In that case, all SU(9) × SU(3) representations break down to singlets which make the Higgs effect possible. This implies, that the background produced by the Higgs mechanism is a E_8 × SU(3) background. As we will see in the following, the modes can no longer be associated to single fixed points.

The $E_8 \times SU(3)$ Nonstandard Embedding

Since we have nine fixed points with one hypermultiplet in the $(\mathbf{9},\mathbf{3})$ of SU(9) × SU(3) each, by giving generic vev to all nine of them SU(9) × SU(3) is completely broken. E_7, however, remains unbroken and there are three hypermultiplets in the $\mathbf{27}$ of E_7. Because there are no charged fields left transforming non-trivially under the gauge shift, this then can be analyzed as the nonstandard embeddings of section 4.1. We calculate the following Dirac indices, together with the number of hypermultiplets and their representations

$$\begin{array}{llllr} \mathrm{SU}(9) & \mathrm{ind}\,\not{D}_{\mathbf{80}} & = -18n_1 + 160 & 9n_1 - 80 & \mathbf{1} \\ & \mathrm{ind}\,\not{D}_{\mathbf{84}} & = -21n_1 + 168 = \mathrm{ind}\,\not{D}_{\overline{\mathbf{84}}} & 21n_1 - 168 & \mathbf{1} \\ \mathrm{SU}(3) & \mathrm{ind}\,\not{D}_{\mathbf{8}} & = -6n_2 + 16 & 3n_2 - 8 & \mathbf{1} \\ & \mathrm{ind}\,\not{D}_{\mathbf{3}} & = -n_2 + 6 = \mathrm{ind}\,\not{D}_{\overline{\mathbf{3}}} & n_2 - 6 & \mathbf{27} \end{array} \tag{4.3.6}$$

Since the spectrum of the higgsed orbifold contains three hypermultiplets in the **27** of E_6, we have $(n_1, n_2) = (15, 9)$. The hypermultiplet spectrum then is

$$\begin{array}{crl} \mathbf{80} & 55 & \mathbf{1} \\ \mathbf{84} & 147 & \mathbf{1} \\ \mathbf{8} & 19 & \mathbf{1} \\ \text{K3 modes} & 20 & \mathbf{1} \\ \hline & 241 & \mathbf{1} \\ & & \\ \mathbf{3} & 3 & \mathbf{27} \end{array} \tag{4.3.7}$$

From the orbifold spectrum (4.3.3) we have 329 hypermultiplets neutral with respect to E_6. Upon higgsing $\mathrm{SU}(9) \times \mathrm{SU}(3)$, 88 of these hypermultiplets get swallowed by the vector multiplets in the adjoint of $\mathrm{SU}(9) \times \mathrm{SU}(3)$ and 241 hypermultiplets remain, as in the nonstandard embedding.

The fact that SU(9) has to be completely broken by the Higgs mechanism suggests that the orbifold provides full E_8 background. Therefore, as already in the case of the standard embedding in section 3.5, index theory should be able to give the correct field content of a single fixed point.

Setting $n_1 = 15/9$ and $n_2 = 9/9 = 1$ for an individual fixed point, we calculate Dirac indices on X_3 for all relevant representations. By (C.3.33), (C.3.35) and (C.3.36), we arrive at the following indices together with the number of multiplets and their representation in E_6

Index		No.	Irrep.
$\mathrm{ind}\,\not{D}_{1/2}$	$= 0$	0	
$\mathrm{ind}\,\not{D}_{3/2}$	$= -2N + 2 = -4$	2	**1**
$\mathrm{ind}\,\not{D}_{\mathbf{248}}$	$= -60n_1 + 56 = -44$	22	**1**
$\mathrm{ind}\,\not{D}_{\mathbf{3}}$	$= -2n_2 + 2 = 0$	0	
$\mathrm{ind}\,\not{D}_{\mathbf{8}}$	$= -6n_2$	3	**1**

(4.3.8)

Therefore, index theory ascribes 22 hypers to the E_8 background, two to the supergravity modes, as expected, and three hypers to the SU(3) adjoint background. This exactly reproduces the 27 hypers of the $(\mathbf{9}, \mathbf{1}, \mathbf{3})$ field content of a single fixed point.

By the above counting of states, modes after higgsing can no longer be ascribed to individual fixed points, as already in the case of standard embedding.

4.4 Locating Massless Modes in M-theory on $S^1/\mathbb{Z}_2$

As we have seen in the last section, after breaking gauge symmetry in orbifold models by giving vevs to fields in the twisted sectors, the models can be identified with smooth K3 non-standard embeddings. Even though the Higgs mechanism will require a mixing of fields from different fixed points, by making the volume of K3 very big, the curvature can be pushed below the string scale and the whole model can effectively be described by supergravity.

In that case, as shown in [106], a valid description in terms of M-theory on $S^1/\mathbb{Z}_2$ can easily be found. As all index theory calculation as performed for K3 as a whole remain fully valid, the assignments of modes to the individual E_8 factors or the supergravity multiplet can still be trusted.

Clearly, every low energy effective action describing an orbifold model from the M-theory on $S^1/\mathbb{Z}_2$ perspective will have to reproduce the above description upon higgsing. However, this implies that we have to ascribe supergravity modes (i.e. $N-1$ hypermultiplets per $\mathbb{Z}_N$ fixed point) to the modes of the fixed points which are charged under the gauge group. From a ten-dimensional perspective, where gravity and gauge degrees of freedom always appear together, this is no problem, but in M-theory on $S^1/\mathbb{Z}_2$, where gravity is assigned to the interior of the interval and gauge degrees of freedom to the ends, this is a priori non-trivial.

One might think that gravitational modes corresponding to fixed point modes might be localized on one end of the interval at the orbifold point and spread to the interior during the Higgs effect. However, as shown in [106], when making the length of the interval very long compared to the volume of K3, the metric of K3 scales roughly as $(x^{11})^{2/3}$. Even more, this behavior of the metric is controlled by source terms at the ends of the interval and hence is not affected by going to the orbifold limit. Since there is no scale in the expression $y^{2/3}$, gravity modes cannot get localized to one end of the interval. Therefore, in the orbifold limit, gravitational modes have to live on the whole of the interval $S^1/\mathbb{Z}_2$ while they are localized at the orbifold points of K3. Hence, they must be contained in supermultiplets of a seven-dimension supersymmetric theory.

As already explained in section 3.5, this is fully consistent with the assumption that M-theory on a $\mathbb{Z}_N$ orbifold singularity can effectively be described by a $SU(N)$ supersymmetric gauge theory. This theory, in seven dimensions, contains only vector multiplets which are comprised of five vector degrees of freedom and three scalar degrees of freedom. After blowing up (or turning on B-field backgrounds), this theory is higgsed to $U(1)^{N-1}$. Then the vector multiplets contain exactly those degrees of freedom expected from M-theory on X_N: $N-1$ vectors from the three-index tensor $C_3 = C_1^{IJ} \wedge (\Omega^I - \Omega^J)$ and three scalars corresponding to the gravitational modes of the $\vec{r}_I - \vec{r}_J$ (see section 3.5).

Since the only gauge fields appearing in heterotic orbifold models are those of the E_8 symmetries, it has been proposed in [62, 49] that some of these modes are actually diagonal modes of subgroups of the E_8 factors and subgroups of the $SU(N)$ factors[4] of the individual fixed points. This, in turn, solves the problem of fields charged under both ends of the interval, since the modes of the diagonal symmetries now effectively live on the seven-dimensional planes ascribed to the fixed points *as well as* on the whole ten-dimensional plane at the end of the interval to which the gauge symmetry appearing the in orbifold model corresponds.

However, the detailed assignment of fields made in [62, 49] is not compatible with the above analysis, as we will show in the following for the example of the $SU(9) \times E_6 \times SU(3)$ orbifold as treated in the last section. The authors of [62, 49] chose the SU(3) symmetry to be the diagonal of $SU(3) \subset E_8$ and the SU(3) symmetries of the nine fixed points. However, the $(\mathbf{9}, \mathbf{1}, \mathbf{3})$ twisted fields was also expected to live on the boundary corresponding to $SU(9) \subset E_8$, which is clearly not possible by the above analysis, since we showed in the last section that those fields must contain two hypermultiplets of supergravity modes in the smooth K3 compactification. Even more, since these modes must appear as scalars in the vector multiplets of the seven-dimensional theory, they must transform under the adjoint[5] $\mathbf{8}$ of SU(3). In conclusion, the modes $(\mathbf{9}, \mathbf{1}, \mathbf{3})$ must result from a yet to be determined coupling of bulk modes in the $\mathbf{8}$ to modes living at the ends of the interval.

As the authors of [62, 49] have verified their result using anomaly cancellation, we have to comment on that point. In [62, 49], cancellation of anomalies has been shown explicitly including the gravitational anomalies coming from eleven-dimensional bulk fields and ten-dimensional gauge fields. However, as these anomalies come from integrals over the ten-planes at both ends of the interval and have been shown to be canceled locally from an eleven-dimensional viewpoint in [57, 56, 26], they no longer play a role for local anomaly cancellation on the six-planes located at orbifold fixed points and the ends of the interval. Therefore, in the analysis of [62, 49] anomalies were canceled in a non-local way (from an eleven-dimensional perspective) and hence can not be used as a definite criterion on where fields have to be located from an eleven-dimensional viewpoint.

As we have seen in the beginning of section 4.3, the problem of supergravity modes appearing as charged modes in the orbifold limit is already present in the example of the standard embedding. There one E_8 remains completely unbroken

[4] In [62, 49] these have been termed nonperturbative $SU(N)$ symmetries.

[5] By supersymmetry, scalars as well a spinors appearing in vector multiplets are always transforming in the adjoint.

which, by the charge of the supergravity modes, suggests the more that the SU(2) of the first E_8 extends to the interior of the interval.

In fact, this a general problem of all orbifold models treated in this work, since the rank of the gauge group (sixteen from $E_8 \times E_8$) is not reduced. Therefore, as smooth backgrounds always imply rank reduction, since some subgroup of $E_8 \times E_8$ has to support a smooth instanton background, the moduli controlling the blowups have to be charged to make a Higgs mechanism possible.

Summing up, we conclude that the assignment of modes as suggested by index theory (as in in the example of (4.3.8) or orbifolds with standard embedding) indeed can be trusted, even from an eleven-dimensional viewpoint. Therefore, the gravitational moduli of the fixed points have to appear as scalars in vector multiplets of an seven-dimensional SU(N) supersymmetric gauge theory assigned to the interior of the interval located at the fixed points of the orbifold singularities. By supersymmetry, the scalars are always transforming in the adjoint of the SU(N) gauge theory. The only way to explain the observed field content at the fixed points is to demand that this field content results from a yet unknown coupling to boundary fields, including the possibility of additional multiplets sitting at the boundary on orbifold fixed points as suggested by index calculations.

The crucial unknown to be determined in future work is to work out the precise form of the couplings of bulk modes to boundary modes. These couplings must especially explain how the bulk SU(N) gauge theories mix with the boundary E_8 gauge theories in a supersymmetric way. As such, this problem should be accessible from a viewpoint of purely six-dimensional $\mathcal{N} = (0,1)$ supersymmetric gauge theory.

Chapter 5

Conclusion and Outlook

We have devoted our work to a detailed study of orbifolds and Kaluza-Klein-monopoles of heterotic $E_8 \times E_8$ string theory.

Analyzing heterotic orbifolds in the operator approach, we have shown that there is a strong as well as a weak version of the level matching condition where the strong one is essential for the correct transformation properties of states in the twisted sectors. Furthermore, only the weak level matching condition is relevant for the classification of orbifold models and is equivalent to a condition on fractional instanton numbers of $E_8 \times E_8$ instantons sitting on blown up orbifold singularities. By this relation, all orbifolds considered in this work can be classified by flat $E_8 \times E_8$ bundles on the orbifolds with the fixed points taken out, under the only constraint that the fractional parts of gravitational and gauge instanton numbers match. This directly carries over to M-theory on $S^1/\mathbb{Z}_2$.

For Kaluza-Klein-monopoles we constructed solutions in background Wilson lines and verified our results by calculating quantum numbers and comparing them to those of KK-monopoles in toroidal compactifications. We developed the t'Hooft ansatz in the background of Kaluza-Klein-monopoles to study non-abelian instantons on Kaluza-Klein-monopoles and Eguchi-Hanson spaces, including the case of Wilson line backgrounds. We proposed that the moduli space of a single SU(2) non-abelian instanton on a Kaluza-Klein-Monopole background is given by the t'Hooft ansatz as in flat space and explicitly showed that instantons can become pointlike at orbifold singularities even though their scale parameter remains finite. Only the case of sending the scale parameter to zero corresponds to a small instanton singularity. These results were used to show that orbifold models with standard embedding can locally be analyzed by studying Kaluza-Klein-monopoles with non-abelian instantons. This implies that heterotic $E_8 \times E_8$ orbifold models with standard embedding contain small instanton singularities in their moduli spaces where tensionless strings appear. On the other hand, we argued that N Kaluza-Klein monopoles with N SU(2) non-abelian instantons can for general N be described by heterotic string theory on $\mathbb{C}^2/\mathbb{Z}_N$.

Applying the above results to orbifold models, we gave new evidence that higgsing of the models leads to smooth K3 compactifications of heterotic $E_8 \times E_8$ string theory. Our arguments are based on index calculations as well as details of the Higgs mechanism in $\mathcal{N} = (0,1)$, $D = 6$ supersymmetry. In particular, we give evidence that massless modes ascribed to the supergravity multiplet in ten dimensions such as the geometric moduli of K3 have to appear in twisted sectors as massless modes charged with respect to gauge groups of the orbifold models. From the perspective of M-theory on $S^1/\mathbb{Z}_2$, these modes are eleven-dimensional bulk modes which have to live in the interior of the interval, as expected arguments on M-theory on $\mathbb{Z}_N$ orbifold singularities. Especially, these modes cannot be localized on the ends of the eleven-dimensional interval as suggested in the previous literature. We point out that this problem is a general problem of the orbifold models treated in this work and especially is already present in models with standard embedding.

Based on these observations, there is a number of possibilities for future research. First of all, it is desirable to develop the correct low energy effective action for heterotic orbifolds from the ten-dimensional viewpoint and study the moduli spaces of individual fixed points, perhaps on the lines of [61, 19]. It is to be expected that supersymmetry, as already in the discovery of M-theory on $S^1/\mathbb{Z}_2$ in [57, 56], plays a leading role in extending the action to eleven dimensions. Of course, the same applies to local anomaly cancellation, which has not been carried out fully, neither ten-dimensional nor eleven-dimensional.

In case of non-abelian instantons on Kaluza-Klein-monopoles, it might be interesting to determine the full classical instanton moduli space, perhaps based on the methods of [65]. Furthermore, as suggested by the relation of Kaluza-Klein-monopoles to orbifold models, the moduli space of instantons on Eguchi-Hanson spaces combined with the moduli space of Eguchi-Hanson spaces themselves should be described by the gauge theory living on the corresponding orbifold singularity.

Another possible way to shed light on the quantum moduli space of instantons on Kaluza-Klein-Monopoles as well as the corresponding heterotic orbifolds might be to try and use $\mathrm{Sl}(2,\mathbb{Z})$ S-duality in $\mathcal{N} = 4$, $D = 4$ supersymmetry. By the same arguments as in [86], these moduli spaces should be related to the moduli spaces of elementary string states appearing in toroidal compactifications.

Finally, it remains to discuss four-dimensional orbifold models from the M-theory perspective. These models should also be related to the recent work on manifolds of G_2 holonomy (see [108, 1] and references therein).

Appendix A

Fractional Instanton Numbers of E_8

We consider the situation of an E_8 instanton localized in the interior of a real four-manifold[1] M with boundary L, that is, we treat the gauge bundle as being flat on L. We start by showing that the fractional part of the instanton number depends only on the isomorphism class of the bundle on L. After that we construct one explicit element of that class corresponding to the given data and finally compute the fractional instanton number of that element[2].

The first two steps are mostly standard (see for example [93, 94]) whereas the third one requires a bit more technology (see especially [21] chapter III and [20]). The calculation of this appendix has been previously published in [27].

A.1 Basic Facts

The instanton number is defined as (a discussion of the normalization can be found in [5])

$$I = -\frac{1}{60}\frac{1}{8\pi^2}\int_M \operatorname{Tr} F^2 \tag{A.1.1}$$

with the trace in the adjoint representation. Since E_8 is semi-simple the first nontrivial homotopy group is $\pi_3(E_8) = \mathbb{Z}$. This implies, due to the presence of the boundary, that the bundle is trivial on M. This can be seen as follows: a possible obstruction in constructing a section of a principal E_8-bundle on M (and thereby showing triviality of the bundle) is given by an nonzero element of $H^4(M, \mathbb{Z})$. Since M is an orientable and compact manifold with boundary, we

[1]To be precise, we require M to be orientable, compact and connected with a connected boundary.

[2]More mathematically speaking, in step two we construct a bundle map from the Hopf fibration $S^3 \longrightarrow S^2$ to $E_8 \longrightarrow E_8/T^8$ with T^8 a maximal torus of E_8 and in step three we calculate the image of the fundamental cycle of S^3 under that map by a spectral sequence.

have by duality $H^4(M,\mathbb{Z}) \simeq H_0(M,\partial M,\mathbb{Z}) = 0$. (see for example [94] part III, or [103] for a nice introduction) (A.1.1) now reads

$$I = -\frac{1}{30}\frac{1}{8\pi^2}\int_L \mathrm{Tr}\left(AF - \frac{1}{3}A^3\right) = \frac{1}{3\cdot 60}\frac{1}{8\pi^2}\int_L \mathrm{Tr}\, A^3 \tag{A.1.2}$$

To make contact to the situation studied in section 2.5 we have to take $L = S^3/\mathbb{Z}_N$ (a lens space) with $\pi_1(L) = \mathbb{Z}_N$. By the same reasoning as at the end of section 2.5, S^3 is the covering space of L and (r,γ) specifies a flat bundle on L. Moreover, by pulling back the bundle via the covering map $\pi : S^3 \to S^3/\mathbb{Z}_N$ we get a bundle on S^3 on which,since $\pi_1(S^3) = 0$, A can be gauge transformed to $A' = 0$. Denoting the gauge transformation by $g : S^3 \to \mathrm{E}_8$ we get $A = g^{-1}A'g + g^{-1}\mathrm{d}g = g^{-1}\mathrm{d}g$. Plugging that into (A.1.2) we get $I = I_S/N$ with I_S defined by

$$I_S = \frac{1}{3\cdot 60}\frac{1}{8\pi^2}\int_{S^3} \mathrm{Tr}(g^{-1}\mathrm{d}g)^3 \tag{A.1.3}$$

Of course g represents an element of $\pi_3(\mathrm{E}_8)$ which is identified with $\mathbb{Z}$ by (A.1.3) (see again [5]). This especially shows that I is a multiple of $1/N$.

To show that I_S changes by a multiple of N when the bundle on L is gauge transformed we consider two flat connections A_1 and A_2 on L, both corresponding to the same generator (r,γ). Since both bundles are isomorphic we have $A_2 = h^{-1}A_1h + h^{-1}\mathrm{d}h$ for some $h : L \to \mathrm{E}_8$. Now h can be lifted to S^3 giving $h_S = h \circ \pi$ and we get $A_2 = g_2^{-1}\mathrm{d}g_2 = h_S^{-1}g_1^{-1}(\mathrm{d}g_1)h_S + h_S^{-1}\mathrm{d}h_S = (g_1h_S)^{-1}\mathrm{d}(g_1h_S)$. But since the pointwise product $g \cdot g'$ for two elements of π_3 of a Lie group G corresponds to the group addition of the elements, we have $g_2 = g_1 + h_S$.

To proceed, we note that, since π_3 is the first nontrivial homotopy class of E_8, by the Hurewicz isomorphism $\pi_3(\mathrm{E}_8)$ is isomorphic to $H_3(\mathrm{E}_8,\mathbb{Z})$ and further by the universal coefficient theorem to $H^3(\mathrm{E}_8,\mathbb{Z})$ because, by the same argument, we have $H^1 = H^2 = H_1 = H_2 = 0$. Therefore the instanton number I_S is given by the pullback via g of the generator ω of $H^3(\mathrm{E}_8,\mathbb{Z})$ evaluated on the fundamental cycle C_S of S^3:

$$I_S = g^*\omega(C_S) \tag{A.1.4}$$

Since $\pi : S^3 \to L$ is N to one we have $\pi_*C_S = NC_L$ with C_L the fundamental cycle of L. This immediately yields $h_S^*\omega(C_S) = \pi^*h^*\omega(C_S) = h^*\omega(\pi_*C_S) = Nh^*\omega(C_L) \in N\mathbb{Z}$.

A.2 Construction of the Bundle

We now construct a bundle on S^3 simply by constructing the map g, which we require to obey $g(rx) = \gamma g(x)$. This bundle will obviously be the pullback of a bundle on L that fulfills our requirements.

To construct g (and for the final step) we will need some basic facts about lens spaces (see [21], § 18). As above we define L by the fibration $\mathbb{Z}_N \longrightarrow S^3 \stackrel{\pi}{\longrightarrow} L$ where S^3 is identified with the unit sphere in $\mathbb{C}^2$ and the generator of $\mathbb{Z}_N$ acts on S^3 like[3]

$$e^{2\pi i/N} : (Z^1, Z^2) \mapsto (e^{2\pi i/N} Z^1, e^{2\pi i/N} Z^2) \tag{A.2.1}$$

This action is of course compatible with the U(1) action on S^3 (where U(1) is identified with the unit circle in $\mathbb{C}$)

$$(Z^1, Z^2) \mapsto (\lambda Z^1, \lambda Z^2) \qquad \lambda \in S^1 \subset \mathbb{C} \tag{A.2.2}$$

and we get the Hopf-fibration $S^1 \longrightarrow S^3 \stackrel{\pi_S}{\longrightarrow} \mathbb{C}P^1 \simeq S^2$. Now since $\mathbb{Z}_N \subset \mathrm{U}(1)$ the U(1) action descends to an action on L and we have

$$\begin{array}{ccccc} S^1 & \longrightarrow & S^3 & \stackrel{\pi_S}{\longrightarrow} & S^2 \\ & & \pi \downarrow & & 1\!\!1 \downarrow \\ S^1 & \longrightarrow & L & \stackrel{\pi_L}{\longrightarrow} & S^2 \end{array} \tag{A.2.3}$$

In this diagram π is a bundle map which is N to one on the standard fibre.

This can be made more explicit by identifying S^2 with a cylinder $I \times S^1$ where at both ends of the interval S^1 is identified to a point. We parametrize $I \times S^1$ by (ϱ, Ψ) where $\varrho \in \left[0, \frac{\pi}{2}\right], \Psi \in [0, 2\pi[$. To write down the bundle explicitly we divide S^2 into upper (D^2_+) and lower (D^2_-) hemisphere. Using $(Z^1, Z^2) = (\cos\varrho\, e^{i\phi_1}, \sin\varrho\, e^{i\phi_2})$ we write

$$\begin{array}{llll} D^2_+ : & \varrho \geq \frac{\pi}{4} & \phi_2 = \phi_+ & \phi_1 = \phi_+ - \Psi \\ D^2_- : & \varrho \leq \frac{\pi}{4} & \phi_1 = \phi_- & \phi_2 = \phi_- + \Psi \end{array} \tag{A.2.4}$$

Therefore $\lambda = e^{i\phi} \in \mathrm{U}(1)$ acts like $\phi_- \mapsto \phi_- + \phi$, $\phi_+ \mapsto \phi_+ + \phi$ and the fibre S^1 is parametrized by $\phi_-, \phi_+ \in [0, 2\pi[$.

Since on the equator $\phi_+ = \phi_- + \Psi$ the sphere S^3 as a bundle ist made of two trivial S^1 bundles on the hemispheres clutched together by the generator of $\pi_1(S^1)$. Analogously, by (A.2.3), the clutching function of L is $N \in \pi_1(S^1)$.

To simplify the construction of g we insert a cylinder $C = I \times S^1$ parametrized by (x, Ψ), $x \in [0,1], \Psi \in [0, 2\pi[$ between the hemispheres by attaching D^2_+ at $x = 1$ and D^2_- at $x = 0$. This is extended to the bundle by parametrizing the fibre as ϕ_+ as on D^2_+. Therefore the bundle is now clutched non-trivially at $x = 0$ and trivially at $x = 1$.

Finally we define $g(\varrho, \Psi, \phi_+) = g_+(\varrho, \Psi, \phi_+) = e^{iqH\phi_+}$ on D^2_+ and $g(\varrho, \Psi, \phi_-) = g_-(\varrho, \Psi, \phi_-) = e^{iqH\phi_-}$ on D^2_-. This yields

$$\begin{array}{lll} g(x = 0, \Psi, \phi_+) & = g_-(\varrho = \frac{\pi}{4}, \Psi, \phi_+ - \Psi) & = e^{iqH(\phi_+ - \Psi)} \\ g(x = 1, \Psi, \phi_+) & = g_+(\varrho = \frac{\pi}{4}, \Psi, \phi_+) & = e^{iqH\phi_+} \end{array} \tag{A.2.5}$$

[3]Here we have chosen the generator to act like $e^{2\pi i/N}$ on Z^2, as opposed to orbifolds where it acts like $e^{-2\pi i/N}$. This corresponds to taking the opposite orientation and therefore the instanton number changes sign.

on the cylinder C and g can be extended to $g(x,\Psi,\phi_+) = e^{iqH\phi_+}$ at $\Psi = 0, 2\pi$.

g at $\phi_+ = 0$ is now defined on the boundary of the square $0 \le x \le 1,\ 0 \le \Psi \le 2\pi$ (the identification of $\Psi = 0$ with $\Psi = 2\pi$ plays no role in the consideration) and the only nontrivial step in the construction is to extend this definition to the interior of the square. Topologically this is the same as to extend a map $g : S^1 \to \mathrm{U}(1) \subset \mathrm{E}_8$ with $S^1 = \partial D^2$ to the whole of D^2. Therefore we consider the fibration $\mathrm{U}(1) \longrightarrow \mathrm{E}_8 \longrightarrow \mathrm{E}_8/\mathrm{U}(1)$ where $\mathrm{U}(1)$ is the subgroup of E_8 generated from $e^{iqH\phi}$. The relative homotopy sequence of this fibration contains the following part

$$\ldots \longrightarrow \pi_2(\mathrm{E}_8) = 0 \longrightarrow \pi_2(\mathrm{E}_8, \mathrm{U}(1)) \xrightarrow{\partial} \pi_1(\mathrm{U}(1)) \longrightarrow \pi_1(\mathrm{E}_8) = 0 \longrightarrow \ldots \tag{A.2.6}$$

Since the sequence is exact the map ∂ is an isomorphism and g can be extended to $\phi = 0$ in C. Finally we define g on the whole of C by $g(x,\Psi,\phi_+) = e^{iqH\phi_+} g(x,\Psi,\phi_+ = 0)$.

A.3 Calculation of the Instanton Number

As the map g constructed in the last section actually fulfills $g(e^{i\phi}Z^i) = e^{iqH\phi} g(Z^i)$ it is a bundle map from S^3 to E_8 which especially provides a homomorhism from the (cohomology) spectral sequence of the former to that of the latter[4].

We start by writing down the standard example of the spectral sequence of the Hopf fibration. The E_2 term is given by $E_2^{p,q} = H^p(S^2) \otimes H^q(S^1)$, explicitly:

$$E_2 = \begin{array}{c|ccc} 1 & \mathbb{Z}/a_S & 0 & \mathbb{Z}/a_S b \\ 0 & \mathbb{Z}/1 & 0 & \mathbb{Z}/b \\ \hline & 0 & 1 & 2 \end{array} \tag{A.3.1}$$

where p, q label columns and rows and the diagonal arrow denotes the map d_2. By H/h we denote the group H generated by h. As the sequence stops at E_3 we have $E_3 \simeq H^*(S^3) = (\mathbb{Z}/1, 0, 0, \mathbb{Z}/c_S)$ with $c_S = a_S b$. This implies that d_2 is an isomorphism from $E_2^{0,1}$ to $E_2^{2,0}$.

We now turn to the spectral sequence of the fibration $T^8 \longrightarrow \mathrm{E}_8 \longrightarrow \mathrm{E}_8/T^8$ where T^8 is a maximal torus of E_8 containing the $\mathrm{U}(1)$ generated from the elements $e^{iqH\phi}$. First we need to verify that the base is simply connected. This is clear from the homotopy sequence

$$\ldots \longrightarrow \pi_1(\mathrm{E}_8) = 0 \longrightarrow \pi_1(\mathrm{E}_8/T^8) \longrightarrow \pi_0(T^8) = \{*\} \longrightarrow \cdots \tag{A.3.2}$$

[4]It can be easily seen that we could restrict ourselves to a map to Spin(16) at this point: by acting with the Weyl group we can map q into the Spin(16) sublattice of E_8. Since $\pi_1(\mathrm{Spin}(16)) = 0$ all steps of the last section apply as before and we are left with a pure Spin(16) bundle. This then allows to compute the instanton number with the formulas given in [61, 2]. However, we will proceed in a different way, since our calculation gives the instanton number (including the integer part) for gauge bundles satisfying $g(e^{i\phi}Z^i) = e^{iqH\phi} g(Z^i)$.

(π_0 is not a group here and consists only of the (arbitrary) base point $*$). Furthermore, as shown by Morse theoretic methods in [20], the base is torsion free and $H^{2n+1}(\mathrm{E}_8/T^8) = 0$ for $n \in \mathbb{Z}$. With this information the E_2 term is[5]

$$E_2 = \quad \begin{array}{c|ccccc} 3 & 56\mathbb{Z} & & & & \\ 2 & 28\mathbb{Z} \searrow & 0 & & & \\ 1 & 8\mathbb{Z} \searrow & 0 & 64\mathbb{Z} \searrow & & \\ 0 & \mathbb{Z} & 0 & 8\mathbb{Z} & 0 & 35\mathbb{Z} \\ \hline & 0 & 1 & 2 & 3 & 4 \end{array} \tag{A.3.3}$$

This can be seen as follows: Firstly, we note that $H^*(T^8) = (\mathbb{Z}, 8\mathbb{Z}, 28\mathbb{Z}, 56\mathbb{Z}, \ldots)$. Secondly, since $H^*(\mathrm{E}_8) = (\mathbb{Z}/1, 0, 0, \mathbb{Z}/\omega, \text{higher groups})$ by Hurewicz E_3 must be trivial for degree 1 and 2. Therefore, again $\mathrm{d}_2 : E_2^{0,1} \to E_2^{2,0}$ is an isomorphism. This implies, since by the Künneth formula $H^*(T^8) = (H^*(S^1))^8$ (with respect to $\otimes$), that all maps d_2 from the first to the third column are invertible.

Moreover, since d_3 maps everything to zero all elements of $H^*(\mathrm{E}_8)$ up to degree three are given by E_3 up to degree three. However, because d_2 from the first to the third column is invertible, the only nonvanishing element up to degree three of E_3 must be $E_3^{2,1} = \mathbb{Z}/\omega$. This implies $E_2^{4,0} = 35\mathbb{Z}$.

This can independently be verified by calculating $H^*(\mathrm{E}_8/T^8)$ as described in [20]: the dimension of $H^{2n}(\mathrm{E}_8/T^8)$ is given by the number of elements of the Weyl group which change the sign of precisely q of the positive roots. This can be computed easily, since, for a given Weyl reflection σ, q is given by the length of σ (see for example [59] section 10.3), i.e. the (minimal) number of simple Weyl reflections σ can be composed of $\sigma = \sigma_{\alpha_1} \cdot \ldots \cdot \sigma_{\alpha_q}$. For $q = 1$ there are 8 simple roots so $H^2(\mathrm{E}_8/T^8) = 8\mathbb{Z}$. For $q = 2$ there are $8 \cdot 7/2 + 7 = 35$ combinations, because simple Weyl reflections of two simple roots which are not connected by a line in the Dynkin diagram commute. So $H^4(\mathrm{E}_8/T^8) = 35\mathbb{Z}$.

Explicitly we denote the generators of $H^1(T^8)$ by $a^i, i = 1, \ldots, 8$ and those of $H^2(\mathrm{E}_8/T^8)$ by $b^i = \mathrm{d}_2 a^i$. As $E_2^{0,2} = H^2(T^8)$ is generated by the 28 elements $a^i a^j$ with $i < j$ their image under d_2 in $E_2^{2,1}$ is given by the elements $\mathrm{d}_2(a^i a^j) = b^i a^j - a^i b^j = -a^i b^j + a^j b^i$.

To calculate ω we write all elements in terms of an euclidian basis A_I of the real homology $H_1(T^8, \mathbb{R})$ and its dual a^I in $H^1(T^8, \mathbb{R})$. We choose the basis such that it is compatible with the lattice Λ_8 given in the introduction. Then ω can be written as $\omega = w_{IJ} a^I b^J$ where w_{IJ} is a symmetric matrix, because the image of d_2 consists of all elements $w_{IJ} a^I b^J$ with antisymmetric w_{IJ}. Under an element T of the Weyl group w is transformed to $T^{-1} w T$ (T is an orthogonal matrix). As w must be invariant under the Weyl group, which acts irreducible on a vector, by Schur's lemma, $\omega = w \delta_{IJ} a^I b^J$.

[5] Addition and multiplication of groups are written with respect to the operations $\oplus$ and $\otimes$ on abelian groups, i.e. $2\mathbb{Z} = \mathbb{Z} \oplus \mathbb{Z}$, $\mathbb{Z}^2 = \mathbb{Z} \otimes \mathbb{Z}$.

We now turn to the calculation of the instanton number. By construction g maps the fundamental cycle A_S of the S^1 fiber of the Hopf fibration to $q^I A_I$. This implies

$$g^* a^I(A_S) = a^I(g_* A_S) = a^I(q^J A_J) = q^I \tag{A.3.4}$$

and therefore $g^* a^I = q^I a_S$. Finally we have

$$\begin{aligned} I_S = g^*\omega(C_S) &= w\delta_{IJ}\ (g^* a^I\ g^* b^J)(C_S) = w\delta_{IJ}\ (g^* a^I\ g^*(\mathrm{d}_2 a^J))(C_S) \\ &= w\delta_{IJ}\ (g^* a^I\ \mathrm{d}_2(g^* a^J))(C_S) = w\delta_{IJ}\ (q^I a_S\ q^J \mathrm{d}_2 a_S)(C_S) \\ &= w\delta_{IJ}\ q^I q^J\ (a_S b)(C_S) = w q^2 \end{aligned} \tag{A.3.5}$$

To normalize this equation we compare to the standard embedding $I_S = 1$, $q^I = (1, 1, 0, \ldots)$ resulting in $w = 1/2$. Therefore we have

$$I = \frac{I_S}{N} = \frac{1}{2N} q^2 = \frac{N}{2}\beta^2 \pmod 1 \tag{A.3.6}$$

Appendix B

Group Theory

This appendix is devoted to sketches of some of the group theoretical calculations used in the main chapters. A good introduction, from a physicist's point of view, is given in [91]. More mathematical treatments can be found in [29, 59]. A collection of useful material can be found in chapter 13 of [42].

B.1 E_8 and its Lattice

On the euclidian space $\mathbb{R}^8$ the lattice of the group E_8 can be given as

$$\Gamma_8 = \left\{ (n^I), (\tfrac{1}{2} + n^I) \;\middle|\; n^I \in \mathbb{Z},\ \textstyle\sum_{I=1}^{8} n^I = 0 \pmod 2 \right\} \tag{B.1.1}$$

where the roots of E_8 are precisely those vectors with length $p^I p^I = 2$:

$$\begin{array}{rl} (\pm 1, \pm 1, 0^6) & \text{and permutations} \\ ((\pm\frac{1}{2})^8) & \text{even number of } - \text{ signs} \end{array} \tag{B.1.2}$$

We introduce an ordering by saying that a positive vector is given by a vector whose first nonvanishing entry is positive. Therefore a root α is bigger than α' if the difference $\alpha - \alpha'$ is positive. Such an ordering corresponds to simple roots α^i of E_8: since all positive roots $\alpha = a_i \alpha^i$ must have positive coefficients a_i in the simple roots, we start from the lowest positive root and proceed to the higher roots, writing down only those which are linearly independent of those we already have. This gives in ascending order

$$
\begin{array}{cccccccccc}
(& 0 & ,0 & ,0 & ,0 & ,0 & ,0 & ,+1 & ,-1 &) \\
(& 0 & ,0 & ,0 & ,0 & ,0 & ,0 & ,+1 & ,+1 &) \\
(& 0 & ,0 & ,0 & ,0 & ,0 & ,+1 & ,-1 & ,0 &) \\
(& 0 & ,0 & ,0 & ,0 & ,+1 & ,-1 & ,0 & ,0 &) \\
(& 0 & ,0 & ,0 & ,+1 & ,-1 & ,0 & ,0 & ,0 &) \\
(& 0 & ,0 & ,+1 & ,-1 & ,0 & ,0 & ,0 & ,0 &) \\
(& 0 & ,+1 & ,-1 & ,0 & ,0 & ,0 & ,0 & ,0 &) \\
(& +\frac{1}{2} & ,-\frac{1}{2} & ,-\frac{1}{2} & ,-\frac{1}{2} & ,-\frac{1}{2} & ,-\frac{1}{2} & ,-\frac{1}{2} & ,+\frac{1}{2} &) \\
\\
(& +1 & ,+1 & ,0 & ,0 & ,0 & ,0 & ,0 & ,0 &)
\end{array}
\tag{B.1.3}
$$

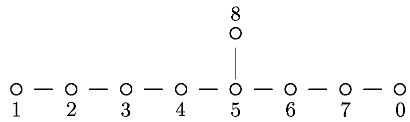
where the last vector shows the highest root θ. We calculate the scalar products $A^{ij} = \alpha^i \cdot \alpha^j$ and, after some reordering (given below), arrive at the Cartan matrix

$$
A = \begin{pmatrix}
2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\
-1 & 2 & -1 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 2 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & 2 & -1 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & 2 & -1 & 0 & -1 \\
0 & 0 & 0 & 0 & -1 & 2 & -1 & 0 \\
0 & 0 & 0 & 0 & 0 & -1 & 2 & 0 \\
0 & 0 & 0 & 0 & -1 & 0 & 0 & 2
\end{pmatrix}
\tag{B.1.4}
$$

which uniquely identifies the group E_8. In addition, this shows that the lattice is even and self-dual, since $A^{ii} = 2$ and $\det A = 1$. The roots are given as follows:

$$
\begin{array}{lcccccccccc}
\alpha^0 = -\theta & = (& -1 & ,-1 & ,0 & ,0 & ,0 & ,0 & ,0 & ,0 &) \\
\\
\alpha^1 & = (& 0 & ,+1 & ,-1 & ,0 & ,0 & ,0 & ,0 & ,0 &) \\
\alpha^2 & = (& 0 & ,0 & ,+1 & ,-1 & ,0 & ,0 & ,0 & ,0 &) \\
\alpha^3 & = (& 0 & ,0 & ,0 & ,+1 & ,-1 & ,0 & ,0 & ,0 &) \\
\alpha^4 & = (& 0 & ,0 & ,0 & ,0 & ,+1 & ,-1 & ,0 & ,0 &) \\
\alpha^5 & = (& 0 & ,0 & ,0 & ,0 & ,0 & ,+1 & ,-1 & ,0 &) \\
\alpha^6 & = (& 0 & ,0 & ,0 & ,0 & ,0 & ,0 & ,+1 & ,-1 &) \\
\alpha^7 & = (& +\frac{1}{2} & ,-\frac{1}{2} & ,-\frac{1}{2} & ,-\frac{1}{2} & ,-\frac{1}{2} & ,-\frac{1}{2} & ,-\frac{1}{2} & ,+\frac{1}{2} &) \\
\alpha^8 & = (& 0 & ,0 & ,0 & ,0 & ,0 & ,0 & ,+1 & ,+1 &)
\end{array}
\tag{B.1.5}
$$

where we have also included the lowest root $\alpha^0 = -\theta$. Its only nonvanishing scalar product with another root is given by $\alpha^7 \cdot \alpha^0 = -1$. Therefore, we arrive at the extended Dynkin diagram of E_8



(B.1.6)

Here we have chosen the labeling of the roots such that it is most compatible with the standard labeling of the SO(16) subgroup of E_8. Since this is the subgroup which is manifest in fermionic formulations of the heterotic string, quantum numbers will be canonical with this labeling.

The fundamental weights γ_i of E_8 are defined to be the generators of the lattice dual to the weight lattice (which is the same as the root lattice for E_8). Therefore $\gamma_i \cdot \alpha^j = \delta_i^j$ and the dual metric is given as $A_{ij} = (A^{-1})_{ij}$. As raising and lowering is achieved using the metric (and A is symmetric, as for all ADE groups) we have

$$\gamma_i = (A^{-1})_{ij}\alpha^j \tag{B.1.7}$$

This gives

$$A^{-1} = \begin{pmatrix} 2 & 3 & 4 & 5 & 6 & 4 & 2 & 3 \\ 3 & 6 & 8 & 10 & 12 & 8 & 4 & 6 \\ 4 & 8 & 12 & 15 & 18 & 12 & 6 & 9 \\ 5 & 10 & 15 & 20 & 24 & 16 & 8 & 12 \\ 6 & 12 & 18 & 24 & 30 & 20 & 10 & 15 \\ 4 & 8 & 12 & 16 & 20 & 14 & 7 & 10 \\ 2 & 4 & 6 & 8 & 10 & 7 & 4 & 5 \\ 3 & 6 & 9 & 12 & 15 & 10 & 5 & 8 \end{pmatrix} \tag{B.1.8}$$

We will denote vectors in the basis of the simple roots by $\{\ldots\}$. The coefficients in the dual basis γ_i, which is called Dynkin basis, will be denoted by $[\ldots]$. The coefficients themselves are called Dynkin labels.

The coefficients of the highest root $\theta = n_i\alpha^i$ in terms of the simple roots are called marks and can be easily computed

$$n_i = \{2, 3, 4, 5, 6, 4, 2, 3\} \tag{B.1.9}$$

Setting $n_0 = 1$, the $n_{\hat{\imath}}$ where $\hat{\imath} = 0, 1, \ldots, 8$ spawn the kernel of the extended Cartan matrix $A^{\hat{\imath}\hat{\jmath}} = \alpha^{\hat{\imath}} \cdot \alpha^{\hat{\jmath}}$. This is clear from

$$\begin{aligned} A^{0\hat{\imath}}n_{\hat{\imath}} &= (-\theta)\cdot(-\theta) + (-\theta)\cdot\alpha^i n_i = \theta\cdot\theta + -\theta\cdot\theta = 0 \\ A^{j\hat{\imath}}n_{\hat{\imath}} &= \alpha^j\cdot(-\theta) + \alpha^j\cdot\alpha^i n_i = -\alpha^j\cdot\theta + \alpha^j\cdot\theta = 0 \end{aligned} \tag{B.1.10}$$

B.2 Classifying $\mathbb{Z}_N$ Shift Vectors of E_8

To classify shift vectors for heterotic orbifolds in section 2.4 we need to find all possible group elements of E_8 corresponding to vectors β such that $s = N\beta \in \Gamma_8$ and $N > 1$. We restrict our search to those β for which N is the smallest such integer.

A classification of such shifts was given in [63] (see references therein) and will be described in the following. Since the Weyl group partitions the whole lattice into equivalent chambers, we can restrict ourselves to the fundamental chamber $\mathcal{F}$ (this means that we restrict to weights $s = N\beta$ with $s^i \geq 0$). But even more, since we are looking for group elements $\exp(2\pi i\beta^I H^I)$, lying in a maximal torus and the extended Weyl group partitions that torus into cells (see for instance [20]), we can restrict to the fundamental cell $\Delta_\mathcal{F}$ of the extended Weyl group. This cell is bounded by the plane $\{x\,|\theta x = 1\}$ and contains the origin. Therefore we have

$$\theta\frac{s}{N} = \frac{m_i s^i}{N} \leq 1 \tag{B.2.1}$$

and we define

$$s^0 = N - m_i s^i \tag{B.2.2}$$

which implies

$$N = \sum_{\hat{i}=0}^{8} m_{\hat{i}} s^{\hat{i}} \tag{B.2.3}$$

Therefore the classification is given by all possible ways to fulfill (B.2.3) using non-negative integers $s^{\hat{i}}$.

This classification allows us furthermore to read of the gauge group left unbroken by β from the extended Dynkin diagram (B.1.6). Simply any dot with $s^{\hat{i}} \neq 0$ has to be deleted leaving an unbroken gauge group. For n deleted dots one gets $n-1$ unbroken U(1) factors in addition.

For $N = 2, 3, 4$, the results of this procedure are shown in table B.1.

B.3 Equivalences of Vectors

In the literature quite often shift vectors appear which are not located in the fundamental Weyl cell. However, there are a few easy rules to produce equivalent vectors. Since the E_8 Dynkin diagram has no symmetries, there are no outer automorphisms. Therefore, all equivalences either stem from Weyl reflections or from lattice translations.

s^i	s^I	$s^I s^I$	unbroken group
	$N = 2$		
$[2,0,0,0,0,0,0,0,0]$	$= (0^8)$	0	E_8
$[0,0,0,0,0,0,0,0,1]$	$= (2,0^7)$	4	SO(16)
$[0,0,1,0,0,0,0,0,0]$	$= (1,1,0^6)$	2	$E_7 \times SU(2)$
	$N = 3$		
$[3,0,0,0,0,0,0,0,0]$	$= (0^8)$	0	E_8
$[1,0,0,0,0,0,0,0,1]$	$= (2,0^7)$	4	$SO(14) \times U(1)$
$[1,0,1,0,0,0,0,0,0]$	$= (1,1,0^7)$	2	$E_7 \times U(1)$
$[0,0,0,1,0,0,0,0,0]$	$= (2,1,1,0^7)$	6	$E_6 \times SU(3)$
$[0,1,0,0,0,0,0,0,0]$	$= \frac{1}{2}(5,1^7)$	8	SU(9)
	$N = 4$		
$[4,0,0,0,0,0,0,0,0]$	$= (0^8)$	0	E_8
$[2,0,0,0,0,0,0,0,1]$	$= (2,0^7)$	4	$SO(14) \times U(1)$
$[2,0,1,0,0,0,0,0,0]$	$= (1,1,0^7)$	2	$E_7 \times U(1)$
$[1,0,0,1,0,0,0,0,0]$	$= (2,1,1,0^7)$	6	$E_6 \times SU(2)$
$[1,1,0,0,0,0,0,0,0]$	$= \frac{1}{2}(5,1^7)$	8	$SU(8) \times U(1)$
$[0,0,1,0,0,0,0,0,1]$	$= (3,1,0^6)$	10	$SO(12) \times SU(2) \times U(1)$
$[0,0,0,0,0,0,0,1,0]$	$= \frac{1}{2}(7,1^6,-1)$	14	$SU(8) \times SU(2)$
$[0,0,0,0,1,0,0,0,0]$	$= (3,1^3,0^4)$	12	$SO(10) \times SU(4)$

Table B.1: Classification of E_8 shift vectors up to $N = 4$ and the corresponding unbroken gauge groups, derived from the marks $m_{\hat{i}} = \{1,2,3,4,5,6,4,2,3\}$ of E_8.

Weyl Reflections

The Weyl group of the lattice Γ_8 is generated[1] by reflections at the planes perpendicular to the roots of E_8. Therefore, for any root α we have the reflection

$$S_\alpha q = S_{-\alpha} q = q - 2\frac{\alpha \cdot q}{\alpha^2}\alpha = q - (\alpha \cdot q)\alpha \tag{B.3.1}$$

As reflections preserve length, it is enough to study the action on $s = m\beta$. We begin by the roots of the form

$$\alpha^I = (+1,+1,0^6) \qquad \text{and permutations} \tag{B.3.2}$$

We get

$$(S_\alpha s)^I = s^I - (\alpha \cdot s)\alpha^I = s^I - (s^1 + s^2)\alpha^I \tag{B.3.3}$$

[1]To be precise, the Weyl group is generated by the reflections of the simple roots.

therefore

$$\begin{aligned}(S_\alpha s)^1 &= -s^2\\ (S_\alpha s)^2 &= -s^1\end{aligned} \tag{B.3.4}$$

and the remaining components are unchanged.

Analogously, for the roots of the form

$$\alpha^I = (+1, -1, 0^6) \qquad \text{and permutations} \tag{B.3.5}$$

we have

$$\begin{aligned}(S_\alpha s)^1 &= s^2\\ (S_\alpha s)^2 &= s^1\end{aligned} \tag{B.3.6}$$

where, again, the remaining components are unchanged. This shows that one can sort the components of β in arbitrary order and, given one component is zero, turn all to positive value.

Furthermore, we have the possibility of generating as much as possible zeros in s^I by Weyl-reflecting with $\alpha = (\frac{1}{2}^8)$:

$$(S_\alpha s) = s^I - \frac{1}{4}\left(\sum_J s^J\right) \tag{B.3.7}$$

Therefore, by changing signs till we arrive at $\sum_I s^I = 4$ we can bring ones in s to zero. Of course, this method will generate ones from zeros and hence works only if there are more ones than zeros.

Of course, there is still the possibility of Weyl reflecting by other half-integer roots, but this is mostly a matter of trial an error. For example, reflecting the vector $\frac{1}{2}(5, 1^7)$ by $\alpha = \frac{1}{2}(1^4, (-1)^4)$ we get

$$\frac{1}{2}\left((5, 1^7) - \frac{1}{4}(5+3-4)(1^4, (-1)^4)\right) = \frac{1}{2}(4, 0^3, 2^4) = (2, 1^4, 0^3) \tag{B.3.8}$$

B.4 Some Subgroups

The Shift $\beta = (1/N, 1/N, 0^6)$

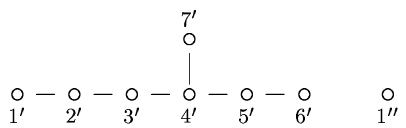
We start by setting $N = 2$. In this case the following positive roots of E_8 are invariant under $\exp(2\pi i \beta^I H^I)$

$$\begin{aligned}&(+1, +1, 0^6) && \\ &(+1, -1, 0^6) && \\ &(0^2, +1, +1, 0^4) && \text{and permutations in } q^{3\ldots 8}\\ &(+\tfrac{1}{2}, -\tfrac{1}{2}, (\pm\tfrac{1}{2})^6) && \text{odd '}-\text{'}\end{aligned} \tag{B.4.1}$$

collecting simple roots we get

$$
\begin{array}{lllllllllll}
\alpha'_1 & = (& 0 & ,0 & ,+1 & ,-1 & ,0 & ,0 & ,0 & ,0 &) \\
\alpha'_2 & = (& 0 & ,0 & ,0 & ,+1 & ,-1 & ,0 & ,0 & ,0 &) \\
\alpha'_3 & = (& 0 & ,0 & ,0 & ,0 & ,+1 & ,-1 & ,0 & ,0 &) \\
\alpha'_4 & = (& 0 & ,0 & ,0 & ,0 & ,0 & ,+1 & ,-1 & ,0 &) \\
\alpha'_5 & = (& 0 & ,0 & ,0 & ,0 & ,0 & ,0 & ,+1 & ,-1 &) \\
\alpha'_6 & = (& +\frac{1}{2} & ,-\frac{1}{2} & ,-\frac{1}{2} & ,-\frac{1}{2} & ,-\frac{1}{2} & ,-\frac{1}{2} & ,-\frac{1}{2} & ,+\frac{1}{2} &) \\
\alpha'_7 & = (& 0 & ,0 & ,0 & ,0 & ,0 & ,0 & ,+1 & ,+1 &) \\
 & & & & & & & & & & \\
\alpha''_1 & = (& +1 & ,+1 & ,0 & ,0 & ,0 & ,0 & ,0 & ,0 &)
\end{array}
\tag{B.4.2}
$$

corresponding to the group $E_7 \times SU(2)$



(B.4.3)

For SU(2) we have the Cartan matrix $A'' = (2)$ with $A''^{-1} = (1/2)$. Therefore we have the highest weights

$$
\begin{array}{llll}
\mathbf{2} & = [1] & = \frac{1}{2}\alpha''_1 & \alpha^1 \\
\mathbf{3} & = [2] & = \alpha''_1 & \alpha^0
\end{array}
\tag{B.4.4}
$$

and their transformation properties under β in powers of $\alpha = \exp(2\pi i/2)$.

For E_7 the highest weights can be read off from

$$
A'^{-1} = \begin{pmatrix}
3/2 & 2 & 5/2 & 3 & 2 & 1 & 3/2 \\
2 & 4 & 5 & 6 & 4 & 2 & 3 \\
5/2 & 5 & 15/2 & 9 & 6 & 3 & 9/2 \\
3 & 6 & 9 & 12 & 8 & 4 & 6 \\
2 & 4 & 6 & 8 & 6 & 3 & 4 \\
1 & 2 & 3 & 4 & 3 & 2 & 2 \\
3/2 & 3 & 9/2 & 6 & 4 & 2 & 7/2
\end{pmatrix}
\tag{B.4.5}
$$

with the lowest dimensional representations

$$
\begin{array}{lll}
\mathbf{56} & = [1,0,0,0,0,0,0] & = (+\frac{1}{2}, -\frac{1}{2}, +1, 0^5) \\
\mathbf{133} & = [0,0,0,0,0,1,0] & = (+1, -1, 0^6)
\end{array}
\tag{B.4.6}
$$

where **133** is the adjoint representation. We have the following decomposition

$$
\begin{array}{lll}
E_8 & \rightarrow & E_7 \times SU(2) \\
\mathbf{248} & \rightarrow & \underset{\alpha^0}{(\mathbf{133}, \mathbf{1})} + \underset{\alpha^0}{(\mathbf{1}, \mathbf{3})} + \underset{\alpha^1}{(\mathbf{56}, \mathbf{2})}
\end{array}
\tag{B.4.7}
$$

Turning to $N > 2$, we recognize that all E_7 roots remain invariant whereas the root of SU(2) transforms as α^2 with $\alpha = \exp(2\pi i/N)$. Therefore the invariant group is now $E_7 \times U(1)$. The generator H'' of U(1) is easy to find

$$H'' = H^1 + H^2 \qquad \text{(B.4.8)}$$

as its charge is zero for all roots of E_7 and it is normalized such that

$$\begin{aligned} \mathrm{SU}(2) &\to \mathrm{U}(1) \\ \mathbf{2} &\to \mathbf{1}_{+1} + \mathbf{1}_{-1} \\ \mathbf{3} &\to \mathbf{1}_{+2} + \mathbf{1}_{0} + \mathbf{1}_{-2} \end{aligned} \qquad \text{(B.4.9)}$$

and states transform as $\alpha^{q''}$ where q'' is the U(1) charge. But since the length of H'' is two, a state corresponding to U(1) charge q'' is given as

$$:e^{i\frac{q''}{2}H''}:\big|0\big\rangle \qquad \text{(B.4.10)}$$

because such a state has the charge vector

$$q^I = (q''/2, q''/2, 0^6) \qquad \text{(B.4.11)}$$

Finally, we have the following decomposition

$$\begin{array}{lcl} \mathrm{E}_8 & \to & \mathrm{E}_7 \times \mathrm{U}(1) \\ \mathbf{248} & \to & \underset{\alpha^0}{\mathbf{133}_0} + \underset{\alpha^0}{\mathbf{1}_0} + \underset{\alpha^2}{\mathbf{1}_{+2}} + \underset{\alpha^{-2}}{\mathbf{1}_{-2}} + \underset{\alpha^1}{\mathbf{56}_{+1}} + \underset{\alpha^{-1}}{\mathbf{56}_{-1}} \end{array} \qquad \text{(B.4.12)}$$

The Shift $\beta = \frac{1}{3}(2, 1^2, 0^6)$

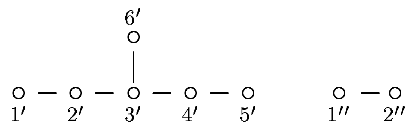
The following positive roots of E_8 are invariant under $\exp(2\pi i \beta^I H^I)$

$$\begin{array}{rl} (0, +1, -1, 0^5) & \\ (+1, +1, 0, 0^5) & \\ (+1, 0, +1, 0^5) & \\ (0^3, +1, -1, 0^3) & \text{and permutations in } q^{4\ldots8} \\ (+\frac{1}{2}, (-\frac{1}{2})^2, (\pm\frac{1}{2})^5) & \text{even '}-\text{'} \end{array} \qquad \text{(B.4.13)}$$

collecting simple roots gives

$$
\begin{array}{llllllllll}
\alpha'_1 & =(& +\frac{1}{2} & ,-\frac{1}{2} & ,-\frac{1}{2} & ,-\frac{1}{2} & ,-\frac{1}{2} & ,-\frac{1}{2} & ,-\frac{1}{2} & ,+\frac{1}{2} \;) \\
\alpha'_2 & =(& 0 & ,0 & ,0 & ,0 & ,0 & ,0 & ,+1 & ,-1 \;) \\
\alpha'_3 & =(& 0 & ,0 & ,0 & ,0 & ,0 & ,+1 & ,-1 & ,0 \;) \\
\alpha'_4 & =(& 0 & ,0 & ,0 & ,0 & ,+1 & ,-1 & ,0 & ,0 \;) \\
\alpha'_5 & =(& 0 & ,0 & ,0 & ,+1 & ,-1 & ,0 & ,0 & ,0 \;) \\
\alpha'_6 & =(& 0 & ,0 & ,0 & ,0 & ,0 & ,0 & ,+1 & ,+1 \;) \\
\\
\alpha''_1 & =(& 0 & ,+1 & ,-1 & ,0 & ,0 & ,0 & ,0 & ,0 \;) \\
\alpha''_2 & =(& +1 & ,0 & ,+1 & ,0 & ,0 & ,0 & ,0 & ,0 \;)
\end{array}
\tag{B.4.14}
$$

corresponding to the group $E_6 \times SU(3)$



(B.4.15)

For E_6 the highest weights can be read off from

$$
A'^{-1} = \frac{1}{3}\begin{pmatrix} 4 & 5 & 6 & 4 & 2 & 3 \\ 5 & 10 & 12 & 8 & 4 & 6 \\ 6 & 12 & 18 & 12 & 6 & 9 \\ 4 & 8 & 12 & 10 & 5 & 6 \\ 2 & 4 & 6 & 5 & 4 & 3 \\ 3 & 6 & 9 & 6 & 3 & 6 \end{pmatrix}
\tag{B.4.16}
$$

with the lowest dimensional representations

$$
\begin{array}{lll}
\mathbf{27} & = [1,0,0,0,0,0] & = \frac{1}{3}(2,(-2)^2,0^5) \\
\overline{\mathbf{27}} & = [0,0,0,0,1,0] & = \frac{1}{3}(1,(-1)^2,3,0^4) \\
\mathbf{78} & = [0,0,0,0,0,1] & = (\frac{1}{2},(-\frac{1}{2})^2,(\frac{1}{2})^5)
\end{array}
\tag{B.4.17}
$$

where **78** is the adjoint representation and we note that all representations of E_6 are invariant under β. For SU(3) we have

$$
A''^{-1} = \frac{1}{3}\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}
\tag{B.4.18}
$$

With some of the highest weight representations

$$
\begin{array}{lllll}
\mathbf{3} & = [1,0] & = \frac{1}{3}(2\alpha''_1 + \alpha''_2) & = \frac{1}{3}(1,2,-1,0^5) & \alpha^1 \\
\bar{\mathbf{3}} & = [0,1] & = \frac{1}{3}(\alpha''_1 + 2\alpha''_2) & = \frac{1}{3}(2,1,+1,0^5) & \alpha^2 \\
\mathbf{8} & = [1,1] & = \frac{1}{3}(3\alpha''_1 + 3\alpha''_2) & = \frac{1}{3}(1,1,0,0^5) & \alpha^0
\end{array}
\tag{B.4.19}
$$

and their transformation properties under β in powers of $\alpha = \exp(2\pi i/3)$. We have the following decomposition

$$\begin{array}{lcl} \mathrm{E}_8 & \rightarrow & \mathrm{E}_6 \times \mathrm{SU}(3) \\ \mathbf{248} & \rightarrow & \underset{\alpha^0}{(\mathbf{78},\mathbf{1})} + \underset{\alpha^0}{(\mathbf{1},\mathbf{8})} + \underset{\alpha^1}{(\mathbf{27},\mathbf{3})} + \underset{\alpha^2}{(\overline{\mathbf{27}},\bar{\mathbf{3}})} \end{array} \tag{B.4.20}$$

This is clear since the highest root of $(\mathbf{27},\mathbf{3})$ is $(1,0,-1,0^5) \in \Gamma_8$.

The Shift $\beta = \frac{1}{6}(5,1^7)$

The following positive roots of E_8 are invariant under $\exp(2\pi i \beta^I H^I)$

$$\begin{array}{rl} (0,+1,-1,0^5) & \text{and permutations in } q^{2\ldots 8} \\ (+\frac{1}{2},(\pm\frac{1}{2})^7) & 0 \text{ or } 6 \text{ '}-\text{' signs} \end{array} \tag{B.4.21}$$

collecting simple roots we get

$$\begin{array}{llllllllll} \alpha'_1 & =(& 0 & ,+1 & ,-1 & ,0 & ,0 & ,0 & ,0 & ,0 \ \ ) \\ \alpha'_2 & =(& 0 & ,0 & ,+1 & ,-1 & ,0 & ,0 & ,0 & ,0 \ \ ) \\ \alpha'_3 & =(& 0 & ,0 & ,0 & ,+1 & ,-1 & ,0 & ,0 & ,0 \ \ ) \\ \alpha'_4 & =(& 0 & ,0 & ,0 & ,0 & ,+1 & ,-1 & ,0 & ,0 \ \ ) \\ \alpha'_5 & =(& 0 & ,0 & ,0 & ,0 & ,0 & ,+1 & ,-1 & ,0 \ \ ) \\ \alpha'_6 & =(& 0 & ,0 & ,0 & ,0 & ,0 & ,0 & ,+1 & ,-1 \ \ ) \\ \alpha'_7 & =(& +\frac{1}{2} & ,-\frac{1}{2} & ,-\frac{1}{2} & ,-\frac{1}{2} & ,-\frac{1}{2} & ,-\frac{1}{2} & ,-\frac{1}{2} & ,+\frac{1}{2} \ \ ) \\ \alpha'_8 & =(& +\frac{1}{2} & ,+\frac{1}{2} & ,+\frac{1}{2} & ,+\frac{1}{2} & ,+\frac{1}{2} & ,+\frac{1}{2} & ,+\frac{1}{2} & ,+\frac{1}{2} \ \ ) \end{array} \tag{B.4.22}$$

corresponding to the group SU(9)

$$\underset{1'}{\circ} - \underset{2'}{\circ} - \underset{3'}{\circ} - \underset{4'}{\circ} - \underset{5'}{\circ} - \underset{7'}{\circ} - \underset{8'}{\circ} \tag{B.4.23}$$

For SU(9) the highest weights can be read off from

$$A'^{-1} = \frac{1}{9} \begin{pmatrix} 8 & 7 & 6 & 5 & 4 & 3 & 2 & 1 \\ 7 & 14 & 12 & 10 & 8 & 6 & 4 & 2 \\ 6 & 12 & 18 & 15 & 12 & 9 & 6 & 3 \\ 5 & 10 & 15 & 20 & 16 & 12 & 8 & 4 \\ 4 & 8 & 12 & 16 & 20 & 15 & 10 & 5 \\ 3 & 6 & 9 & 12 & 15 & 18 & 12 & 6 \\ 2 & 4 & 6 & 8 & 10 & 12 & 14 & 7 \\ 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \end{pmatrix} \tag{B.4.24}$$

with the some of the highest weight representations

$$\begin{array}{llll} \mathbf{9} & = [1,0,0,0,0,0,0,0] & = \frac{1}{6}(1,5,(-1)^6) & \alpha^{1/3} \\ \bar{\mathbf{9}} & = [0,0,0,0,0,0,0,1] & = \frac{1}{6}(5,1^7) & \alpha^{-1/3} \\ \mathbf{80} & = [1,0,0,0,0,0,0,1] & = (1,1,0^6) & \alpha^0 \\ \mathbf{84} & = [0,0,1,0,0,0,0,0] & = \frac{1}{2}(1^4,(-1)^4) & \alpha^1 \\ \overline{\mathbf{84}} & = [0,0,0,0,0,1,0,0] & = (1,0^6,-1) & \alpha^{-1} \end{array} \tag{B.4.25}$$

where **80** is the adjoint representation and transformation properties are given under β in powers of $\alpha = \exp(2\pi i/3)$. We have the following decomposition

$$\begin{array}{lll} \mathrm{E}_8 & \to & \mathrm{SU}(9) \\ \mathbf{248} & \to & \underset{\alpha^0}{\mathbf{80}} + \underset{\alpha^1}{\mathbf{84}} + \underset{\alpha^{-1}}{\overline{\mathbf{84}}} \end{array} \tag{B.4.26}$$

Appendix C

Properties of the Kaluza-Klein-Monopole Solutions

This appendix is devoted to the more technical details and conventions related to Kaluza-Klein-monopole solutions (3.1.1).

C.1 The Multi KK-Monopole Solution

Metric and Vielbein

We start from the metric (3.1.1)

$$\mathrm{d}s^2 = U^{-1}(\mathrm{d}x^4 + \vec{\omega}\cdot\mathrm{d}\vec{r})^2 + U\mathrm{d}\vec{r}^{\,2} = G_{ab}\mathrm{d}x^a \otimes \mathrm{d}x^b \tag{C.1.1}$$

where $a, b, c, \ldots$ indices range from 1 to 4. We let $i, k, l, \ldots$ indices range from 1 to 3 and use these for "vectored" things such as $\vec{\omega} = (\omega_i)$ and $\mathrm{d}\vec{r} = (\mathrm{d}r^i) = (\mathrm{d}x^i)$. Squares of vectors $\vec{\omega}^2 = \omega^2 = \omega_i\omega_j\delta^{ij}$ are understood with respect to the the standard euclidian metric δ_{ij}. The epsilon tensor with three indices *always* is defined as the totally antisymmetric object ϵ_{ijk} with $\epsilon_{123} = +1$. Square brackets on indices denote antisymmetrization without dividing by the number of permutations: $[12] = 12 - 21$.

We then have the metric G_{ab} and its inverse G^{ab}

$$\begin{aligned} G_{ab} &= \begin{pmatrix} U + U^{-1}\vec{\omega}\vec{\omega}^T & U^{-1}\vec{\omega} \\ U^{-1}\vec{\omega}^T & U^{-1} \end{pmatrix} \\ &= \begin{pmatrix} U^{1/2} & U^{-1/2}\vec{\omega} \\ \vec{0}^T & U^{-1/2} \end{pmatrix} \cdot \begin{pmatrix} U^{1/2} & \vec{0} \\ U^{-1/2}\vec{\omega}^T & U^{-1/2} \end{pmatrix} \\ G^{ab} &= \begin{pmatrix} U^{-1} & -U^{-1}\vec{\omega} \\ -U^{-1}\vec{\omega}^T & U + U^{-1}\vec{\omega}^2 \end{pmatrix} \\ &= \begin{pmatrix} -U^{-1/2} & \vec{0} \\ U^{-1/2}\vec{\omega}^T & U^{1/2} \end{pmatrix} \cdot \begin{pmatrix} -U^{-1/2} & U^{-1/2}\vec{\omega} \\ \vec{0}^T & U^{1/2} \end{pmatrix} \end{aligned} \tag{C.1.2}$$

and

$$\det G = U^2 \qquad \det G^{-1} = U^{-2} \tag{C.1.3}$$

As vielbeine have to be related to the metric by

$$\begin{aligned} G_{ab} &= e_a{}^{\underline{a}} e_b{}^{\underline{b}} \delta_{\underline{ab}} &= \left((e_a{}^{\underline{a}})(e_b{}^{\underline{b}})^T\right)_{ab} \\ G^{ab} &= e^a{}_{\underline{a}} e^b{}_{\underline{b}} \delta^{\underline{ab}} &= \left((e^a{}_{\underline{a}})(e^b{}_{\underline{b}})^T\right)^{ab} \end{aligned} \tag{C.1.4}$$

we can read them off from (C.1.2)

$$\begin{aligned} e_i{}^{\underline{j}} &= U^{1/2}\delta_i^j & e_i{}^{\underline{4}} &= U^{-1/2}\omega_j \\ e_4{}^{\underline{j}} &= 0 & e_4{}^{\underline{4}} &= U^{-1/2} \\ \\ e^i{}_{\underline{j}} &= -U^{-1/2}\delta_i^j & e^i{}_{\underline{4}} &= 0 \\ e^4{}_{\underline{j}} &= U^{-1/2}\omega_j & e^4{}_{\underline{4}} &= U^{1/2} \end{aligned} \tag{C.1.5}$$

The one-form vielbein then is

$$\begin{aligned} e^{\underline{j}} &= U^{1/2}\mathrm{d}r^j = U^{1/2}\delta_i^j \mathrm{d}r^i \\ e^{\underline{4}} &= U^{-1/2}\,\vec{\omega}\mathrm{d}\vec{r} + U^{-1/2}\mathrm{d}x^4 \end{aligned} \tag{C.1.6}$$

Using the vielbein, we have for the epsilon tensor

$$\begin{aligned} \epsilon_{abcd} &= U\epsilon_{\underline{abcd}} & \epsilon_{\underline{1234}} &= +1 \\ \epsilon_{ijk}{}^4 &= (U^2 + \omega^2)\epsilon_{ijk} \end{aligned} \tag{C.1.7}$$

Connection and Curvature

The exterior derivative of the vielbein is given by

$$\begin{aligned} \mathrm{d}e^{\underline{4}} &= \frac{1}{2}U^{-3/2}\partial_{[i}\omega_{j]}e^{\underline{i}} \wedge e^{\underline{j}} - \frac{1}{2}U^{-3/2}\partial_i U e^{\underline{i}} \wedge e^{\underline{4}} \\ \mathrm{d}e^{\underline{i}} &= -\frac{1}{2}U^{-3/2}\partial_j U e^{\underline{i}} \wedge e^{\underline{4}} \end{aligned} \tag{C.1.8}$$

From this one gets the connection one-forms (see, for example, sections 2 and 3 of [37])

$$\begin{aligned} \omega_{\underline{4i}} &= \frac{1}{2}U^{-3/2}\left[\partial_{[i}\omega_{j]}e^{\underline{j}} - \partial_i U e^{\underline{4}}\right] \\ \omega_{\underline{jk}} &= \frac{1}{2}U^{-3/2}\left[\delta_{ij}\,\partial_k U e^{\underline{i}} - \delta_{ki}\,\partial_j U e^{\underline{i}} - \partial_{[j}\omega_{k]}e^{\underline{4}}\right] \end{aligned} \tag{C.1.9}$$

From these, the curvature tensor $R_{\underline{ab}} = \mathrm{d}\omega_{\underline{ab}} + \omega_{\underline{ac}} \wedge \omega_{\underline{cb}}$ is easily calculated

$$\begin{aligned} R_{\underline{4i}} = U^{-3}\Big[&\frac{5}{4}\partial_i U\partial_k U e^{\underline{k}} \wedge e^{\underline{4}} - \frac{1}{4}\partial_k U\partial_k U e^{\underline{i}} \wedge e^{\underline{4}} - \frac{1}{2}\partial_k\partial_i U e^{\underline{k}} \wedge e^{\underline{4}} \\ &\frac{1}{4}\partial_{[k}\omega_{j]}\partial_{[i}\omega_{k]}e^{\underline{j}} \wedge e^{\underline{4}} + \frac{1}{2}U\partial_k\partial_i\omega_j e^{\underline{k}} \wedge e^{\underline{i}} \\ &- \frac{1}{2}\partial_{[i}\omega_{j]}\partial_k U - \frac{1}{2}\partial_{[k}\omega_{j]}\partial_i U e^{\underline{k}} \wedge e^{\underline{j}} - \frac{1}{2}\partial_{[k}\omega_{j]}\partial_k U e^{\underline{j}} \wedge e^{\underline{i}}\Big] \end{aligned} \tag{C.1.10}$$

and

$$\begin{aligned} R_{\underline{ij}} = \frac{1}{4}U^{-3}\Big[&(-2\partial_j U\partial_m U e^{\underline{m}} \wedge e^{\underline{i}} - (ij)) \\ &\partial_k U\partial_k U e^{\underline{i}} \wedge e^{\underline{j}} \\ &+ \partial_{[j}\omega_{k]}\partial_k U e^{\underline{i}} \wedge e^{\underline{4}} + (\partial_{[j}\omega_{k]}\partial_i U e^{\underline{k}} \wedge e^{\underline{4}} - (ij)) \\ &\partial_{[i}\omega_{j]}\partial_k U e^{\underline{k}} \wedge e^{\underline{4}} \\ &- \partial_{[i}\omega_{m]}\partial_{[j}\omega_{n]}e^{\underline{m}} \wedge e^{\underline{n}} + \partial_{[i}\omega_{j]}\partial_{[m}\omega_{n]}e^{\underline{m}} \wedge e^{\underline{n}} \\ &(2U\partial_k\partial_j U e^{\underline{k}} \wedge e^{\underline{i}} - (ij)) \\ &+ 2U(\partial_j\partial_k\omega_i - \partial_i\partial_k\omega_j)e^{\underline{k}} \wedge e^{\underline{4}}\Big] \end{aligned} \tag{C.1.11}$$

where (ij) denotes the same formula with i and j exchanged.

The four dimensional Laplace-operator $\Box = G^{ab}\nabla_a\nabla_b\Phi$ can be calculated by the standard formula

$$\Box\Phi = (\det G)^{-1/2}\partial_a\Big((\det G)^{1/2}G^{ab}\partial_b\Phi\Big) \tag{C.1.12}$$

and is

$$\begin{aligned} \Box\Phi = U^{-1}\big(&\Delta\Phi - (\vec{\partial}\cdot\vec{\omega})\Phi' - 2\vec{\omega}\cdot\vec{\partial}\Phi' - \vec{\omega}'\cdot\vec{\partial}\Phi \\ &+ (2UU' + 2\vec{\omega}\cdot\vec{\omega}')\Phi' + (U^2 + \omega^2)\Phi''\big) \end{aligned} \tag{C.1.13}$$

where a prime denotes differentiation with respect to x^4 and $\Delta = \partial_i \partial_i$. Assuming $\vec{\omega}' = 0$, $U' = 0$ and $\vec{\partial} \cdot \vec{\omega} = 0$, this reduces to

$$\Box\Phi = U^{-1}\big(\Delta\Phi - 2\vec{\omega}\cdot\vec{\partial}\Phi' + (U^2+\omega^2)\Phi''\big) \tag{C.1.14}$$

Self-Duality

The self dual and anti-self-dual parts of the curvature tensor are defined as

$$R^{\pm}_{ab} = \pm\frac{1}{2}\epsilon_{ab}{}^{cd}R^{\pm}_{cd} \tag{C.1.15}$$

where self-duality corresponds to the + sign. In flat coordinates, this amounts to

$$\begin{aligned} R^{\pm}_{\underline{ij}} &= \mp\epsilon_{ijk}R^{\pm}_{\underline{4k}} \\ R^{\pm}_{\underline{0i}} &= \mp\frac{1}{2}\epsilon_{ijk}R^{\pm}_{\underline{jk}} \end{aligned} \tag{C.1.16}$$

From the curvature tensor (C.1.10) and (C.1.10) the K_N solution is anti-self-dual. The anti-self-duality condition is

$$\begin{aligned} \partial_i U &= \epsilon_{ijk}\partial_j\omega_k = \frac{1}{2}\epsilon_{ijk}\partial_{[j}\omega_{k]} \\ \partial_{[p}\omega_{q]} &= \epsilon_{ipq}\partial_i U \end{aligned} \tag{C.1.17}$$

In out conventions, some useful formulas for the Hodge-star operator $*$ are

$$\begin{aligned} *(e^{\underline{i}}\wedge e^{\underline{j}}) &= \epsilon^{ijk}e^{\underline{k}}\wedge e^{\underline{4}} \\ *(e^{\underline{i}}\wedge e^{\underline{4}}) &= \frac{1}{2}\epsilon^{ijk}e^{\underline{j}}\wedge e^{\underline{k}} \end{aligned} \tag{C.1.18}$$

The Hyperkähler Structure

We define the following three two-forms

$$s^{\underline{i}} = e^{\underline{i}}\wedge e^{\underline{4}} + \frac{1}{2}\epsilon^{ijk}e^{\underline{j}}\wedge e^{\underline{k}} \tag{C.1.19}$$

By (C.1.18), we have

$$*s^{\underline{i}} = \frac{1}{2}\epsilon^{ijk}e^{\underline{j}}\wedge e^{\underline{k}} + \frac{1}{2}\epsilon^{ijk}\epsilon^{jkl}e^{\underline{l}}\wedge e^{\underline{4}} = s^{\underline{i}} \tag{C.1.20}$$

and the $s^{\underline{i}}$ are self-dual. Using (C.1.8), a short calculation shows that they are closed

$$\mathrm{d}s^{\underline{i}} = 0 \tag{C.1.21}$$

In flat coordinates, we have

$$s_{\underline{kl}}{}^{\underline{i}} = s^{\underline{i}}(e_{\underline{k}}, e_{\underline{l}}) = \epsilon_{ikl} \qquad s_{\underline{j4}}{}^{\underline{i}} = -s_{\underline{4j}}{}^{\underline{i}} = s^{\underline{i}}(e_{\underline{j}}, e_{\underline{4}}) = \delta^i_j \tag{C.1.22}$$

Explicitly, $s_{\underline{ab}}{}^{\underline{1}}$ is given by

$$J^{\underline{1}} = (s^{\underline{a}}{}_{\underline{b}}{}^{\underline{1}}) = \begin{pmatrix} & & & +1 \\ & & +1 & \\ & -1 & & \\ -1 & & & \end{pmatrix} \qquad (J^{\underline{1}})^2 = 1 \tag{C.1.23}$$

Therefore, as we can rotate the $s^{\underline{i}}$ into each other by a SO(3) rotation, all $J^{\underline{i}}$ square to one. Since these are self-dual, they must transform as $(\mathbf{3},\mathbf{1})$ under $\mathrm{Spin}(4) = \mathrm{SU}(2)_1 \times \mathrm{SU}2_2$ and therefore provide us with a triplet of Kähler structures on K_N with respect to $\mathrm{SU}(2)_1$. This is known as hyperkähler structure.

A General Ansatz for Anti-Self-Duality

To find potentials of (anti-)self-dual two-forms, we try the ansatz[1] $U^\alpha e^{\underline{4}}$

$$\mathrm{d}(U^\alpha e^{\underline{4}}) = U^{\alpha-3/2}\Big((\alpha - \tfrac{1}{2})\partial_k U e^{\underline{k}} \wedge e^{\underline{4}} + \tfrac{1}{2}\epsilon_{mnk}\partial_k U e^{\underline{m}} \wedge e^{\underline{n}}\Big) \tag{C.1.24}$$

Hodge dualizing gives

$$*\mathrm{d}(U^\alpha e^{\underline{4}}) = U^{\alpha-3/2}\Big(\partial_k U e^{\underline{k}} \wedge e^{\underline{4}} + \tfrac{1}{2}(\alpha - \tfrac{1}{2})\epsilon_{mnk}\partial_k U e^{\underline{m}} \wedge e^{\underline{n}}\Big) \tag{C.1.25}$$

This is anti-self-dual for $\alpha = -1/2$ and self-dual for $\alpha = +3/2$. Since we are looking for anti-self-dual solutions which are living in the adjoint $\mathbf{3}$ of SU(2), a general ansatz is given by

$$A^{\underline{i}} = a^{\underline{i}}\, U^{-1/2} e^{\underline{4}} + a^{\underline{ij}}\, U^{-1/2} e^{\underline{j}} \tag{C.1.26}$$

($U^{-1/2}e^{\underline{i}}$ is closed). After some algebra, we arrive at the anti-self-duality condition for the field strength $F^{\underline{i}} = \mathrm{d}A^{\underline{i}} + \frac{1}{2}\epsilon^{ijk} A^{\underline{j}} \wedge A^{\underline{j}} = - * F^{\underline{i}}$

$$\begin{aligned} &U^{-1}\Big(\partial_m a^{\underline{in}}\, \epsilon^{mnj} - \omega_m \partial_4 a^{\underline{in}}\, \epsilon^{mnj} + \frac{1}{2}\epsilon^{ipq} a^{\underline{pm}}\, a^{\underline{qn}}\, \epsilon^{mnj}\Big) \\ &= U^{-1}\Big(- \partial_j a^{\underline{i}} + \omega_j \partial_4 a^{\underline{i}} + U\partial_4 a^{\underline{ij}} + \epsilon^{ipq} a^{\underline{p}}\, a^{\underline{qj}}\Big) \end{aligned} \tag{C.1.27}$$

[1] This is motivated by the arguments given in section 3.3, where an anti-self-dual U(1) instanton is expected to exist for K_N which approaches a Wilson line background at $r \to \infty$.

C.2 The Single KK-Monopole Solution

Normalizability of Modes

As can be seen from the explicit form of the metric of the single KK-monopole solution (3.1.8) and the coordinates (3.1.10), the direction ∂_r is always perpendicular to the three-sphere S^3_r defined by $\varrho^2 = 2Rr = \text{const}$. Therefore the induced metric h on S^3_r is simply

$$\mathrm{d}s_h^2 = U^{-1}(\mathrm{d}x^4 + R\,\tfrac{1}{2}(\cos\vartheta - 1)\,\mathrm{d}\varphi)^2 + Ur^2(\mathrm{d}\vartheta^2 + \sin^2\vartheta\mathrm{d}\varphi^2) \tag{C.2.1}$$

with

$$\det h = Ur^4\sin^2\vartheta \tag{C.2.2}$$

Therefore, the volume is given by

$$V_r = \int_0^{2\pi R}\mathrm{d}x^4 \int_0^{2\pi}\mathrm{d}\varphi \int_0^{\pi}\mathrm{d}\vartheta\,\sqrt{Ur^4\sin^2\vartheta} = 8\pi^2 RU^{1/2}r^2 \tag{C.2.3}$$

In the limit $r \to 0$ with $\varrho^2 = 2Rr$ we have

$$V_r = 2\pi^2 R\sqrt{\frac{R}{2r}}r^2 = 2\pi^2\varrho^3 \tag{C.2.4}$$

as it should, since $2\pi^2$ is the volume of the unit S^3. Since the integral

$$\int_{\varrho_-}^{\infty}\mathrm{d}\varrho\; 2\pi^2\varrho^3\;\varrho^{2\alpha} \tag{C.2.5}$$

converges only for $\alpha < -3/2$. Square integrability on X_1 is guaranteed if the modes fall faster than

$$\varrho^{-3/2} \qquad \text{or} \qquad r^{-3/4} \tag{C.2.6}$$

In the limit $r \to \infty$ we have $U \to 1$ and hence

$$V_r = 2\pi R\; 2\pi\; r^2 \tag{C.2.7}$$

which is nothing but the unit volume or S^2 scaled by r^2 times the volume of the Kaluza-Klein S^1. Therefore, since the integral

$$\int_{r_-}^{\infty}\mathrm{d}r\; 2\pi r^2\; r^{2\alpha} \tag{C.2.8}$$

converges only for $\alpha < -1$, square integrability on K_1 is guaranteed if the modes fall faster than

$$r^{-1} \qquad \text{or} \qquad \varrho^{-2} \tag{C.2.9}$$

C.3 Index Calculations on X_N

In the following, we will calculate Dirac indices for twisted[2] Dirac operators on X_N. For Dirac operators without gauge interactions, this has been carried out in [47]. By the same arguments as in section 4.1, this directly gives the supermultiplets of normalizable modes. A general overview of the needed mathematical technology can be found in [37], chapters 7 and 8. For a detailed account, the reader is referred to the original literature: index theory as such is introduced in [10, 9, 11, 12, 13]; index theory for manifolds with boundary was developed in [6, 7, 8].

The general formula for an index of a Dirac operator $\not{D}$ on a manifold X with boundary $Y = \partial X$ is given by

$$\operatorname{ind}\not{D} = \int_X P + \xi(0) \tag{C.3.1}$$

where P is an invariant polynomial in gauge and Lorentz-curvature tensors and $\xi(0)$ is a correction depending only on the Dirac operator restricted to the boundary Y. Here we have already specialized on cases where the metric on X approaches a product metric near the boundary Y (see [37], section 8.1). For the Eguchi-Hanson multi-center gravitational instanton X_N this is the case, whereas for KK-monopoles K_N it is not.

For spinors in a representation R of some gauge group G, the integral is given by

$$\int_{X_N} -\frac{1}{8\pi^2}\operatorname{tr}_R F^2 + \frac{\operatorname{Dim}_{\mathbb{C}} R}{12\cdot 16\pi^2}\operatorname{tr} R^2 \;=\; c_R I_{YM} - \frac{\operatorname{Dim}_{\mathbb{C}} R}{12} I_L \tag{C.3.2}$$

The constant c_R only depends on the chosen representation and will be determined below.

In the case of X_N, the expression of $\xi(0)$ can be calculated easily. In general, $\xi(0)$ is given as

$$\xi(0) = \frac{h + \eta(0)}{2} \tag{C.3.3}$$

where h is the dimension of the space of harmonic spinors on the boundary Y and $\eta(0)$ is the eta-invariant of Atiyah, Patodi and Singer (see [6], theorem 4.2). In case of X_N, however, $Y = \partial X$ is a lens space $L = S^3/\mathbb{Z}_N$ (compare to appendix A.2) and hence the scalar curvature is just that of a sphere S^3, which is strictly positive. For the square of the untwisted Dirac operator for spin 1/2, we have

$$-\not{D}^2_{1/2} = -\mathrm{D}_a\mathrm{D}^a + R/4 \tag{C.3.4}$$

where R is the scalar curvature (compare to [52], eq. (15.5.5)). The operator $-\mathrm{D}_a\mathrm{D}^a$, however, is positive semidefinite, which implies for a harmonic spinor

[2] In the mathematical literature, a twisted Dirac operator is a Dirac operator for fermions coupled to gauge symmetries.

($\not{D}_{1/2}\Psi = 0$)

$$
\begin{aligned}
0 &= \int \bar{\Psi}(-\not{D}^2_{1/2})\Psi = \int \bar{\Psi}(-\mathrm{D}_a\mathrm{D}^a + R/4)\Psi \\
&= \int \overline{\mathrm{D}_a\Psi}\mathrm{D}_a\Psi + R/4\bar{\Psi}\Psi
\end{aligned}
\tag{C.3.5}
$$

Since $\int \overline{\mathrm{D}_a\Psi}\mathrm{D}_a\Psi$ is positive semidefinite and $\int \overline{\Psi}\Psi$ is strictly positive, this is a contradiction if R is strictly positive and there are no harmonic spinors on compact spaces with positive scalar curvature. This is Lichnerowicz's theorem (see [37], section 10.4.3).

However, since we are to calculate indices of twisted Dirac operators, we have to take the gauge curvature into account. In this case, the square of the Dirac operator is

$$
-\not{D}^2_{1/2} = -\mathrm{D}_a\mathrm{D}^a + R/4 - \Gamma^{ab}F_{ab}/4 \tag{C.3.6}
$$

(see eq. (15.5.17) of [52]) and the above argument no longer works in general. Here supersymmetry comes to our rescue, which restricts the gauge curvature to be (anti-) self-dual as shown in section 3.2. As explained in [52], in the paragraph after equation (15.7.4), SU(2) holonomy implies that the gauge curvature drops out of (C.3.6) and hence, there are no harmonic spinors on $Y = \partial X_N$.

It remains to calculate the eta invariant of the Lens space L. The method was introduced in proposition 2.12 of [7], applied to X_N in [47] and uses the Lefschetz number as for example given in theorem 3.1 of [9].

We take X_N in the orbifold limit where it is $\mathbb{C}^2/\mathbb{Z}_N$. Then the action of the orbifold twist has precisely one fixed point at the origin. Since we consider twisted Dirac operators, spinors are in a representation $S_\pm \otimes R$ of $\mathrm{Spin}(4) \times G$ where $S_\pm$ denotes the vector space of positive or negative chirality neutral spinors. Then the complex E of vector bundles relevant for the Dirac operator is

$$
0 \longrightarrow E^R_0 \longrightarrow E^R_1 \longrightarrow 0 \tag{C.3.7}
$$

with

$$
\begin{aligned}
E^R_0 &= P \times_{\mathrm{Spin}(4)\times G} S_+ \otimes R \\
E^R_1 &= P \times_{\mathrm{Spin}(4)\times G} S_- \otimes R
\end{aligned}
\tag{C.3.8}
$$

the bundles of positive and negative chirality spinors associated to a principal bundle P. Then the Lefschetz number for $g \in \mathbb{Z}_N$ (which for more general spaces is a sum over all fixed points of the generator of $\mathbb{Z}_N$) is given as

$$
L(g, E) = \frac{\sum_{i=0}^{1}(-1)^i \operatorname{tr}(g|E_{i,0})}{\det(1 - g|T_0)} \tag{C.3.9}
$$

Here $E_{i,0}$ denotes the fiber of E_i at the origin 0 and T_0 denotes the tangential space at the origin 0.

We begin with neutral spinors as in [47]. Then by (2.6.1), positive chirality spinors are invariant under the twist whereas negative chirality ones have one component transforming as α^1 and another transforming as α^{-1}. Then $\theta_n = 2\pi n/N \in \mathbb{Z}_N$ acts like

$$\begin{pmatrix} 1 & & & \\ & 1 & & \\ & & e^{2\pi i n/N} & \\ & & & e^{-2\pi i n/N} \end{pmatrix} \tag{C.3.10}$$

on a four-spinor. Therefore

$$\begin{aligned} \sum_{i=0}^{1} (-1)^i \operatorname{tr}(\theta_n|E_{i,0}) &= 2 + e^{i\theta_n} + e^{-i\theta_n} = 2 + 2\cos\theta_n \\ &= 4\sin^2\theta_n/2 \end{aligned} \tag{C.3.11}$$

In the denominator, θ_n acts like the orbifold twist

$$\begin{pmatrix} D_{\theta_n} & \\ & D_{-\theta_n} \end{pmatrix} \qquad \text{with} \qquad D_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \tag{C.3.12}$$

on a four dimensional vector. Since

$$\det(\mathbb{1} - D_\theta) = \begin{pmatrix} 1-\cos\theta & -\sin\theta \\ \sin\theta & 1-\cos\theta \end{pmatrix} = 4\sin^2\theta/2 \tag{C.3.13}$$

we have

$$\det(1 - \theta_n|T_0) = 16\sin^4\theta_n/2 \tag{C.3.14}$$

and the Lefschetz number is

$$L(\theta_n, E) = \frac{1}{4\sin^2\theta_n/2} \tag{C.3.15}$$

The eta invariant for $\not{D}_{1/2}$ can now be calculated as in proposition 2.12 of [7]:

$$\begin{aligned} \xi_{1/2}(0) &= \frac{\eta_{1/2}(0)}{2} = \frac{1}{N}\sum_{n\neq 0} L_{1/2}(\theta_n, E) = \frac{1}{N}\sum_{n\neq 0}\frac{1}{4\sin^2\theta_n/2} \\ &= \frac{N^2-1}{12N} = \frac{1}{12} I_L \end{aligned} \tag{C.3.16}$$

(the last equivalence was given in eq. (23) of [47]).

Turning to twisted Dirac operators, we start by the Rarita-Schwinger operator which is just a Dirac operator twisted by the four-dimensional vector representation of SO(4) as above. Since the neutral spinors are in a basis of eigenstates of θ_n, we have

$$\mathrm{tr}(\theta_n|E^R_{i,0}) = \mathrm{tr}(\theta_n|E_{i,0})\,\mathrm{tr}(\theta_n|R) \tag{C.3.17}$$

With

$$\mathrm{tr}(\theta_n|R) = 4\cos\theta_n \tag{C.3.18}$$

from above, we arrive at

$$\begin{aligned}\xi_{3/2}(0) &= \frac{1}{N}\sum_{n\neq 0}\frac{4\cos\theta_n}{4\sin^2\theta_n/2} = \frac{1}{N}\sum_{n\neq 0}\frac{1-2\sin^2\theta_n/2}{\sin^2\theta_n/2}\\ &= 4\xi_{1/2}(0) - 2 + \frac{2}{N}\end{aligned} \tag{C.3.19}$$

For spinors in the **2** and the **3** of SU(2), by (B.4.9), we have the following decompositions

$$\begin{aligned}\mathrm{SU}(2) &\to \mathrm{U}(1)\\ \mathbf{2} &\to \underset{\alpha^1}{\mathbf{1}_{+1}} + \underset{\alpha^{-1}}{\mathbf{1}_{-1}}\\ \mathbf{3} &\to \underset{\alpha^2}{\mathbf{1}_{+2}} + \underset{\alpha^0}{\mathbf{1}_{0}} + \underset{\alpha^{-2}}{\mathbf{1}_{-2}}\end{aligned} \tag{C.3.20}$$

with

$$\begin{aligned}\mathrm{tr}(\theta_n|\mathbf{2}) &= e^{i\theta_n} + e^{-i\theta_n} = 2\cos\theta_n\\ \mathrm{tr}(\theta_n|\mathbf{3}) &= 1 + e^{2i\theta_n} + e^{-2i\theta_n} = 1 + 2\cos 2\theta_n\end{aligned} \tag{C.3.21}$$

and finally

$$\begin{aligned}\xi_{\mathbf{2}}(0) &= \frac{1}{N}\sum_{n\neq 0}\frac{2\cos\theta_n}{4\sin^2\theta_n/2} = 2\xi_{1/2} - 1 + \frac{1}{N}\\ \xi_{\mathbf{3}}(0) &= \frac{1}{N}\sum_{n\neq 0}\frac{1+2\cos 2\theta_n}{4\sin^2\theta_n/2} = 3\xi_{1/2} - 2 + \frac{4}{N}\end{aligned} \tag{C.3.22}$$

In addition, we will treat the special cases of E_8 broken to SU(9) and E_8 broken to $\mathrm{E}_7\times\mathrm{SU}(3)$ in $\mathbb{Z}_3$ orbifolds. In the first case, we have the decomposition (B.4.26)

$$\begin{aligned}\mathrm{E}_8 &\to \mathrm{SU}(9)\\ \mathbf{248} &\to \underset{\alpha^0}{\mathbf{80}} + \underset{\alpha^1}{\mathbf{84}} + \underset{\alpha^{-1}}{\overline{\mathbf{84}}}\end{aligned} \tag{C.3.23}$$

which leads to

$$\mathrm{tr}(\theta_n|\mathbf{248}) = 80 + 84\,e^{i\theta_n} + 84\,e^{-i\theta_n} = 80 + 2\cdot 84\cos 2\pi i/3 \tag{C.3.24}$$

and

$$\xi_{\mathbf{248}}(0) = \frac{1}{3} \cdot 2 \cdot \frac{80 + 2 \cdot 84 \cos 2\pi i/3}{4 \sin^2 2\pi/6} = -\frac{8}{9} \tag{C.3.25}$$

In the second case, we are interested in a background of SU(3) with unbroken E$_7$ (see section 4.3). Then, by (B.4.20), **8** is invariant whereas **3** and $\bar{\mathbf{3}}$ transform as α^1 and α^{-1}, respectively. Therefore, we have

$$\begin{aligned} \mathrm{tr}(\theta_n|\mathbf{8}) &= 8 \\ \mathrm{tr}(\theta_n|\mathbf{3} \oplus \bar{\mathbf{3}}) &= 3\, e^{i\theta_n} + 3\, e^{-i\theta_n} = 6 \cos \theta_n \end{aligned} \tag{C.3.26}$$

where we have looked at the representation $\mathbf{3} \oplus \bar{\mathbf{3}}$ under which the hypermultiplets will transform. This yields

$$\begin{aligned} \xi_{\mathbf{8}}(0) &= 8\xi_{1/2}(0) \\ \xi_{\mathbf{3}\oplus\bar{\mathbf{3}}}(0) &= 6\xi_{1/2}(0) + \frac{3}{2}\left(-2 + \frac{2}{3}\right) = 6\xi_{1/2}(0) - 2 \end{aligned} \tag{C.3.27}$$

The constant c_R can be computed from representation theory. The most important relation here is that the instanton number I_{YM} in the adjoint representation of a group G is given as

$$I_{YM} = \frac{1}{2h} \int_{X_N} -\frac{1}{8\pi^2} \mathrm{tr}_{\mathrm{Adj.}} F^2 \tag{C.3.28}$$

where h is the dual Coxeter number of G (see [5]). Since one can always embed an instanton of instanton number I_{YM} in an arbitrary SU(2) subgroup of G, (C.3.28) is equivalent to

$$\mathrm{tr}_{\mathrm{Adj.\ of}\ G} F^2 = 2h\, \mathrm{tr}_{\mathbf{2}\ \mathrm{of\ SU(2)}} F^2 \tag{C.3.29}$$

if the whole background is confined in a SU(2) subgroup of G. Therefore, we have

$$c_{\mathrm{Adj.}} = 2h \tag{C.3.30}$$

Applied to a fundamental **N** of SU(N), embedding the gauge background into a **2** of SU(2) yields

$$\begin{array}{lcl} \mathrm{SU}(N) & \to & \mathrm{SU}(2) \\ \mathbf{N} & \to & \mathbf{2} + (N-2)\mathbf{1} \\ \mathrm{tr}_{\mathbf{N}} F^2 & = & \mathrm{tr}_{\mathbf{2}} F^2 + (N-2) \cdot 0 = \mathrm{tr}_{\mathbf{2}} F^2 \end{array} \tag{C.3.31}$$

Since the dual Coxeter number of SU(N) is N, we have

$$c_{\mathrm{Adj.\ of\ SU}(N)} = 2N \qquad c_{\mathrm{Fund.\ of\ SU}(N)} = 1 \tag{C.3.32}$$

Group	h	Adjoint	Fundamental
SU(N)	N	$\mathrm{tr}_{\mathbf{N^2-1}}\, F^2$	$= 2N\,\mathrm{tr}_{\mathbf{N}}\, F^2$
SO(N)	$N-2$	$\mathrm{tr}_{\mathbf{N(N-1)/2}}\, F^2$	$= (N-2)\,\mathrm{tr}_{\mathbf{N}}\, F^2$
Sp(N)	$N+1$	$\mathrm{tr}_{\mathbf{N(2N+1)}}\, F^2$	$= 2(N+1)\,\mathrm{tr}_{\mathbf{2N}}\, F^2$
G_2	4	$\mathrm{tr}_{\mathbf{14}}\, F^2$	$= 4\,\mathrm{tr}_{\mathbf{7}}\, F^2$
F_4	9	$\mathrm{tr}_{\mathbf{52}}\, F^2$	$= 3\,\mathrm{tr}_{\mathbf{26}}\, F^2$
E_6	12	$\mathrm{tr}_{\mathbf{78}}\, F^2$	$= 4\,\mathrm{tr}_{\mathbf{27}}\, F^2$
E_7	18	$\mathrm{tr}_{\mathbf{133}}\, F^2$	$= 3\,\mathrm{tr}_{\mathbf{56}}\, F^2$
E_8	30	$\mathrm{tr}_{\mathbf{248}}\, F^2$	$= \mathrm{tr}_{\mathbf{248}}\, F^2$

Table C.1: Dual Coxeter numbers and traces of adjoint and fundamental representations of simple groups

By this method, one can easily determine the very useful relations given in table C.1 (mostly[3] taken from [36]).

To collect our results, we have the indices for the untwisted Dirac operator and the Rarita-Schwinger operator

$$\begin{aligned} \operatorname{ind} \not{D}_{1/2} &= -\frac{1}{12} I_L + \xi_{1/2}(0) = 0 \\ \operatorname{ind} \not{D}_{3/2} &= c_R I_{YM} - \frac{4}{12} I_L + 4\xi_{1/2}(0) - 2 + \frac{2}{N} \\ &= 2I_L - 2 + \frac{2}{N} = 2N - 2 \end{aligned} \tag{C.3.33}$$

by $I_{YM} = I_L = N - \frac{1}{N}$ and $c_R = 2$. For SU(2) backgrounds on X_N, we have

$$\begin{aligned} \operatorname{ind} \not{D}_{\mathbf{2}} &= I_{YM} - 1 + \frac{1}{N} \\ \operatorname{ind} \not{D}_{\mathbf{3}} &= 4I_{YM} - 2 + \frac{4}{N} \end{aligned} \tag{C.3.34}$$

On X_3, for $E_8 \to SU(9)$ backgrounds

$$\operatorname{ind} \not{D}_{\mathbf{248}} = 60 I_{YM} - 56 \tag{C.3.35}$$

and for SU(3) backgrounds

$$\begin{aligned} \operatorname{ind} \not{D}_{\mathbf{3}\oplus\bar{\mathbf{3}}} &= 2I_{YM} - \frac{6}{12} I_L + 6\xi_{1/2}(0) - 2 = 2I_{YM} - 2 \\ \operatorname{ind} \not{D}_{\mathbf{8}} &= 6 I_{YM} \end{aligned} \tag{C.3.36}$$

[3] In case of the group E_8 much care has to be taken. Very often one defines the symbol $\mathrm{tr} = \frac{1}{30}\,\mathrm{Tr}$ where Tr is the trace in the adjoint of E_8. If a gauge background is confined to the SO(16) subgroup of E_8, tr becomes the trace in the fundamental of SO(16). However, the fundamental of E_8 is *identical* to the adjoint of E_8 and therefore tr is *not* the trace in the fundamental of E_8 (as stated in [36]).

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