Large N

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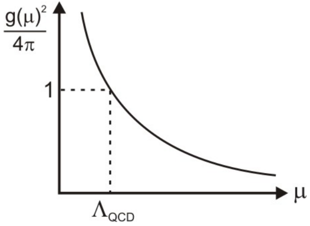
Seminar Relativistic Quantum Field Theory
05/11/2006

Contents

Motivation

Looking for an approximation scheme for QCD ...

- coupling constant g not good expansion parameter in low energy regime μ



Suggestion by 't Hooft:

- generalize SU(3) with 3 colors to SU(N) with N colors
- hope that theory simplifies for large N
- obtain new expansion parameter: $1/N$

Large N QCD

Consider QCD Lagrangian with SU(N) gauge group:

$$\mathcal{L} = -\frac{1}{2}\mathrm{tr}\, F_{\mu\nu}F^{\mu\nu} + \sum_{f=1}^{N_F} (\bar{q}_i)_f (\mathrm{i}\not{D} - m_f)^i_j (q^j)_f$$

- $D_\mu = \partial_\mu + \mathrm{i}g A_\mu$
- $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + \mathrm{i}g[A_\mu, A_\nu]$
- N_F flavor (anti-)**quark fields** q^i ($\bar{q}_i$) in fundamental representation ($i = 1, \ldots, N$)
- **gluon field** $(A_\mu)^i_j = A^a_\mu (T^a)^i_j$: hermitian traceless $N \times N$ matrix ($a = 1, \ldots, N^2 - 1$)

But so far no explicit N dependence . . .

Large N QCD

Hint: consider renormalization group flow of QCD:

$$\mu \frac{dg}{d\mu} = (-\frac{11}{3} N + \frac{2}{3} N_F) \frac{g^3}{16\pi^2} + \mathcal{O}(g^5)$$

$\Rightarrow$ does not have a sensible large N limit

Solution

- replace:

$$\boxed{g \longrightarrow \frac{g}{\sqrt{N}}}$$

Obtain:

$$\mu \frac{dg}{d\mu} = \left(-\frac{11}{3} + \frac{2}{3} \frac{N_F}{N}\right) \frac{g^3}{16\pi^2} + \mathcal{O}(g^5)$$

Large N QCD

Replace $g \to g/\sqrt{N}$ in $\mathcal{L}$, and for convenience, rescale the fields:

- $A_\mu \longrightarrow \frac{\sqrt{N}}{g} A_\mu$
- $q \longrightarrow \sqrt{N} q$

SU(N) Lagrangian:

$$\boxed{\mathcal{L} = N \left[-\frac{1}{2g^2} \mathrm{tr}\, F_{\mu\nu} F^{\mu\nu} + \bar{q}_i (\mathrm{i}\not{D} - m)^i_j q^j \right]}$$

Note: g does not occur in D_μ and $F_{\mu\nu}$ anymore

Counting rules:

Read off N-dependence of vertices and propagators:

- all **vertices** $\propto N$
- all **propagators** $\propto \frac{1}{N}$

Double-Line Notation

Reorganize Feynman diagrams to visualize color flow

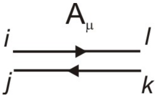
Propagators:

- quark: $\langle \mathcal{T} q^i(x)\bar{q}_j(y)\rangle = \delta^i_j\, S_F(x-y)$

- gluon: $\langle \mathcal{T} A_{\mu}{}^i_j(x) A_{\nu}{}^k_l(y)\rangle = \langle \mathcal{T} A^a_\mu(x) A^b_\nu(y)\rangle\,(T^a)^i_j (T^b)^k_l$

$$= \langle \mathcal{T} A^a_\mu(x) A^b_\nu(y)\rangle\, \delta^{ab} (T^a)^i_j (T^b)^k_l = (\delta^i_l \delta^k_j - \underbrace{\frac{1}{N}\delta^i_j \delta^k_l}_{(*)})\, D_{\mu\nu}(x-y)$$

$(*)$ drops out for U(N)



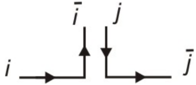
group theoretically: $A_{\mu}{}^i_j$ transforms as $q^i \bar{q}_j$

for simplicity: from now on **consider U(N)** instead of SU(N)!

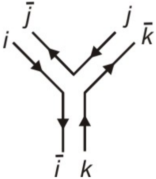
Double-Line Notation

Vertices

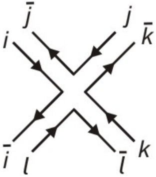
- quark-gluon: $\bar{q}_i \gamma^\mu q^j A_{\mu}{}^i_j$



- 3-gluons: $A_{\mu}{}^i_j A_{\nu}{}^j_k \partial_\mu A_{\nu}{}^k_i$

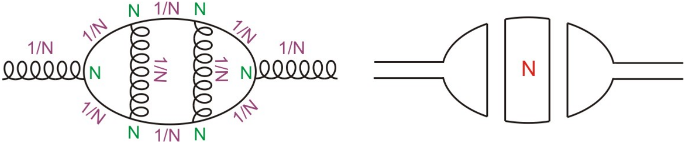


- 4-gluons: $A_{\mu}{}^i_j A_{\nu}{}^j_k A_{\mu}{}^k_l A_{\nu}{}^l_i$

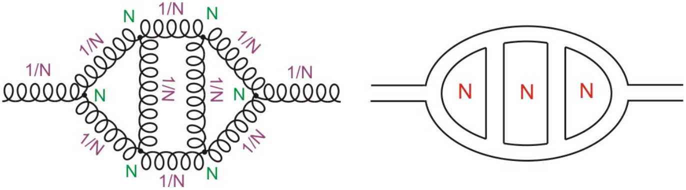


Double-Line Notation - Examples

Can now determine N-dependence of an arbitrary Feynman diagram:



$\propto \frac{1}{N^3}$



$\propto \frac{1}{N}$

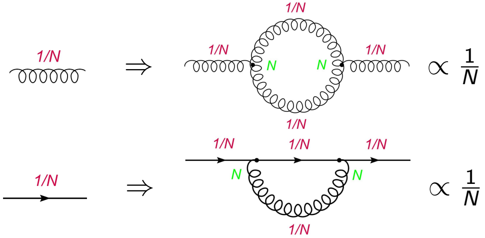
$\rightarrow$ Basic reason: N times more intermediate gluon states than quark states to sum over

Diagram Rules

How does this nontrivial N dependence help simplifying QCD analysis?

Given an arbitrary diagram, one can see...

1. additional internal gluon lines don't change N dependence



2. internal quark loops are suppressed by $\frac{N_F}{N}$

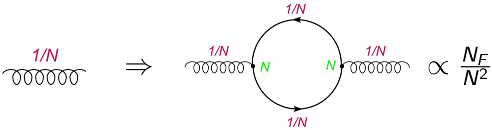
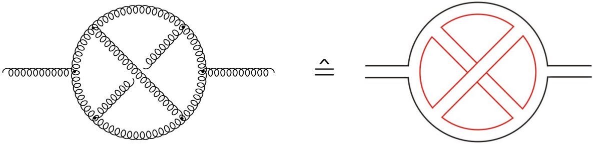


Diagram Rules

3. non-planar diagrams are suppressed by $\frac{1}{N^2}$



$\longrightarrow$ fewer index loops compared to corresponding planar diagram!

Graph Topology

Consider first only vacuum-to-vacuum graphs
Denote:

L	no. of index loops
P	no. of quark and gluon propagators
V	no. of vertices

Then

$$\boxed{\mathcal{O}(\text{Graph}) \sim N^{L-P+V} \equiv N^{\chi}}$$

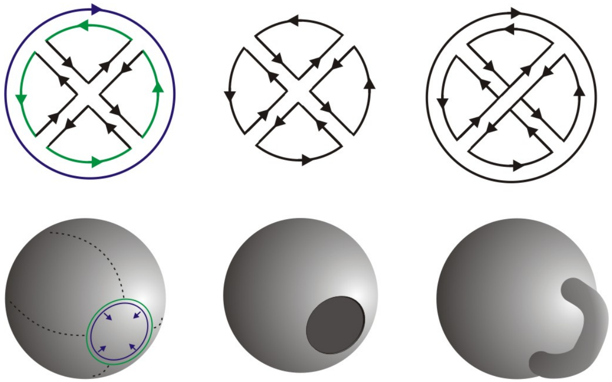
Construct **2d orientable surface** from a double-line graph:

1. loops → faces, propagators → edges, vertices → vertices
2. identify edges when on the same double-line propagator
3. give orientation according to arrows on perimeter

Thus χ is the **Euler characteristic**

Graph Topology

Every 2d orientable surface is topologically equivalent to a 2-sphere with holes and handles:



Graph Topology

Therefore: $\chi = 2 - 2H - B$

with
- H no. of handles stuck onto the sphere
- B no. of boundaries (holes) in the sphere

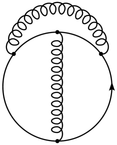
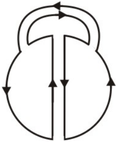
But also: $B =$ no. of quark loops

Conclusion:

$$\boxed{\mathcal{O}(\text{Graph}) \sim N^{2-2H-B}}$$

- **l.o. graphs**: $H = 0 \Leftrightarrow$ planar, $B = 0 \Leftrightarrow$ no quark loops
- **l.o. graphs** with **quark dependence**: $H = 0 \Leftrightarrow$ planar, $B = 1 \Leftrightarrow$ one single quark loop on the outer edge

Why only on the outer edge?

Because:  $\hat{=}$  would be "non-planar" too

Mesons

To create a meson: apply to the vacuum a quark bilinear B

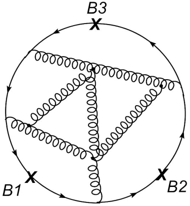
$$B \in \{q\bar{q},\ q\gamma^\mu\bar{q},\ qF_{\mu\nu}\bar{q}, \ldots\}$$

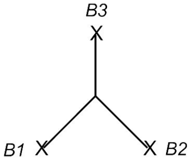
Interactions of n mesons $\rightarrow$ conn. Greens function $\langle B_1 \ldots B_n \rangle$

To use our previous counting rules...

- replace action: $S \rightarrow S + N \sum_i b_i B_i$
- then $\langle B_1 \ldots B_n \rangle = \frac{1}{(\mathrm{i}N)^n} \frac{\partial^n W}{\partial b_1 \ldots \partial b_n}\Big|_{b_i=0}$

  with $W = \sum$(connected vacuum-to-vacuum graphs)

Example: $\langle B_1 B_2 B_3 \rangle \sim$ 

$\hat{=}$ 

Mesons - Diagram Rules

Conclude:

l.o. interaction graphs $=$ l.o. vacuum graphs with bilinears inserted into quark loop

$\Rightarrow$ order of a graph now: $\langle B_1 \dots B_n \rangle \propto N^{(1-n)}$

Assumption:

QCD shows confinement for arbitrary large N

- all states made by the B_i's are **SU(N) singlets**

Transition amplitude $\langle BB \rangle$ should be $\sim \mathcal{O}(1)$ for arbitrary N

$\Rightarrow$ use properly normalized operators $B_i' = N^{\frac{1}{2}} B_i$

Finally

$$\boxed{\langle B_1' \dots B_n' \rangle \propto N^{1-\frac{n}{2}}}$$

Mesons - Diagram Rules

Claim:

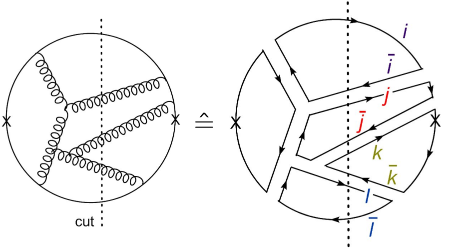
To leading order in $1/N$,

$$\boxed{\langle B'_1 \dots B'_n \rangle = \sum (\text{meson tree diagrams})}$$

$\Rightarrow$ a B'_i creates only a single particle

Heuristical understanding:

Look at intermediate states in a planar diagram:



$\sim \bar{q}_l A^l_k A^k_j A^j_i q^i$

cannot be broken up to color singlets

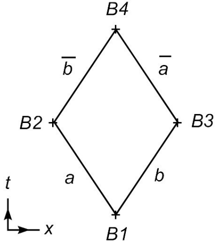
Mesons - Diagram Rules

Proof (by contradiction):

We know: "a B'_i creates only a single particle"
$\Leftrightarrow$ "the only singularities of $\langle B'_1 \dots B'_n \rangle$ are simple poles"

Consider 2-point function $\langle B'_i B'_j \rangle$:

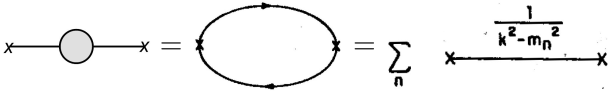
1. assume B'_1 creates two color-singlet particles (a,b), amplitude $\sim \mathcal{O}(1)$
2. particles reflected (B'_2, B'_1) and finally absorbed (B'_4)
   all amplitudes also $\sim \mathcal{O}(1)$ by crossing symmetry
3. thus get singularity of $\sim \mathcal{O}(1)$ in 4-point function
4. but $\langle B'_1 \dots B'_4 \rangle \sim N^{1-\frac{4}{2}} = \frac{1}{N}$



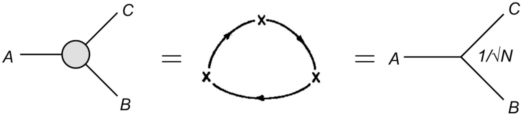
Mesons - Interactions

By reduction formula: $\mathcal{S}_{\{\text{n particles}\}} \propto \langle B_1' \dots B_n' \rangle \propto N^{1-\frac{n}{2}}$

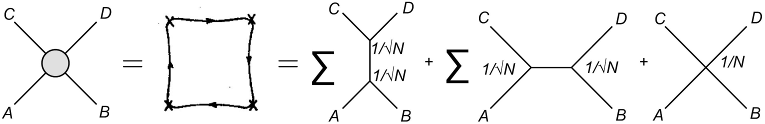
- Leading order 2-point function: $\sim \mathcal{O}(1)$



- Leading order 3-point function: $\sim \mathcal{O}(1/\sqrt{N})$



- Leading order 4-point function: $\sim \mathcal{O}(1/N)$



Mesons - Phenomenology

It seems: we have rewritten QCD as a effective theory of weakly interacting hadrons...

- effective coupling constant $\sim \boxed{1/\sqrt{N}}$
- l.o. in $1/N$ is tree approximation to this theory

Behavior for **large N**

- Mesons stable and noninteracting for $N \to \infty$
- infinite number of mesons

  Why? $\to$ Can expand 2-point function as sum over 1-particle resonances:

$$\int d^x e^{iqx} \langle B_1'(x) B_2'(0) \rangle = \sum_i \frac{Z_i}{q^2 - m_i^2}$$

  Now: l.h.s. is known $\sim \log(q^2) \Rightarrow$ r.h.s. must be infinite sum

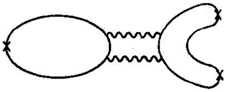
Mesons - Phenomenology

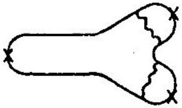
Predictions for reality **(N=3)**

- leading order scattering amplitudes $= \sum$ tree diagrams with physical hadrons exchanged
  $\rightarrow$ similarity to successful "Regge phenomenology"
- multiparticle decays of unstable mesons preferably through two body states
- suppression of the $q\bar{q}$ sea in mesons
- suppression of $q\bar{q}q\bar{q}$ exotics

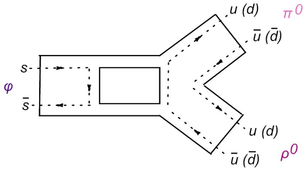
Mesons - Phenomenology

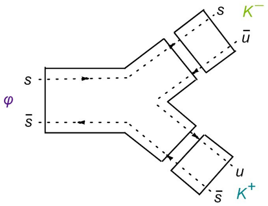
Justification for OZI (Zweig) rule:
"flavor disconnected processes are suppressed"


suppressed


allowed

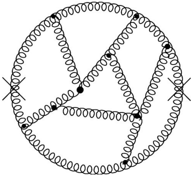
Looks in double line notation:





Count color loops $\Rightarrow$ expect branching ratio: $\frac{\Gamma_{\text{OZI suppressed}}}{\Gamma_{\text{OZI allowed}}} \propto \frac{1}{N^2}$

Glue states



Same analysis can be applied to glueballs:
Let

$$G_i \in \{\mathrm{tr}\, F_{\mu\nu}F^{\mu\nu}, \mathrm{tr}\, F_{\mu\nu}(*F)^{\mu\nu}\}$$

Facts:

- pure glue states: l.o. graphs = planar, no quark loops
  $\Rightarrow \langle G_1 \ldots G_n \rangle \sim N^{2-n}$ (already properly normalized)
- mixed glueball-meson states: l.o. graphs = planar, one quark loop at boundary
  $\Rightarrow \langle B'_1 \ldots B'_m \, G_1 \ldots G_n \rangle \sim N^{1-\frac{m}{2}-n}$

Conclusions:

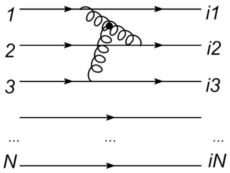
- glueball interaction constant $\sim \frac{1}{N}$ $\Rightarrow$ weakly interacting
- meson-glueball coupling $\langle GB' \rangle \sim \frac{1}{\sqrt{N}}$ $\Rightarrow$ mixing suppressed

Baryons - Counting Rules

Baryon $=$ N quark color singlet state $\sim \epsilon_{i_1 \cdots i_N} q^{i_1} \cdots q^{i_N}$

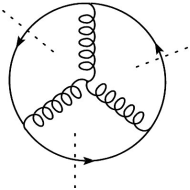
Have well-defined large N limit although no. of quarks diverges!

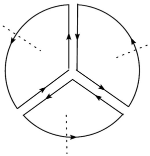
Propagator:



Any connected k-body interaction subgraph $\hat{=}\ \mathcal{O}(N)$ meson graph:

- cut k fermion lines
- but you loose k color index sums





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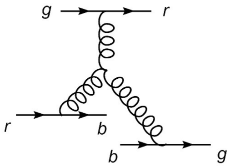

Therefore: k-particle interactions $\sim N^{1-k}$

Baryon Masses

$\mathcal{O}(N^k)$ ways of choosing k quarks from an N baryon
$\Rightarrow$ net effect k particle interaction $\sim \mathcal{O}(N)$

In general: diagram with l disconnected pieces is $\sim \mathcal{O}(N^l)$

Baryon mass:

$$M_B = Nm_q + NT_q + \frac{1}{2}N^2\left(\frac{1}{N}V_{qq}\right) \sim \mathcal{O}(N)$$

Now baryon propagator

$$e^{-\mathrm{i}M_B t} = 1 - \mathrm{i}M_B t - \frac{1}{2}M_B^2 t^2 + \dots$$

lth term represents l disconnected subdiagrams
$\Rightarrow$ disconnected graph $\sim \mathcal{O}(M_B^l)$

Baryons as Solitons

We have seen:

- QCD coupling constant: $g_s \sim \frac{1}{\sqrt{N}}$
- Baryon mass: $\boxed{M_B \sim 1/g_s^2 \to \infty}$ for $g_s \to 0$

Compare with 't Hooft-Polyakov monopole: $\boxed{M_{\text{monopole}} \sim \mathcal{O}(1/\alpha)}$

More analogs:

for **large** N respectively **small** $\alpha \ldots$

- baryon size and shape independent of N
  $\Leftrightarrow$ size and shape of the monopole independent of α
- mesons become non-interacting, baryons still interact
  $\Leftrightarrow$ e^+, e^- non-interacting, but m.-m. and m.-$e^\pm$ interaction still possible

Baryons $\sim$ solitons in weakly coupled theory of strong interactions

Justification for $1/N$ expansion

Why consider at all large N?

- understand tree approximation ($=$ large N) of QCD first before study loop corrections ($=$ finite N)

How good is $1/N$ expansion for $N = 3$?

- depends on coefficients of the suppressed terms:
  - quark loops $\mathcal{O}(1/N)$ often unimportant in phenomenology

  $\Rightarrow$ expansion really in terms of $1/N^2 = 1/9$ (!)
- explains a lot of observations

Compare to QED:

- electric charge actually is $e = \sqrt{4\pi\alpha} \approx 0.3$
- correct expansion parameter found to be $\frac{e^2}{4\pi}$

Lastly: $1/N$ is the only known expansion parameter of QCD in low energy regime

Master Field

As far, only studied overall N dependence of the theory

- want to calculate at least leading term in 1/N
- sum all planar diagrams? $\rightarrow$ hopeless

Hint:

consider large N behavior of

$$\boxed{G = \text{gauge invariant operator made up of gauge fields}}$$

- remember: $\langle G_1 \ldots G_n \rangle_C \propto N^{2-n}$
- $\boxed{G' \equiv G/N}$ has well defined v.e.v. for $N \rightarrow \infty$: $\langle G' \rangle_C \propto 1$

Compute variance of G':

$$\begin{aligned}
\langle (G' - \langle G' \rangle)^2 \rangle &= \langle G' G' \rangle - \langle G' \rangle \langle G' \rangle \\
&= \langle G' G' \rangle_C \\
&= \mathcal{O}(1/N^2) \xrightarrow{N \rightarrow \infty} 0
\end{aligned}$$

Master Field

What does this imply?

Path integral for pure U(N) gauge theory:

$$\langle G_1' \ldots G_n' \rangle = \frac{1}{Z} \int \mathcal{D}A_\mu e^{-N \int d^4x \left[\frac{1}{4g^2} \mathrm{tr} F_{\mu\nu} F^{\mu\nu}\right]} G_1' \ldots G_n'$$

Compare:

If $f(x)$ has minimum at a: $f(x) = f(a) + \frac{1}{2} f''(a)(x-a)^2 + \mathcal{O}\left((x-a)^3\right)$
Then for large N:

$$\int dx\, e^{-N f(x)} = e^{-N f(a)} \left(\frac{2\pi}{N f''(a)}\right)^{1/2} e^{-\mathcal{O}(1/\sqrt{N})}$$

Master field $\bar{A}_\mu$:

For $N \to \infty$: path integral determined by extremal field configuration $\bar{A}_\mu \in \left\{ U A_\mu U^{-1} - iU\partial_\mu U^{-1} | U \in U(N) \right\}$

$$\Rightarrow \boxed{\langle G_1'(A_\mu(x_1)) \ldots G_n'(A_\mu(x_n)) \rangle = G_1'(\bar{A}_\mu(x_1)) \ldots G_n'(\bar{A}_\mu(x_n))}$$

Master Field

Properties of $\bar{A}_\mu$:

- Four hermitian '$\infty \times \infty$'matrices
- Expected to be spacetime independent!
  Reason: action and measure are translationally invariant.

$$\bar{A}_\mu(x) = e^{iP\cdot x}\bar{A}_\mu(0)e^{-iP\cdot x}$$

Perform gauge transformation $U = e^{iP\cdot x}$:

$$\Rightarrow \bar{A}_\mu(x) = \bar{A}_\mu(0) + P_\mu$$

Also: $\bar{F}_{\mu\nu} = [\bar{A}_\mu, \bar{A}_\nu]$

Solution of large N QCD by finding four '$\infty \times \infty$' matrices!

Matrix Model

How to deal with '$\infty \times \infty$' matrices?

Solvable model:

QCD in $0+0$ dimensional spacetime

$\rightarrow$ Evaluate in large N limit:

$$\langle \mathrm{tr}\, g(A) \rangle = \frac{1}{Z} \int dA\, e^{-N\, \mathrm{tr} V(A)}\, \mathrm{tr}\, g(A)$$

where

- A: hermitian $N \times N$ matrix
- $g(A)$ gauge invariant function of A
- $dA = \prod dA_{ab}, \quad a, b = 1 \ldots N$
- $dA_{ab} dA_{ba} = d(\mathrm{Re} A_{ab}) d(\mathrm{Im} A_{ba})$
- $V(A)$ gauge inv. function of A (e.g. $V(A) = \frac{1}{2} M^2 A^2 + A^4$)

Matrix Model

Procedure:

1. write $A = U^\dagger \Lambda U$, with $\Lambda = \mathrm{diag}(\lambda_1, \ldots, \lambda_N)$
2. use $dA = dU(\prod_k d\lambda_k) \prod_{i \neq j}(\lambda_i - \lambda_j)$

Arrive at:

$$\langle \mathrm{tr}\, g(A) \rangle = \frac{1}{Z} \int (\prod_i d\lambda_i)(\sum_j g(\lambda_j)) e^{-N \sum_k V(\lambda_k) + \sum_{m \neq n} \log|\lambda_m - \lambda_n|}$$

Remark (Dyson): $Z =$ partition function of classical 1-dimensional gas with particle positions λ_i

Consider further:

$$S_{\mathrm{eff}} = N \sum_k V(\lambda_k) - \sum_{m \neq n} \log|\lambda_m - \lambda_n|$$

Matrix Model

Density of eigenvalues: $\rho(\lambda) \equiv \frac{1}{N}\sum_i \delta(\lambda - \lambda_i)$, $\int d\lambda \rho(\lambda) = 1$

Rewrite S_{eff}:

$$S_{\text{eff}} = N^2 \left[\int d\lambda \rho(\lambda) V(\lambda) - \int d\lambda d\lambda' \rho(\lambda)\rho(\lambda') \log|\lambda - \lambda'| \right]$$

For $N \to \infty$: $\langle \operatorname{tr} g(A) \rangle$ is dominated by the minimal ρ:

$$V'(\lambda) = 2\,\mathrm{P} \int d\lambda' \frac{\rho(\lambda')}{\lambda - \lambda'}$$

- solve for ρ to get d.o.e. for the master matrix
- to first order then $\langle \operatorname{tr} g(A) \rangle = N \int d\lambda \rho(\lambda) g(\lambda)\ \bar{A}$

E.g. for $V(A) = \frac{1}{2}m^2 A^2 \to$ Wigner Semi-Circle distribution:

$$\rho(\lambda) = \frac{2}{\pi(4/m^2)^2}\sqrt{(4/m)^2 - \lambda^2}$$

Summary

- Right QCD expansion parameter is $1/N$
- Large N QCD gets simple (tree graphs)
- Mesons appear as particles, Baryons as solitons
- Explains many of strong interaction phenomena (often the only known general explanation)
- For $N = 3$ it might be not such a bad approximation
- Summation of planar diagrams seems not feasible
- Master field is hopeful solution candidate

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