Magnetic Monopoles

Roman Schmitz

Seminar on Theoretical Particle Physics
University of Bonn

May 4th 2006

Motivation: Why magnetic monopoles ?

- First idea from Dirac in 1931 (symmetric form of Maxwell-Equations)
- Appear in non-abelian gauge theories with symmetry breakdown
- possibly particles not yet observed, no experimental evidence up to now!

The Maxwell Equations in terms of the Dual Tensor

Take Maxwell Equations:

$$\partial_\nu F^{\mu\nu} = -j^\mu \qquad dF = 0$$

In terms of the **Dual Tensor** defined as

$$\tilde{F}^{\mu\nu} = \frac{1}{2}\epsilon^{\mu\nu\rho\sigma}F_{\rho\sigma}$$

with components:

$$\tilde{F^{0i}} = \frac{1}{2}\epsilon^{0ijk}F_{jk} = -\frac{1}{2}\epsilon^{0ijk}\epsilon_{jkl}B^l = B^i$$

$$\tilde{F^{ik}} = \frac{1}{2}\epsilon^{ij\mu\nu}F_{\mu\nu} = \frac{1}{2}(\epsilon^{ijk0}F_{k0} + \epsilon^{ij0l}F_{0l} = e^{ijk}E^k)$$

Maxwell's equations read:

$$\partial_\nu F^{\mu\nu} = -j^\mu \qquad \partial_\nu \tilde{F}^{\mu\nu} = 0$$

Extension of Maxwell's Equations

To describe Monopoles construct a magnetic 4-current in analogy to the electrical:

$$k^\mu = (\sigma, \vec{k})$$

Now Maxwell's Equations read

$$\partial_\nu F^{\mu\nu} = -j^\mu \qquad \partial_\nu \tilde{F}^{\mu\nu} = -k^\mu$$

in a nice symmetric form and are invariant under the so-called **Duality Transformation**:

$$F^{\mu\nu} \mapsto \tilde{F}^{\mu\nu} \qquad \tilde{F}^{\mu\nu} \mapsto -F^{\mu\nu} \qquad j^\mu \mapsto k^\mu \qquad k^\mu \mapsto -j^\mu$$

Magnetic monopole field

Magnetic field for a point-source with magnetic charge g:

$$\vec{B}(\vec{r},t) = \frac{g}{4\pi r^2} \cdot \frac{\vec{r}}{r}$$

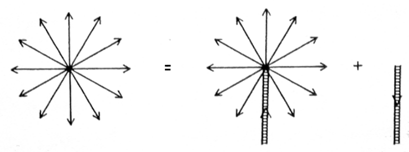
Problem:

$$div\vec{B} = \vec{\nabla} \cdot \frac{g}{4\pi} \cdot \underbrace{\frac{\vec{r}}{r^3}}_{=\vec{\nabla}(-\frac{1}{r})} = -\frac{g}{4\pi}\Delta\frac{1}{r} = -\frac{g}{4\pi}\delta(r) \neq 0$$

$$\Rightarrow \nexists\vec{A} \quad s.t. \quad \vec{B} = rot\vec{A}$$

Is this the end of magnetic monopoles ?

Solution: The Dirac string



Add an infinetely small, infinitely extended solenoid field (e.g. along the negative z-axis):

$$\vec{B}_{sol} = \frac{g}{4\pi r^2}\hat{r} + g \cdot \Theta(-z)\delta(x)\delta(y)\hat{z}$$

Verify that the flux is zero by integrating and using Gauss' theorem. Now:

$$\vec{B}_{Monopole} = rot\vec{A}_{sol} - g \cdot \Theta(-z)\delta(x)\delta(y)\hat{z}$$

Dirac's Quantization of Magnetic Charge

Motion of charged particle in Monopole field:

$$\frac{d}{dt}L = m[\vec{r} \times \ddot{\vec{r}}] = \frac{gq}{4\pi r^3}[\vec{r} \times [\dot{\vec{r}} \times \vec{r}]] = \frac{gq}{4\pi} \underbrace{(\frac{\dot{\vec{r}}}{r} - \frac{\vec{r}(\dot{\vec{r}} \cdot \vec{r})}{r^3})}_{= \frac{d}{dt}\frac{\vec{r}}{r}} = \frac{d}{dt}\frac{gq\vec{r}}{4\pi r}$$

Angular momentum of Electromagnetic field:

$$L_{EM} = \int d^3x[\vec{r} \times [\vec{E} \times \vec{B}]] = \int d^3x \frac{\vec{E}}{r} - \frac{\vec{r}}{r^3}(\vec{r} \cdot \vec{E}) = \int d^3x E^i (\nabla^i \hat{r}) = -\frac{gq\vec{r}}{4\pi r}$$

In QM quantized angular momenta in units of $\frac{n}{2}$:

$$\frac{eg}{4\pi} = \frac{n}{2}$$

Remark: Dirac string unobservable

Aharonov-Bohm-Effect changes phase factors of wavefunctions if A given, but $B = 0$. Consider 2 paths around Dirac string. Condition to obsereve no Dirac string:

$$|\psi_1 + \psi_2|^2 = \left| \exp\left(iq \int_1 \vec{A} \ \vec{dl}\right) \cdot \psi_1 + \exp\left(iq \int_2 \vec{A} \ \vec{dl}\right) \cdot \psi_2 \right|^2$$

interference terms with exponents can differ by $n2\pi$:

$$2\pi n = (iq \int_1 \vec{A} \ \vec{dl}) - (iq \int_2 \vec{A} \ \vec{dl}) = q \oint \vec{A} \ \vec{dl} = qg$$

That means the Dirac string is unobservable because of the quantization condition.

Magnetic coupling strength

From the quantization condition

$$\frac{qg}{4\pi} = \frac{n}{2}$$

one can estimate the magnetic coupling constant.
Coupling of 2 monopoles will be $\sim g^2$, so:

$$\sim g^2 \sim \frac{n^2}{4} \cdot \frac{(4\pi)^2}{q^2} \sim \frac{n^2}{4} q^2 \underbrace{\left(\frac{4\pi}{q^2}\right)^2}_{1/\alpha^2}$$

The means the magnetic coupling between two monopoles is about 10^4 times stronger than the electrical coupling.

Electromagnetic Duality

In vacuum ($j^\mu = 0$) the "old" Maxwell-Equations are symmetric under the **Duality Transformation**

$$F^{\mu\nu} \mapsto \tilde{F}^{\mu\nu} \qquad \textit{and} \qquad \tilde{F}^{\mu\nu} \mapsto -F^{\mu\nu}$$

which is equivalent to

$$E \mapsto B \qquad \textit{and} \qquad B \mapsto -E$$

With magnetic charges we have a symmetric form that is invariant under the Duality Transformation if

$$j^\mu \mapsto k^\mu \qquad \textit{and} \qquad k^\mu \mapsto -j^\mu$$

Summary

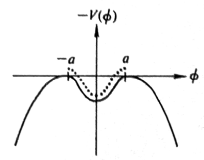
- symmtreic form of Maxwell Equations, duality transformation
- describes quantization of electric/magnetic charges by QM
- still have to deal with point-sources and singularities
- Dirac string unobservable
- no mass predictions

What are Solitons ?

Solitary waves or so-called Solitons are static finite-energy solutions to the equations of motion that appear in most field theories.

Example: 1+1-dimensional scalar fields with potential

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi)^2 - V(\phi) \qquad \textit{with} \qquad V(\phi) = \frac{\lambda}{4}(\phi^2 - a^2)^2$$



· from $\mathcal{L}$: equivalent
to motion of particle in Potential $-V(\phi)$

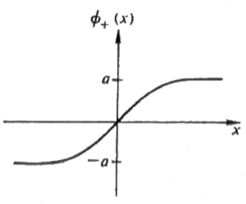
· $E < \infty \Rightarrow \phi \to \pm a,\ T \to 0$ for $x \to \infty$

Solitons in 1+1d

Energy conservation: $\frac{1}{2}\left(\frac{d\phi}{dx}\right)^2 = V(\phi)$ leads to solutions

$$\phi_{\pm}(x) = \pm a \cdot tanh(\mu x)$$

(kink, anti-kink) with mass $\mu = \sqrt{\lambda}a$ (symmetry breaking).



Stability and topological conservation law

Soliton mass scale $\sim$ symmetry breaking scale $\rightarrow$ heavy, unstable ?

$$\phi(\infty) - \phi(-\infty) = n \cdot 2a \quad with \quad n = 0, \pm 1$$

define current by $j_\mu(x) = \epsilon_{\mu\nu}\partial^\nu \phi$

$$\rightarrow Q = \int_{-\infty}^{\infty} j_0(x)dx = \int_{-\infty}^{\infty} (\partial_x \phi(x))dx = n(2a)$$

is the topologically conserved charge. Hence, n is conserved and there should be no transitions between the states and no decay of the soliton to the vacuum.

Generalization to 3+1 dimensions

- "sphere" of minima: $\mathcal{M}_0 = \{\phi_i = \eta_i | V(\eta_i) = 0\}$
- finite energy: $\phi_i^\infty = \lim\limits_{R\to\infty} \phi_i(R\hat{r}) \epsilon \mathcal{M}_0$
- $H = \int d^3x \; [\frac{1}{2}(\partial_0\phi_i)^2 + \frac{1}{2}(\nabla\phi_i)^2 + V(\phi_i)]$ should converge !
- $(\nabla\phi)^2 = (\frac{\partial\phi}{\partial r}\hat{r} + \frac{1}{r}\frac{\partial\phi}{\partial\varphi}\hat{\varphi} + \frac{1}{r\cdot\sin\varphi}\frac{\partial\phi}{\partial\theta}\hat{\theta})^2$ *transverse* $\sim r^-2$
- add gauge fileds s.t. $D_i\phi \sim r^{-2}$ and $A_i^a \sim \phi_i \sim r^{-1}$ makes integral convergent (Derrick 1964)

The Georgi-Glashow-SO(3) model

Consider SO(3)-model with Higgs-Triplet in adj. representation:

$$\mathcal{L} = \frac{1}{2}(D^\mu \phi)^a (D_\mu \phi)^a - \frac{1}{4} F_a^{\mu\nu} F_{\mu\nu}^a - V(\phi)$$

with the potential, fields and cov. derivatives given by:

$$F_{\mu\nu}^a = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a - e\epsilon^{abc} A_\mu^b A_\nu^c$$

$$(D_\mu \phi)^a = \partial_\mu \phi^a - e\epsilon^{abc} A_\mu^b \phi^c$$

$$V(\phi) = \frac{\lambda}{4}(\phi^2 - a^2)^2$$

Breakdown $SO(3) \sim SU(2) \to SO(2) \sim U(1)$ via ground state:

$$\phi = (0, 0, a)$$

gives 2 massive gauge bosons and a massles (photon). Now identify:

$$F_3^{0i} = E^i \qquad F_3^{ij} = -\epsilon^{ijk} B^k$$

Minima of potential form a sphere:

$$\mathcal{M}_0 = \{\phi = \eta | \eta^2 = a^2\}$$

t'Hooft-Polyakov Ansatz

We need $D_i\phi \sim r^{-2}$ and $A_i^a \sim \phi_a \sim r^{-1}$ for H to converge.
In addition we want $\phi_i^\infty = \lim_{R\to\infty} \phi_i = \eta_i = a \cdot \hat{r}$

Ansatz by 't Hooft and Polyakov (1974):

$$\phi_b = \frac{r^b}{er^2} H(aer) \qquad A_b^i = -\epsilon_{bij}\frac{r^j}{er^2}(1 - K(aer)) \qquad A_b^0 = 0$$

Energy of system given by Hamiltonian:

$$E = \frac{4\pi a}{e}\int_0^\infty \frac{d\xi}{\xi^2}\left[\xi^2\left(\frac{dK}{d\xi}\right)^2 + \frac{1}{2}\left(\xi\frac{dH}{d\xi} - H\right)^2 + \frac{1}{2}(K^2-1)^2 + K^2H^2 + \frac{\lambda}{4e^2}(H^2-\xi^2)^2\right]$$

with $\xi = aer$.

$$E = \frac{4\pi a}{e}\int_0^\infty \frac{d\xi}{\xi^2}\left[\xi^2\left(\frac{dK}{d\xi}\right)^2 + \frac{1}{2}\left(\xi\frac{dH}{d\xi} - H\right)^2 + \frac{1}{2}(K^2-1)^2 + K^2H^2 + \frac{\lambda}{4e^2}(H^2-\xi^2)^2\right]$$

determine EOM for H,K by variation of E w.r.t H and K:

$$\xi^2\frac{d^2K}{d\xi^2} = KH^2 + K(K^2-1) \qquad \xi^2\frac{d^2H}{d\xi^2} = 2K^2H + \frac{\lambda}{e^2}H(H^2-\xi^2)$$

Our asymptotic condition $\phi_i^\infty = \eta_i = a\cdot\hat{r}$ implies:

$$H \sim \xi \ \ for \ \ \xi \rightarrow \infty$$

The terms (K^2H^2) and $\frac{1}{\xi^2}(K^2-1)^2$ imply:

$$K \rightarrow 0 \ \ for \ \ \xi \rightarrow \infty \quad and \quad H \leq O(\xi) \quad K^2 - 1 \leq O(\xi)$$

The mass of this (static) solution is given by its Energy, the integral can be solved numerically and is ≈ 1, so we have:

$$M \approx \frac{4\pi a}{e}$$

so the mass is set by vev of the scalar field which is also a scale for symmetry breaking $SO(3) \to SO(2)$.

The magnetic field

Plugging the Ansatz into F_a^{ij} we get after several ϵ^{ijk}-Terms cancel out:

$$F_a^{ij} = \epsilon^{ijk} \frac{r^k r^a}{er^4} = \frac{1}{aer^3} \epsilon^{ijk} r^k \phi_a \qquad with \qquad \phi_a = \frac{ar^a}{r}$$

so at large distances we get:

$$\vec{B} = \frac{g}{4\pi} \frac{\vec{r}}{r^3} \qquad with \qquad g = -\frac{4\pi}{e}$$

Size of the monopole

The monopole has finite size as can be seen below. For large ξ we have $H \to \xi$ and $K \to 0$:

$$\xi^2 \frac{d^2K}{d\xi^2} = \underbrace{KH^2}_{\to K\xi^2} + \underbrace{K(K^2-1)}_{\to 0} \Rightarrow \frac{d^2K}{d\xi^2} = K$$

$$\xi^2 \frac{d^2H}{d\xi^2} = \underbrace{2K^2H}_{\to 0} + \frac{\lambda}{e^2} \underbrace{H(H^2-\xi^2)}_{\to 2\xi^2 h} \Rightarrow \frac{d^2h}{d\xi^2} = \frac{2\lambda}{e^2} h \qquad h = H - \xi$$

with solutions:

$$K \sim e^{-\xi} = e^{-(ea)r} \qquad H - \xi \sim e^{-(2\lambda)^{\frac{1}{2}} ar}$$

The prefactors are the masses $\mu = (2\lambda)^{\frac{1}{2}} a$ of the scalar and $M = ea$ of the gauge bosons after breaking the symmetry.

Summary

- At large distances behaves like Dirac monopole
- finite size and smooth structure, no point charge
- classical mass of order of symmetry breaking
- no singularities like Dirac string