Exercises on 'Elementary Particle Physics'

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1. *Nonrelativistic perturbation theory*

   Consider an electron scattering in some interaction potential $V(\vec{x},t)$ in times $-T/2 < t < T/2$. We neglect spin. It was shown in the lecture that the ansatz

   $$\psi = \sum_n a_n(t)\phi_n(\vec{x})e^{-iE_nt}$$

   solves *Schroedinger*'s equation $(H_0 + V(\vec{x},t)\psi = i\partial_t\psi$ if the $a_n(t)$ fullfill

   $$i\sum_n \frac{da_n}{dt}\phi_n(\vec{x})e^{-iE_nt} = \sum_n V(\vec{x},t)a_n\phi_n(\vec{x})e^{-iE_nt} \ ,$$

   where the ϕ_n are solutions of the unperturbed problem and orthonormalized to a standard volume V. From this we get

   $$\frac{da_f}{dt} = -i\sum_n a_n(t)\int \phi_f^* V\phi_n d^3x e^{i(E_f-E_n)t} \ . \qquad (1)$$

   Suppose that before the interaction, i.e. at time $t = -T/2$ the particle is in an eigenstate i of H_0, and that (to first order) the a_n's remain constant at all times (This is only valid for a small and transient potential). Then

   $$\frac{da_f(t)}{dt} = -i\int d^3x\phi_f^* V\phi_i e^{i(E_f-E_i)t}$$

   is solved by

   $$a_f(t) = -i\int_{-T/2}^{t} dt' \int d^3x\phi_f^* V\phi_i e^{i(E_f-E_i)t'} \ ,$$

   and i.e. at $t = +T/2$

   $$T_{fi} \equiv a_f(T/2) = -i\int_{-T/2}^{T/2} dt \int d^3x \left[\phi_f(\vec{x})e^{-iE_ft}\right]^* V(\vec{x},t) \left[\phi_i(\vec{x})e^{-iE_it}\right] \ .$$

   Note that from our assumptions the perturbation is only valid for $a_f(t) \ll 1$.

(a) Go through the first order computation for yourself. We are perturbing $a_n(t)$. How is this given to first order, i.e. what is $a_n^{(1)}$?

(b) We now want to go to second order. For which indices n of $a_n(t)$ does a perturbation make sense? Let the potential be independent of time and define $V_{fi} \equiv \int d^3x \phi_f^*(\vec{x}) V(\vec{x}) \phi_i(\vec{x})$. Show that then to second order in perturbation equation (1) becomes

$$\frac{da_f}{dt} = \cdots + (-i)^2 \left[\sum_{n \neq i} V_{ni} \int_{-T/2}^{t} dt' e^{i(E_n - E_i)t'} \right] V_{fn} e^{i(E_f - E_n)t} ,$$

where the dots denote the first order terms already given in equation (1).

(c) Show that the second order correction for T_{fi} (in the limit $T \to \infty$) is given by

$$T_{fi} = \cdots - \sum_{n \neq i} V_{fn} V_{ni} \int_{-\infty}^{\infty} dt e^{i(E_f - E_n)t} \int_{-\infty}^{t} dt' e^{i(E_n - E_i)t'} .$$

The last integral is not meaningful. Why? A way out is to add a so called regulator term $-i\epsilon, \epsilon > 0$ small, to the energy difference $(E_n - E_i)$. Comment on why this is well defined? Compute the regulated second order contribution for V_{fi} and sketch an interpretation of the result schematically.

2. *Spin 1/2 fermions coupled to an electromagnetic field*

Remember the minimal coupling presciption

$$i\partial_\mu \to i\partial_\mu - QA_\mu ,$$

where Q is the charge of the particle, i.e. $Q_{e^-} = -e = -|e|$ and $Q_{e^+} = +e = +|e|$, and $A_\mu = (\phi, \vec{A})_\mu$ is the electromagnetic four-vector potential.

(a) What are the *Dirac* equations for an electron field ψ and a positron field ψ_C, each of them coupled to an electromagnetic field A_μ.

(b) Assume (similar to the *Lorentz* transformations) that a local relation between ψ_C and ψ exists. Show that for a C with

$$-(C\gamma^0)\gamma^{\mu *} = \gamma^\mu (C\gamma^0)$$

this local relation reads $\psi_C = C\bar{\psi}^T$.

(c) Show that in the *Pauli-Dirac* representation of the γ-matrices a possible choice for C is

$$C\gamma^0 = i\gamma^2 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} .$$

(d) Now let $\psi^{(1)} = u^{(1)}(p)e^{-ip\cdot x}$ and $\phi^{(1)} = v^{(1)}(p)e^{ip\cdot x}$ be two of the 'standard' solutions we found in exercise 1. Show that $\psi_C^{(1)} = \phi^{(1)}$. Interpret the result.