Universität Bonn
Physikalisches Institut
Theoretische Physik

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Example sheet 7
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# General Relativity

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## *1. Motion in Schwarzschild solutions*

(a) Write the equations of motion for the Schwarzschild solution.
*(Use last week's Christoffels and keep $A$ and $B$ general for now.)*

(b) Set $\theta = \pi/2$ (why can this be done?), and integrate suitably to get

$$\frac{dt}{dp} = \frac{1}{B(r)}\,, \qquad r^2\,\frac{d\phi}{dp} = J = const\,. \tag{1}$$

*(Multiply the equations for $\phi$ and $t$ in (a) with $dp/d\phi$ and $dp/dt$ respectively.)*
The integral of the equation for $r$ should now give

$$A(r)\left(\frac{dr}{dp}\right)^2 + \frac{J^2}{r^2} - \frac{1}{B(r)} = -E = const\,, \tag{2}$$

where $p$ is the parameter along the worldline.

(c) Show that $d\tau^2 = E\,dp^2$, and that therefore $E = 0$ must hold for photons, while $E > 0$ for other matter.

(d) Eliminate $dp$ from the integrals of motion obtained in (b) to get a relation between $r$ and $\phi$. Show that

$$\phi = \pm\int \frac{\sqrt{A(r)}\,dr}{r^2\sqrt{\frac{1}{J^2\,B(r)} - \frac{E}{J^2} - \frac{1}{r^2}}} \tag{3}$$

is a solution.

***2. Light deflection***

A photon approaches the central mass from infinity along the direction $\phi_\infty = 0$ with impact parameter $b$. We want to calculate the deflection of its trajectory. Let $r_0$ be the radius of closest approach.

(a) Determine the value of $J$ in terms of $r_0$.

(b) Show that (3) now reduces to

$$\phi(r) = \phi_\infty + \int_r^\infty \frac{\sqrt{A(r')}}{\sqrt{\frac{r'^2}{r_0^2}\frac{B(r_0)}{B(r')} - 1}} \frac{dr'}{r'} \,. \tag{4}$$

(c) Show that the Schwarzschild line element we calculated last week can be approximated by

$$A(r) = 1 + \frac{2\,GM}{r}\,, \quad B(r) = 1 - \frac{2\,GM}{r} \tag{5}$$

in regions where Newtonian gravity is valid.

(d) Use (4) to calculate the deflection angle $\Delta\phi = 2\,|\phi(r_0) - \phi_\infty| - \pi$.

*Use this equality which holds to lowest order in* $2\,GM/r$*:*

$$\frac{r^2}{r_0^2}\frac{B(r_0)}{B(r)} - 1 = \left[\frac{r^2}{r_0^2} - 1\right]\left[1 - \frac{2\,GMr}{r_0(r + r_0)}\right]$$

*The following integrals may be useful:*

$$\int \frac{dx}{x\sqrt{x^2 - a^2}} = \frac{1}{a}\arccos\frac{a}{x}\,, \tag{6}$$

$$\int \frac{dx}{x^2\sqrt{x^2 - a^2}} = \frac{\sqrt{x^2 - a^2}}{a^2\,x}\,, \tag{7}$$

$$\int \frac{dx}{(x + a)\sqrt{x^2 - a^2}} = \frac{\sqrt{x^2 - a^2}}{a(x + a)}\,. \tag{8}$$