Universität Bonn
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Theoretische Physik

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Example sheet 8
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General Relativity

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1. *Precession of perihelia*

From the last sheet we know that the shape of Schwarzschild trajectories is given by

$$\frac{A(r)}{r^4}\left(\frac{dr}{d\phi}\right)^2+\frac{1}{r^2}-\frac{1}{J^2\,B(r)}=-\frac{E}{J^2}\,, \tag{1}$$

where E and J^2 are constants of the motion. Also, we find that

$$\phi=\pm\int\frac{\sqrt{A(r)}\,dr}{r^2\sqrt{\frac{1}{J^2\,B(r)}-\frac{E}{J^2}-\frac{1}{r^2}}} \tag{2}$$

is a solution of (1). In weak fields, we can expand

$$A(r)=1+\frac{2\,MG}{r}\,,\quad B(r)=1-\frac{2\,MG}{r}\,. \tag{3}$$

(a) Determine E and J^2 by looking at the aphelion $r=r_+$ and perihelion $r=r_-$ of a planet in bound orbit around the sun, for general $B(r)$. *(At $r_\pm$, $dr/d\phi$ vanishes.)*

(b) Show that the amount of orbital precession per revolution is

$$\Delta\phi=2|\phi(r_+)-\phi(r_-)|-2\pi\,, \tag{4}$$

where $\phi(r_+)-\phi(r_-)$

$$=\int\limits_{r_-}^{r_+}\left[\frac{r_-^2\,(B^{-1}(r)-B^{-1}(r_-))-r_+^2\,(B^{-1}(r)-B^{-1}(r_+))}{r_+^2 r_-^2\,(B^{-1}(r_+)-B^{-1}(r_-))}-\frac{1}{r^2}\right]^{-1/2}\times\frac{\sqrt{A(r)}\,dr}{r^2}\,. \tag{5}$$

(c) Show that for weak fields, we can use

$$B^{-1}(r)=1+\frac{2\,MG}{r}+\frac{4\,M^2G^2}{r^2} \tag{6}$$

which makes the first term in (5) quadratic in $1/r$, and that we can then write

$$\phi(r_+) - \phi(r_-) = \int_{r_-}^{r_+} \left[C \left(\frac{1}{r_-} - \frac{1}{r} \right) \left(\frac{1}{r} - \frac{1}{r_+} \right) \right]^{-1/2} \times \frac{\sqrt{A(r)}\, dr}{r^2} . \tag{7}$$

(d) Determine C in the limit $r \to \infty$. You should get

$$C = 1 - \frac{4\, MG}{L} + \dots , \tag{8}$$

where $\frac{1}{L} = \frac{1}{2}\left(\frac{1}{r_+} + \frac{1}{r_-}\right)$. Show that (5) now reduces to

$$\phi(r_+) - \phi(r_-) = \left(1 + \frac{2\, MG}{L}\right) \times \int_{r_-}^{r_+} \frac{\left(1 + \frac{MG}{r}\right) dr}{r^2 \sqrt{\left(\frac{1}{r_-} - \frac{1}{r}\right)\left(\frac{1}{r} - \frac{1}{r_+}\right)}} \tag{9}$$

(e) Calculate $\Delta\phi$. *(You can approximate the result of the integral with* $\left(1 + \frac{MG}{L}\right)\pi$*.)*

(f) Determine the total precession $\Delta\phi$ for Mercury over the time of a century. (415 revolutions per century; $L = 55.3 \times 10^9$ m; $MG = 1475$ m).
The observed value is (43.11 ± 0.45) arcseconds.