Universität Bonn
Physikalisches Institut
Theoretische Physik

Winter term 2004/05
Example sheet 10
2005-01-21/24

General Relativity

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1. Age of the universe

Consider the Einstein equations for the Robertson-Walker metric:

$$\frac{\dot{a}^2}{a^2} + \frac{\alpha}{a^2} = \frac{8\pi k}{3}\rho \,, \tag{1}$$

$$-\frac{2\ddot{a}}{a} - \frac{\dot{a}^2}{a^2} - \frac{\alpha}{a^2} = 8\pi k\, p \,. \tag{2}$$

(a) Show that

$$\frac{\mathrm{d}}{\mathrm{d}t}(\rho a^3) + 3\, p\, a^2 \dot{a} = 0 \,, \tag{3}$$

$$-\frac{4\pi}{3}(\rho + 3p)k = \frac{\ddot{a}}{a} \,, \tag{4}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}(\rho\, a^{3(1+w)}) = 0 \,, \tag{5}$$

where w in last relation is defined by the equation of state, $p = w\,\rho$.

(b) Discuss the evolution of the scale factor a for $p = \rho/3$ and $\alpha = 0, \pm 1$ with initial conditions $\lim_{t\to 0} a(t) = 0$, $\lim_{t\to 0} \dot{a}(t) > 0$.

(c) Discuss the evolution of the scale factor a for $p = 0$ and $\alpha = 0, \pm 1$ with initial conditions $\lim_{t\to 0} a(t) = 0$, $\lim_{t\to 0} \dot{a}(t) > 0$.

(d) Calculate the age of the universe for $\alpha = 0$ and (i) $p = \rho/3$ and (ii) $p = 0$.

Hint: The following integrals may be useful:

$$\int \mathrm{d}x\, \frac{\sqrt{x}}{\sqrt{b+x}} = \sqrt{x\,(b+x)} - b\, \ln\left(\sqrt{x} + \sqrt{b+x}\right) \,, \tag{6}$$

$$\int \mathrm{d}x\, \frac{\sqrt{x}}{\sqrt{b-x}} = -\sqrt{x\,(b-x)} + b\, \arctan\left(\frac{\sqrt{x}}{\sqrt{b-x}}\right) \,, \qquad (0 < x < b) \,. \tag{7}$$

The present day Hubble expansion rate is

$$H_0 = \left.\frac{\dot{a}}{a}\right|_{\text{today}} \simeq 0.073\,\text{Gyr}^{-1} \,. \tag{8}$$

Informations on the test

- **When: Thursday, February 3^{rd}, at 14^{00}.**
- **Where: Lecture hall HS I, PI.**
- **What else:**
  - **No tools (notes, books etc.) are allowed.**
  - **The test lasts 1h.**
  - **Please come in time!**