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Theoretische Physik

Exercise 4
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SS 08

Exercises on Elementary Particle Physics II

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1. *The Higgs of SU(5)*

It is necessary to generalize the Higgs mechanism of the SM to understand the symmetry breaking of any GUT theory to the SM. Thus, we describe the Higgs mechanism for a field H in an arbitrary representation ρ of a semi-simple Lie algebra $\mathfrak{g}$.

(a) Consider a complex scalar H in the representation ρ of a gauge group G. Assume further that H acquires a vev $\langle H \rangle$ due to some potential. Deduce from the kinetic term[1]

$$\left(D_\mu H\right)^* \left(D^\mu H\right)\big|_{\mathbf{1}} = \left(\partial_\mu H + ig\rho(T^a)A_\mu^a H\right)^* \left(\partial^\mu H + ig\rho(T^b)A^{b\mu} H\right)\big|_{\mathbf{1}} ,$$

that a gauge boson A_μ^a is massless, if $\rho(T^a)\langle H \rangle = 0$. Then, T^a belongs to the unbroken gauge group G'.
Specialize to H in the adjoint representation with the kinetic term $\mathrm{Tr}(D_\mu H)^\dagger (D^\mu H)$. Follow the discussion of part (a) to deduce

$$T^a \in G' \quad \text{if} \quad [T^a, \langle H \rangle] = 0, \qquad T^a \notin G' \quad \text{if} \quad [T^a, \langle H \rangle] \neq 0.$$

Discuss also the case, where some linear combination of generators commutes with the Higgs vev!

Let us apply this for the desired symmetry breaking of Ex. 3.2, (a), by introducing a Higgs field in the adjoint of SU(5), i.e a 5×5 hermitian traceless matrix.[2] We work with a scalar potential invariant under $H \to -H$ of the form

$$V(H) = -m^2 \mathrm{Tr}\left(H^2\right) + \lambda_1 \left(\mathrm{Tr}\left(H^2\right)\right)^2 + \lambda_2 \mathrm{Tr}\left(H^4\right).$$

(b) First, use the results of Ex. 3.2 to argue that a Higgs H precisely in the adjoint **24** is an appropriate choice to break SU(5) to the SM. Which component of **24**

[1] The only restriction on ρ is that $\bar{\rho} \otimes \rho$ should contain $\mathbf{1}$ in order to give rise to a gauge invariant kinetic term for H. The subscript $|_{\mathbf{1}}$ denotes this singlet component. For the adjoint, it is obtained by the trace.

[2] Note that this is not the SM-Higgs field, which is contained in the **5**.

should develope the vev, cf. Ex. 3.2, (d)? Use the gauge symmetry $H \to H' = UHU^\dagger$ to obtain

$$H = \mathrm{diag}(h_1, h_2, h_3, h_4, h_5)$$

and check that the minimum of the scalar potential is determined by the same equation for all h_i:

$$4\lambda_2 h_i^3 + 4\lambda_1 a h_i - 2m^2 h_i - \mu = 0 \qquad \text{with} \quad a = \sum_j h_j^2, \quad \forall i = 1, \ldots, 5. \tag{1}$$

Here μ is a Lagrange multiplier necessary to impose the constraint $\sum_i h_i = 0$.

The cubic equation (1) has at most three roots denoted by ϕ_1, ϕ_2, ϕ_3. Thus, there are at most three different eigenvalues $h_i \in \{\phi_1, \phi_2, \phi_3\}$. Let n_i be the multiplicity of the eigenvalue ϕ_i, $i = 1, 2, 3$, in $\langle H \rangle$:

$$\langle H \rangle := \mathrm{diag}(\phi_1, .., \phi_2, .., \phi_3) \qquad \text{with} \quad n_1\phi_1 + n_2\phi_2 + n_3\phi_3 = 0.$$

(c) Following part (a), what is the most general symmetry breaking of SU(5)? What happens to the rank of the gauge group? Consider also possible U(1) factors.

Depending on the relative magnitude of the parameters λ_1 and λ_2, the combinations $(3, 2, 0)$ or $(4, 1, 0)$ for (n_1, n_2, n_3) minimize the potential. Thus,

$$\mathrm{SU}(5) \to \mathrm{SU}(3) \times \mathrm{SU}(2) \times \mathrm{U}(1) \tag{2}$$

or

$$\mathrm{SU}(5) \to \mathrm{SU}(4) \times \mathrm{U}(1),$$

which gives restrictions on phenomenologically reasonable values of λ_1, λ_2.

(d) Focus on the first case and determine what is the most general form of $\langle H \rangle$. Then, the breaking (2) should be obvious. What is the generator of the U(1)? Compare this to your result for Q in Ex. 3.2, (a).

2. *Higgs mass corrections*

In this exercise we present one of the many motivations for the introduction of supersymmetry. We will see how supersymmetry enables us to solve one crucial problem of the standard model, namely the problem of the Higgs mass correction, without fine tuning!
As supersymmetry introduces superpartners for each SM-particle, we will calculate the corrections to m_H due to fermion- as well as scalar-loops to present the magic of supersymmetry at work. First, the relevant part of the SM Lagrangian reads

$$\mathcal{L}_{\text{Higgs-Top}} = -\left(f \bar{Q}_L H_c u_R + h.c.\right),$$

where f denotes the Yukawa coupling and Q_L and u_R the quarks for which we consider the heaviest quark only, the top quark. After electro-weak symmetry breaking (due to $H = (\phi^0, 0)^T$ for $\phi^0 = \frac{1}{\sqrt{2}}(v + h(x))$, i.e. $\langle H \rangle = v$), the Lagrangian reads,

$$\mathcal{L}_{\text{Higgs-Top}} = -\left(\frac{f}{\sqrt{2}}\bar{t}_L t_R (v + h) + h.c.\right),$$

and the top-mass is defined as $m_t = \frac{fv}{\sqrt{2}}$. Next, we will present the problem itself.

(a) Draw the Feynman diagram for the one-loop correction of the Higgs mass due to a top-loop. Show that the amplitude for this process is given by

$$\Pi_t^{hh}(0) = -6f^2 \int \frac{d^4k}{(2\pi)^4} \left(\frac{1}{k^2 - m_t^2} + \frac{2m_t^2}{(k^2 - m_t^2)^2}\right).$$

Hint: A factor 3 occurs due to three color d.o.f. of the top quark and the minus-sign due to the fermion loop.

This integral diverges quadratically. Hence, the Higgs mass m_H receives a quadratically divergent correction, too. In order to keep the physical Higgs mass at the electro-weak scale ($\approx 100 GeV$) an unnatural cancellation between the cut-off $\Lambda \approx M_{\text{Planck}}$ and the bare Higgs mass m_H has to occur. This problem is solved by the introduction of color-triplets of complex scalars that couple to the Higgs, such that their contributions to m_H cancel the top contribution. Let us denote these scalars by $\tilde{t}_L$ and $\tilde{t}_R$ and call them **s-tops**.[3] The s indicates supersymmetry or superpartner.

(b) Compare the degrees of freedom for both the top and the s-tops. Consider

$$\mathcal{L}_{\text{Higgs-s-top}} = \tilde{\lambda}_t |\phi^0|^2 \left(|\tilde{t}_L|^2 + |\tilde{t}_R|^2\right) + \left(f A_t \phi^0 \tilde{t}_L \tilde{t}_R^* + h.c.\right).$$

to determine the vertex factors for the following couplings: $h\,h\,\tilde{t}_L\,\tilde{t}_L$, $h\,h\,\tilde{t}_R\,\tilde{t}_R$, $h\,\tilde{t}_L\,\tilde{t}_L$, $h\,\tilde{t}_R\,\tilde{t}_R$ and $h\,\tilde{t}_R\,\tilde{t}_L$? Draw all Feynman diagrams that contribute to m_H. *Hint: You should find three different types of diagrams. Note that the diagram that involves a $h\,h\,\tilde{t}_L\,\tilde{t}_L$ vertex gets an additional factor of 2.*

(c) Compute the amplitude $\Pi_{\tilde{t}}^{hh}(0)$ for these diagrams.

(d) Next, assume that the s-top Yukawa coupling $\tilde{\lambda}_t$ depends on the top Yukawa coupling f as $\tilde{\lambda}_t = -f^2$. How divergent is the sum of both contributions (b) and (f) to the Higgs mass?

(e) Show that in the case that the s-top masses $m_{\tilde{t}}$ are equal to the top mass (and additionally $A_t = 0$) the sum of the one-loop correction vanishes.

Precisely the assumptions of (e) and (f) are predicted by supersymmetry. Therefore, supersymmetry solves the hierarchy problem. Even for broken supersymmetry with $m_{\tilde{t}} \neq m_t$ the quadratical divergences are traded for logarithmical ones.

[3]Note that for the s-tops the subscripts L and R do not denote left- and right-chiral fields as there are no chiral bosons (ignoring the notion of chiral bosons in 2-dimensional SCFTs).