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Exercise 6
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SS 08

Exercises on Elementary Particle Physics II

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1. *The SUSY Algebra and the Chiral Representation*

The SUSY algebra is an extension of the Poincaré algebra that circumvents the restrictions of the **Coleman-Mandula theorem**[1]. This is achieved by allowing for **Super-Lie algebras** or **graded Lie algebras** with commutators as well as anti-commutators defining its algebra relations as internal symmetries of the theory. Thus, one introduces generators Q_α, $\bar{Q}_{\dot{\beta}} = (Q_\beta)^*$ transforming in the Weyl-representations D_L, D_R, respectively, obeying the (anti-) commutation relations

$$\{Q_\alpha, \bar{Q}_{\dot{\beta}}\} = 2(\sigma^\mu)_{\alpha\dot{\beta}} P_\mu, \qquad [Q_\alpha, P_\mu] = 0$$
$$[M_{\mu\nu}, Q_\alpha] = i\,(\sigma_{\mu\nu})_\alpha^{\ \beta} Q_\beta, \qquad [M_{\mu\nu}, \bar{Q}^{\dot{\alpha}}] = i\,(\bar{\sigma}_{\mu\nu})^{\dot{\alpha}}_{\ \dot{\beta}} \bar{Q}^{\dot{\beta}}. \tag{1}$$

Additionally, one introduces the Grassmann variables θ_α, $\bar{\theta}_{\dot{\alpha}}$ as free parameters.

(a) Check that
$$[(\theta Q), (\bar{Q}\bar{\theta})] = 2(\theta\sigma^\mu\bar{\theta})P_\mu$$

(b) Considering the SUSY algebra (1) as a Lie algebra of an group with coordinates $(x^\mu, \theta, \bar{\theta})$, defining flat **superspace**, we can express a group element by
$$S(a^\mu, \alpha, \bar{\alpha}) := \exp[\alpha Q + \bar{Q}\bar{\alpha} - i a^\mu P_\mu].$$
Show that $S(a^\mu, \alpha, \bar{\alpha})S(b^\mu, \beta, \bar{\beta})$ is again a group element.

(c) Thus, an element S induces a translation in superspace. This is used to define a representation of the SUSY group on **superfields** $\Phi(x^\mu, \theta, \bar{\theta})$ that transform as
$$S(a^\mu, \alpha, \bar{\alpha})[\Phi(x^\mu, \theta, \bar{\theta})] = \Phi(x^\mu + a^\mu - i\alpha\sigma^\mu\bar{\theta} + i\theta\sigma^\mu\bar{\alpha}, \theta + \alpha, \bar{\theta} + \bar{\alpha}). \tag{2}$$
Note that this is analogous to the transformation of a scalar field under Poincaré transformations.
Use an infinitesimal transformation to show that the SUSY algebra on superfields $\Phi(x^\mu, \theta, \bar{\theta})$ is realised by
$$P_\mu = i\partial_\mu, \qquad Q_\alpha = \partial_\alpha - i(\sigma^\mu)_{\alpha\dot{\beta}}\bar{\theta}^{\dot{\beta}}\partial_\mu, \qquad \bar{Q}_{\dot{\beta}} = -\bar{\partial}_{\dot{\beta}} + i\theta^\alpha(\sigma^\mu)_{\alpha\dot{\beta}}\partial_\mu. \tag{3}$$

[1] The assumptions of the theorem are a local, relativistic quantum field theory in four dimensions with a finite number of different particles and an energy gap between the vacuum and the one-particle states. Then it states that the most general Lie algebra of symmetries of the S-matrix contains P^μ, $M^{\mu\nu}$ as well as a finite number of generators T_i of a compact Lie group, that are Lorentz scalars.

(d) Check that (2) is fulfilled for linear combinations as well as products of superfields. Additionally, check that these operators form a representation of the SUSY algebra by explicitly verifying the algebra relations. Hence, (2) defines a linear representation of (1), i.e. on a vectorspace.

(e) Define a (SUSY) covariant derivative D_α by

$$D_\alpha(\delta_{(\epsilon,\bar{\epsilon})}\Phi) = \delta_{(\epsilon,\bar{\epsilon})}(D_\alpha\Phi), \tag{4}$$

where $\delta_{(\epsilon,\bar{\epsilon})} = \epsilon Q + \bar{Q}\bar{\epsilon}$. Consequently, $D\Phi$ transforms as a superfield, too. Show that the following derivatives obey (4) and are therefore covariant:

$$\begin{aligned} D_\alpha &= \partial_\alpha + i(\sigma^\mu)_{\alpha\dot{\beta}}\bar{\theta}^{\dot{\beta}}\partial_\mu \\ \bar{D}_{\dot{\beta}} &= -\bar{\partial}_{\dot{\beta}} - i\theta^\alpha(\sigma^\mu)_{\alpha\dot{\beta}}\partial_\mu \end{aligned}$$

(f) Next define the **left** and **right chiral representations** by

$$\begin{aligned} S_L(a^\mu,\alpha,\bar{\alpha}) &:= \exp[\alpha Q - ia^\mu P_\mu]\exp[\bar{Q}\bar{\alpha}], \\ S_R(a^\mu,\alpha,\bar{\alpha}) &:= \exp[\bar{\alpha}\bar{Q} - ia^\mu P_\mu]\exp[\alpha Q]. \end{aligned} \tag{5}$$

Concentrate on the left chiral representation and work out its relation to the representation S of (b). Check that $S_L(a^\mu,\alpha,\bar{\alpha})S_L(b^\mu,\beta,\bar{\beta})$ is a group element.

(g) A superfield in the left chiral representation is defined by

$$S_L(a^\mu,\alpha,\bar{\alpha})[\phi_L(x^\mu,\theta,\bar{\theta})] = \phi_L(x^\mu + a^\mu + 2i\theta\sigma^\mu\bar{\alpha}, \theta+\alpha, \bar{\theta}+\bar{\alpha}).$$

Determine the representations of the SUSY generators Q_L and $\bar{Q}_L$.

(h) Check that the following operators define covariant derivatives

$$\begin{aligned} D_{L\alpha} &= \partial_\alpha + 2i(\sigma^\mu)_{\alpha\dot{\beta}}\bar{\theta}^{\dot{\beta}}\partial_\mu \\ \bar{D}_{L\dot{\beta}} &= -\bar{\partial}_{\dot{\beta}}. \end{aligned}$$

by evaluating the commutator with the SUSY transformation S_L.

(i) Next, define **chiral superfields** by the constraints

$$\begin{aligned} &\bar{D}\Phi(x,\theta,\bar{\theta}) = 0, && \text{for \textbf{left chiral sf}} \\ &D\Phi(x,\theta,\bar{\theta}) = 0. && \text{for \textbf{right chiral sf}} \end{aligned}$$

This definition is independent of the representation. Work with the representation S of (2) to check that the component fields are not constraint by differential equations in x. Choose the left chiral representation, i.e. $\bar{D}\Phi = \bar{D}_L\phi_L$, to deduce the general form of a left chiral superfield.
Hint: Make a Taylor expansion in θ, what defines the component fields of Φ.

(j) Consider the infinitesimal SUSY transformation $\delta_{(\epsilon,\bar{\epsilon})}$ of a left chiral superfield ϕ_L. How do the component fields of ϕ_L transform?
Hint: Use the left chiral representation of the SUSY generators Q_L and $\bar{Q}_L$ and assume that the transformation is small: $\delta\theta\sigma^\mu\delta\bar{\theta} \approx 0$.