Physikalisches Institut | Exercise 8
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# Exercises on Elementary Particle Physics II

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1. *Vectorsuperfields for the supersymmetric U(1) gauge theory*

   Global $\mathcal{N} = 1$ supersymmetry allows one more supermultiplet, the **vectormultiplet**, which contains a supersymmetric version of a gauge theory. It consists of the usual spin one gauge boson $V_\mu$ as well as its spin one half superpartner $\lambda$ called the **gaugino**. There also exists a superfield formulation of the vectormultiplet completely analogous to the chiral superfield describing the chiral multiplet $(\varphi, \psi)$. The appropriate superfield $V$ is the **vectorsuperfield** defined by $V = V^\dagger$ with the expansion

$$\begin{aligned} V(x,\theta,\bar\theta) \;=\;& C(x) + i\theta\chi(x) - i\bar\theta\bar\chi(x) + \theta\sigma^\mu\bar\theta V_\mu(x) \\ & + \frac{1}{2} i\theta\theta\left[M(x) + iN(x)\right] - \frac{1}{2} i\bar\theta\bar\theta\left[M(x) - iN(x)\right] \\ & + i\theta\theta\bar\theta\left[\bar\lambda(x) + \frac{i}{2}\bar\sigma^\mu\partial_\mu\chi(x)\right] - i\bar\theta\bar\theta\theta\left[\lambda(x) + \frac{i}{2}\sigma^\mu\partial_\mu\bar\chi(x)\right] \\ & + \frac{1}{2}\theta\theta\bar\theta\bar\theta\left[D(x) - \frac{1}{2}\partial_\mu\partial^\mu C(x)\right]. \end{aligned} \tag{1}$$

   (a) Check that (1) is indeed a vectorsuperfield.

   (b) Compare the expansion of $V$ to the one of the vectorsuperfield defined by $\Lambda + \Lambda^\dagger$ where $\Lambda_L(x,\theta) = \Lambda(x) + \sqrt{2}\theta\psi_\Lambda(x) + \theta\theta F_\Lambda(x)$ is a left-chiral superfield, here given in the left-chiral representation. What is the interpretation of the transformation

$$V \mapsto V + \Lambda + \Lambda^\dagger \;\; ? \tag{2}$$

   *Hint: Work in the left-chiral representation by shifting the argument $x^\mu$ of $\Lambda^\dagger$. How does $V_\mu$ transform?*

   (c) Use an appropriate $\Lambda$ in (2) to transform $V$ into the **Wess-Zumino gauge** $V_{\text{WZ}}$, i.e. to obtain $C(x) = \chi(x) = M(x) = N(x) = 0$. What is the highest non-vanishing power of $V_{\text{WZ}}$? Calculate $V_{\text{WZ}}$, $V_{\text{WZ}}^2$ as well as $V_{\text{WZ}}^3$.

To construct an action exhibiting the symmetry (2) as a gauged symmetry we have to find an adequate gauge invariant quantity. This will be the building block of any gauge invariant action. Exploiting the gauge invariance of the gaugino $\lambda$ w.r.t. (2) in WZ-gauge we define the supersymmetric field strength of $V$ by

$$W_\alpha = -\tfrac{1}{4}\bar{D}\bar{D}D_\alpha V, \qquad \bar{W}_{\dot\alpha} = -\tfrac{1}{4}DD\bar{D}_{\dot\alpha}V. \tag{3}$$

Note that the lowest component (in $\theta$, $\bar\theta$) of $W_\alpha$ is the gauge invariant gaugino $\lambda_\alpha$.

(d) Show that (3) defines a gauge invariant, left-chiral superfield! How do $W_\alpha$, $\bar{W}_{\dot\alpha}$ transform under Lorentz transformations? Expand $W_\alpha$ in its component fields. *Hint: Translate $V_{WZ}$, $D_\alpha$ and $\bar{D}_{\dot\alpha}$ into the left-chiral representation. Finally prove and use $\sigma^\mu\bar\sigma^\nu - \eta^{\mu\nu} = -2i\sigma^{\mu\nu}$.*

(e) The simplest SUSY, gauge as well as Lorentz-invariant action for a vectorsuperfield reads

$$S_{\mathrm{U}(1)} = \int d^4x d\theta^2 W^\alpha W_\alpha. \tag{4}$$

Why is this SUSY-invariant? Determine its component expression

$$\mathcal{L}_{\mathrm{U}(1)} = -\tfrac{1}{2}F^{\mu\nu}F_{\mu\nu} - 2i\lambda\sigma^\mu\partial_\mu\bar\lambda + D^2 + \tfrac{i}{4}\epsilon^{\mu\nu\rho\sigma}F_{\mu\nu}F_{\rho\sigma}, \tag{5}$$

with field strength $F_{\mu\nu}$. The last term is imaginary and absent after adding the hermitian conjugate to (4). Note also the presence of the new auxiliary field $D$. *Hint: Check and use $tr(\sigma^{\mu\nu}) = 0$ and $\sigma^{\mu\nu}\sigma^{\rho\sigma} = \frac{1}{4}(\eta^{\mu\rho}\eta^{\nu\sigma} - \eta^{\mu\sigma}\eta^{\nu\rho}) - \frac{i}{4}\epsilon^{\mu\nu\rho\sigma}$.*

2. *Gauge invariant matter couplings & Super Yang-Mills theories*

So far we have considered a supersymmetric U(1) gauge theory. However, the way matter couplings are realized in supersymmetric actions will guide us to the **non-abelian** generalizations of Ex. 8.1.

Consider a chiral superfields $\Phi$ transforming under a global symmetry

$$\Phi \mapsto \Phi' = e^{-i\lambda^a\rho(T_a)}\Phi, \qquad \lambda^a \in \mathbb{R}, \quad a = 1, \ldots, \dim(\mathfrak{g}), \tag{6}$$

in a representation[1] $\rho$ of a Lie algebra $\mathfrak{g}$ with generators $T_a$. In order to gauge this symmetry consistently the transformed superfield $\Phi'$ has to remain chiral.

(a) Check that (6) respects the chirality of $\Phi$ for $\lambda \in \mathbb{R}$ constant and for $\lambda \equiv \Lambda(x,\theta)$ a complete chiral superfield. Althought $W(\Phi)$ can be arranged to be gauge invariant, $\Phi^\dagger\Phi$ cannot. Determine its transformation behaviour.

(b) In order for this to be gauge invariant introduce a minimal coupling of the vectorsuperfield to the matter contained in the chiral superfield of the form

$$\mathcal{L}_{\mathrm{matter}} \supset \Phi^\dagger e^V \Phi\big|_{\theta^2\bar\theta^2}, \qquad V = V^a T_a, \quad a = 1, \ldots, \dim(\mathfrak{g}). \tag{7}$$

Determine the right transformation property of $e^V$ for gauge invariance. What is the first order transformation of $V$? Can you still perform the WZ-gauge?

[1] In the following we will omit the letter $\rho$ for convenience.

(c) Rewrite (7) in the left-chiral represenation by shifting $x^\mu$. This yields $e^{V-2i\theta\sigma^\mu\bar{\theta}\partial_\mu}$. Why do you expect the covariant derivative? Calculate the D-term ($\theta^2\bar{\theta}^2$-term) of (7) in the WZ-gauge in the left-chiral representation using $(V_{\mathrm{WZ}})^n = 0$ for $n \geq 3$, thus $e^{V_{\mathrm{WZ}}} = 1 + V_{\mathrm{WZ}} + \frac{1}{2}V_{\mathrm{WZ}}^2$. Identify the covariant derivatives.

(d) Turning to the kinetic term of the non-abelian gauge sector we have to generalize (4) further. The non-abelian field strength is defined by

$$W_\alpha = -\tfrac{1}{4}\bar{D}\bar{D}\left(e^{-V}D_\alpha e^V\right), \qquad \bar{W}_{\dot{\alpha}} = \tfrac{1}{4}DD\left(e^V\bar{D}_{\dot{\alpha}}e^{-V}\right). \tag{8}$$

How does $W_\alpha$ transform under a gauge transformation of $e^V$? Insert $e^{V_{\mathrm{WZ}}} = 1 + V_{\mathrm{WZ}} + \frac{1}{2}V_{\mathrm{WZ}}^2$ to deduce

$$W_\alpha = -\tfrac{1}{4}\bar{D}\bar{D}D_\alpha V + \tfrac{1}{8}\bar{D}\bar{D}\left[V, D_\alpha V\right].$$

Compare this to the abelian case (3). Calculate $W_\alpha$ explicitly. You obtain the same result as in Ex. 8.1, (d), replacing ordinary derivatives by covariant ones.

(e) Scale the superfield by $V \mapsto 2gV$, where $g$ denotes the **gauge coupling constant**. Next, introduce a complex coupling constant $\tau = \frac{\Theta}{2\pi} + \frac{4\pi i}{g^2}$ containing the **theta-angel** $\Theta$ and determine the action of the gauge sector given by

$$\mathcal{L}_{\mathrm{gauge}} = \frac{1}{32\pi}\mathrm{Im}\left(\tau\int d^2\theta\,\mathrm{Tr}W^\alpha W_\alpha\right). \tag{9}$$

*Hint: $\mathrm{Tr}W^\alpha W_\alpha$ is identical to (5) with covariant derivatives instead of ordinary ones. Then, multiply this by $\tau$ and determine the imaginary part Im.*

(f) Combine the matter and gauge sector of the action. Integrate out the auxiliary fields to determine the full scalar potential.

The result reads

$$\begin{aligned}
\mathcal{L} &= \mathcal{L}_{\mathrm{gauge}} + \mathcal{L}_{\mathrm{matter}} \\
&= \tfrac{1}{32\pi}\mathrm{Im}(\tau\int d^2\theta\,\mathrm{Tr}W^\alpha W_\alpha) + \int d^2\theta d^2\bar{\theta}^2\Phi^\dagger e^{2gV}\Phi + \left(\int d^2\theta W(\Phi) + \mathrm{h.c.}\right) \\
&= \mathrm{Tr}\left(-\tfrac{1}{4}F^{\mu\nu}F_{\mu\nu} - i\lambda\sigma^\mu D_\mu\bar{\lambda}\right) + \tfrac{\Theta}{32\pi^2}g^2\mathrm{Tr}F_{\mu\nu}\tilde{F}^{\mu\nu} \\
&\quad +(D_\mu\varphi)^\dagger D^\mu\varphi - i\psi\sigma^\mu D_\mu\bar{\psi} + i\sqrt{2}g\varphi^\dagger\lambda\psi - i\sqrt{2}g\bar{\psi}\bar{\lambda}\varphi \\
&\quad -\tfrac{1}{2}\frac{\partial^2 W}{\partial\varphi^i\varphi^j}\psi^i\psi^j - \tfrac{1}{2}\frac{\partial^2\bar{W}}{\partial\bar{\varphi}^i\bar{\varphi}^j}\bar{\psi}^i\bar{\psi}^j - V(\varphi^\dagger,\varphi) + \text{total derivatives},
\end{aligned} \tag{10}$$

with $\tilde{F}^{\mu\nu} = \frac{1}{2}\epsilon^{\mu\nu\rho\sigma}F_{\rho\sigma}$, the scalar potential

$$V(\varphi^\dagger,\varphi) = F^\dagger F + \tfrac{1}{2}D^2 = \sum_i\left|\frac{\partial W}{\partial\varphi^i}\right|^2 + \frac{g^2}{2}\sum_a\left|\varphi^\dagger T_a\varphi\right|^2 \tag{11}$$

and covariant derivatives

$$\begin{aligned}
D_\mu\lambda &= \partial_\mu\lambda - ig\left[V_\mu^b,\lambda\right], \qquad D_\mu\varphi = \partial_\mu\varphi - igV_\mu^a T_a\varphi, \\
D_\mu\lambda &= \partial_\mu\lambda - igV_\mu^a T_a\lambda, \qquad F_{\mu\nu} = \partial_\mu V_\nu - \partial_\nu V_\mu - ig\left[V_\mu, V_\nu\right].
\end{aligned} \tag{12}$$