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Exercise 11
30. Juni 2008
SS 08

Exercises on Elementary Particle Physics II

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1. *The non-linear Sigma Model & Kaehler Geometry*

The most general, globally supersymmetric action S_{chiral} for chiral multiplets Φ^i was given in Ex. 7.1 and is defined in terms of two functions $K((\Phi^i)^\dagger, \Phi^j)$ and $W(\Phi^i)$. Renormalizability constrains them to $K(\Phi^\dagger, \Phi) = \Phi^\dagger\Phi$ and a polynomial of degree 3 for W. However, renormalizability is only a meaningful concept for a fundamental or **microscopic theory**, i.e a theory that is valid at any energy scale. Hence, considering S_{chiral} as an **effective theory** valid only at low energies allows to relax the criterion of renormalizability.[1]

In the following consider an arbitrary **Kaehler potential** K and superpotential W.

(a) Work in the representation S of SUSY to calculate the D-term of $K(\Phi^\dagger, \Phi)$. *Hint: First, you have to expand* $K(\Phi^\dagger, \Phi)$ *around the scalar components* ϕ^i. *Then, you should identify the total derivative* $\partial_\mu\partial^\mu K(\phi^\dagger, \phi)$ *and discard it.*

(b) Argue that the D-term is invariant under

$$K(\Phi^\dagger, \Phi) \mapsto K(\Phi^\dagger, \Phi) + f(\Phi) + \bar{f}(\Phi^\dagger), \qquad \forall f \text{ holomorphic}, \tag{1}$$

descending to the **Kaehler-transformation** $K(\phi^\dagger, \phi) \mapsto K(\phi^\dagger, \phi) + f(\phi) + \bar{f}(\phi^\dagger)$. Introduce the **Kaheler metric**[2] as the second derivative of K, i.e.

$$g_{i\bar{j}}(\phi, \phi^\dagger) := \frac{\partial^2 K}{\partial\phi^i \partial(\phi^j)^\dagger}(\phi, \phi^\dagger). \tag{2}$$

In general, g describes an arbitrary **Kaehler manifold** with complex coordinates given by the scalars $(\phi^i, (\phi^j)^\dagger)$. Historically, theories with scalars taking values on a curved manifold as the target space are known as **non-linear sigma models**. We have proven here one direction of the statement that a globally SUSY-invariant action requires K to describe a Kaehler geometry and vice versa, i.e. the scalar target space has to be a Kaehler manifold.

Let us use this Kaehler geometry of the scalar target to geometrize the action further.

[1] For an effective theory the concept of renormalizability is replaced by considering only terms of leading order in space-time derivatives. Terms with more than two derivatives are supressed by positive powers of the length-scale of the fundamental theory.

[2] A Kaehler metric is just defined as a **hermitian metric** that can be written (locally) as (2). Therefore, K is called the Kaehler *potential*. K is unique up to the transformations (1).

(c) Show that the only non-vanishing Christoffel symbols are

$$\Gamma^i_{jk} = g^{i\bar{\jmath}}\partial_j g_{k\bar{\jmath}}, \qquad \Gamma^{\bar{\imath}}_{\bar{\jmath}\bar{k}} = g^{\bar{\imath}j}\partial_{\bar{\jmath}} g_{\bar{k}j}. \tag{3}$$

Use this to calculate the non-vanishing components of the curvature tensor $(R_{\bar{k}i})^l{}_j = -(R_{i\bar{k}})^l{}_j$ and $(R_{i\bar{k}})^{\bar{l}}{}_{\bar{\jmath}} = -(R_{\bar{k}i})^{\bar{l}}{}_{\bar{\jmath}}$.

(d) Define the (pullback of the) covariant derivatives of the Kaehler manifold by

$$D_\mu\psi^i = \partial_\mu\psi^i + \Gamma^i_{jk}\partial_\mu\phi^j\psi^k, \qquad D_\mu\bar{\psi}^{\bar{\imath}} = \partial_\mu\bar{\psi}^{\bar{\imath}} + \Gamma^{\bar{\imath}}_{\bar{\jmath}\bar{k}}\partial_\mu(\phi^j)^\dagger\bar{\psi}^{\bar{k}} \tag{4}$$

to rewrite the kinetic term of the fermions. Add the superpotential W to eliminate the auxiliary fields F^i by their equation of motion

$$F^i = g^{i\bar{\jmath}}\partial_{\bar{\jmath}}W - \tfrac{1}{2}\Gamma^i_{jk}\psi^j\psi^k. \tag{5}$$

Finally, insert (5) into the full action to derive its complete form.

Thus, the most general globally SUSY-invariant action of an effective theory reads

$$\begin{aligned}\mathcal{L}_{\text{chiral}} &= g_{i\bar{\jmath}}\left(\partial_\mu\phi^i\partial^\mu(\phi^j)^\dagger - iD_\mu\psi^i\sigma^\mu\bar{\psi}^{\bar{\jmath}}\right)\\ &- \tfrac{1}{2}\left(\frac{\partial^2 W}{\partial\phi^i\partial\phi^j} - \Gamma^k_{ij}\frac{\partial W}{\partial\phi^k}\right)\psi^i\psi^j - \tfrac{1}{2}\left(\frac{\partial^2\bar{W}}{\partial(\phi^\dagger)^{\bar{\imath}}\partial(\phi^\dagger)^{\bar{\jmath}}} - \Gamma^{\bar{k}}_{\bar{\imath}\bar{\jmath}}\frac{\partial\bar{W}}{\partial(\phi^\dagger)^{\bar{k}}}\right)\bar{\psi}^{\bar{\imath}}\bar{\psi}^{\bar{\jmath}}\\ &+ \tfrac{1}{4}R_{ij\bar{k}\bar{l}}\psi^i\psi^j\bar{\psi}^{\bar{k}}\bar{\psi}^{\bar{l}} - V(\phi,\phi^\dagger)\end{aligned} \tag{6}$$

with scalar potential given by

$$V(\phi,\phi^\dagger) = g^{i\bar{\jmath}}F_iF^\dagger_{\bar{\jmath}} = g^{i\bar{\jmath}}\,\frac{\partial W}{\partial\phi^i}\frac{\partial\bar{W}}{\partial(\phi^\dagger)^{\bar{\jmath}}}. \tag{7}$$

Note the presence of a four-fermion term due to the curved field space of the scalars.

Let us just note that one can consistently couple this action to an effective SUSY-invariant gauge theory. Then, the minimal coupling is realized as $K(\Phi^i, (\Phi^\dagger e^{2gV})^{\bar{\jmath}})$ and the most general kinetic term of the SUSY Yang-Mills theory is

$$\mathcal{L}_{\text{gauge}} = \frac{1}{8g^2}\text{Re}\left(\int d^2\theta f_{ab}(\Phi^i)W^{a\alpha}W^b_\alpha\right), \tag{8}$$

where the **gauge kinetic function** $f_{ab}(\Phi^i)$ is introduced. We recover the action of Ex. 8.2 for $f_{ab} = \frac{\tau g^2}{4\pi i}\text{Tr}(T_aT_b)$.

2. *Coupling a chiral multiplet to Supergravity*

Here we summarize just the results of the complicated and lengthy procedure of coupling the action of Ex. 11.1 to supergravity (SUGRA). This procedure is just the gauging of global SUSY to a local symmetry.

The entire Lagrangian depends only on the single function sometimes referred to as the Kaehler potential given by

$$G(\Phi^i, \Phi_j^*) = -\frac{K(\Phi^i, \Phi_j^*)}{M_{\rm P}^2} - \log\left(\frac{|W(\Phi^i)|^2}{M_{\rm P}^6}\right). \tag{9}$$

Here, we have made explicit the dependence on the Planck-mass $M_{\rm P}$ that can consistently be suppressed in calculations by putting $M_{\rm P} = 1$.[3] This yields

$$V_{\rm scalar}(\phi^i, (\phi^j)^\dagger) = -e^{-G}\left[3 + G_k \left(G^{-1}\right)_l^k G^l\right] M_{\rm P}^4$$

for the scalar potential of the chiral multiplet with

$$G_i = \frac{\partial G}{\partial \phi^{\dagger i}}, \quad G^j = \frac{\partial G}{\partial \phi_j}, \quad G_i^j = \frac{\partial^2 G}{\partial \phi^{\dagger i}\partial \phi_j}.$$

(a) Check $G_j^i = g_{i\bar{\jmath}}$, thus the scalar kinetic term can be expressed in terms of G.

(b) Show that this can be written as

$$V_{\rm scalar} = e^{K/M_{\rm P}^2}\left(D_i W D_{\bar{\jmath}} \bar{W} g^{i\bar{\jmath}} - 3\left|W\right|^2\right) = F^i F^{\dagger\bar{\jmath}} g_{i\bar{\jmath}} - 3e^{K/M_{\rm P}^2}\left|W\right|^2 \tag{10}$$

using the Kaehler metric $g_{i\bar{\jmath}} = \partial_i \partial_{\bar{\jmath}} K$, the covariant derivative $D_i W = \partial_i W + W\partial_i K/M_{\rm P}^2$ and the auxiliary fields of the chiral superfields

$$F^i = e^{K/2M_{\rm P}^2} g^{i\bar{\jmath}} D_{\bar{\jmath}} \bar{W}. \tag{11}$$

Breaking of local supersymmetry is analogous to the global case realized by a non-vanishing VEV of the auxiliary field F^i. However, the massless Goldstone fermion present in the global situation, compare Ex. 9.4, is eaten by the gravitino that acquires a mass in the SUSY-breaking vacuum given by

$$m_{3/2} = M_{\rm P} e^{\langle G\rangle/2}. \tag{12}$$

The scale of SUSY breaking is defined as the VEV of the auxiliary field F yielding

$$M_S^2 = e^{\langle G\rangle/2}\left(\langle G\rangle^{-1}\right)_l^k \langle G\rangle_k M_{\rm P}. \tag{13}$$

3. *Hidden Sector - the Polonyi Model*

A **hidden sector** is defined as sector of the theory that is coupled to the matter or **observable sector** only by gravitational interactions. This hidden sector is usually used for supersymmetry breaking as we will investigate for the Polonyi model.

[3]The dependence of $M_{\rm P}$ is necessary e.g. to decouple gravity from the theory by $M_{\rm P} \to \infty$. Then, SUGRA reproduces the results of global SUSY.

The hidden sector is defined by a single chiral superfield Φ that is a complete gauge singlet. For simplicity, assume a minimal Kähler potential $K(\Phi, \Phi^*) = \Phi^*\Phi$ and the following **Polonyi superpotential**

$$W(\Phi) = m^2(\Phi + \beta)$$

with β a dimensionful parameter which we will use later to adjust the vacuum energy to zero.

(a) Determine the F-term of Φ. For which values of β is SUSY definitely broken?

(b) For which values of β is there a non-SUSY vacuum with zero energy? Calculate the VEV of ϕ at those vacua.
Hint: The solution should be $\langle\phi\rangle_\pm = \pm(\sqrt{3} - 1)M$ *and* $\beta_\pm = \pm(2 - \sqrt{3})M$.

(c) Calculate the gravitino mass $m_{3/2}$. Discuss its order of magnitude. Express it in terms of the SUSY breaking scale M^2_{SUSY}.

(d) Determine the scalar mass m_ϕ in terms of $m_{3/2}$ by expanding around $\langle\phi\rangle$. Finally, calculate the supertrace, including the gravitino.[4] Compare to F-term breaking in the globally supersymmetric case!

This situation changes dramatically in the case of more than one chiral multiplet. For N chiral multiplets the supertrace reads

$$\text{STr}M^2 = 2(N - 1)m_{3/2}. \tag{14}$$

Thus, the scalar SUSY partners have to be more massive than the fermions in the multiplet what is required by the experimental data about the observed particle spectrum up to now. Furthermore, it is possible, due to Ex. 11.3, (c), to obtain a much smaller gravitino mass $m_{3/2}$ than the SUSY breaking scale M_S. Hence, a mass splitting between bosons and fermions of the right order of magnitude seems possible.

[4]The fermion of the superfield Φ is eaten by the gravitino.