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Exercises on Theoretical Astroparticle Physics

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In this exercise sheet we want to take a look at the Lorentz group and various spinors.

1. **The Lorentz group**
   We first focus on the Lorentz group $SO(1,3)$ and its represenations in some detail. The Lie algebra $\mathfrak{so}(1,3)$ of $SO(1,3)$ is defined by $\lambda^T = -\eta\lambda\eta$ where $\lambda \in \mathfrak{so}(1,3)$ and η is the metric of the Minkowski space. The Lorentz group has six generators: Three rotations J_i and three boosts K_i with the following commutation relations

$$[J_i, J_j] = \mathrm{i}\varepsilon_{ijk} J_k, \quad [K_i, K_j] = -\mathrm{i}\varepsilon_{ijk} J_k, \quad [J_i, K_j] = \mathrm{i}\varepsilon_{ijk} K_j \tag{1}$$

   (a) Derive $\lambda^T = -\eta\lambda\eta$ from the definition of a Lorentz transformation $\eta = \Lambda^T \eta \Lambda$ where $\Lambda = e^{\mathrm{i}\lambda}$.

   (b) Find a (complex) change of the above basis such that the algebra splits into two $\mathfrak{su}(2)$, i.e. $\mathfrak{so}(1,3) \otimes \mathbb{C} \cong \mathfrak{su}(2) \otimes \mathfrak{su}(2)$. Thus, every representation of $\mathfrak{so}(1,3)$ can be characterized by the spins (j_1, j_2) with $j_i \in \mathbb{N}_0/2$.

   (c) The algebra obtained in (b) is the algebra of $\mathfrak{sl}(2,\mathbb{C})$. The fact that the algebras coincide does not mean that the groups coincide also[1]. The groups $SO(1,3)$ and $SL(2,\mathbb{C})$ are *topologically* distinct. To that end, consider a map f from $\mathbb{R}^4$ to the hermitian 2×2 matrices defined by $f(x^\mu) = X = x^\mu \sigma_\mu$ (where $\sigma^0 = \mathbb{1}$). Show $\det X = x^\mu x_\mu$ and argue that $y^\mu = \Lambda^\mu_\nu x^\nu$ with $\Lambda \in SO(1,3)$ induces a map $AXA^\dagger$ for an $A \in SL(2,\mathbb{C})$, i.e. $\det A = 1$.

   (d) Show that $A \in SL(2,\mathbb{C})$ and $(-A)$ give the same Lorentz transformation. This means that $SO(1,3) \cong SL(2,\mathbb{C})/\mathbb{Z}_2$, i.e. $SL(2,\mathbb{C})$ is the double cover[2] of $SO(1,3)$.

2. **Weyl spinors**
   Now, we want to look at the spinor representations of $SL(2,\mathbb{C})$. We have the fundamental representation of $SL(2,\mathbb{C})$ and its complex conjugate defined by

$$\psi_\alpha \mapsto A_\alpha{}^\beta \psi_\beta \quad \text{and} \quad \bar\psi_{\dot\alpha} \mapsto A^*{}_{\dot\alpha}{}^{\dot\beta} \bar\psi_{\dot\beta}\,, \quad A \in SL(2,\mathbb{C}) \tag{2}$$

   where the dotted and undotted spinors ψ and $\bar\psi$ are called *left- and right-chiral Weyl* spinors. The representation carried by ψ_α is denoted by $(1/2, 0)$ and the one carried by $\bar\psi_{\dot\alpha}$ by $(0, 1/2)$. Using the result of (a) and the exponential map, we obtain

$$D_L := A = \exp((a_i + ib_i)\sigma_i) \quad \text{and} \quad D_R := A^* = \exp((a_i - ib_i)\sigma_i^*). \tag{3}$$

[1] As explained in the first exercise sheet, the Lie algebra is the tangent space at the unity of the group. This means the local structure of the groups coincide if the algebras are the same, but not necessarily the global structure.

[2] It's also the universal cover since it is simply connected.

(a) Show either using $\sigma_i^* = -\sigma_2\sigma_i\sigma_2$ or through direct calculation the following

$$D_L^T \sigma_2 D_L = \sigma_2. \tag{4}$$

What does this mean for the representation $\rho(A) = (A^{-1})^T$ for $A \in SL(2,\mathbb{C})$?

(b) Using the result of (a), we define the dual representation ψ^α to ψ_α through $\psi^\alpha = \varepsilon^{\alpha\beta}\psi_\beta$ with $\varepsilon^{\alpha\beta} = i(\sigma^2)^{\alpha\beta}$. Thus, we use $\varepsilon^{\alpha\beta}$ to raise and lower the spinor indices. Find the inverse metric $\varepsilon_{\alpha\beta}$.
The same analysis can be done for $\rho(A) = ((A^*)^{-1})^T$. The result is

$$\bar{\psi}^{\dot{\alpha}} = \varepsilon^{\dot{\alpha}\dot{\beta}}\bar{\psi}_{\dot{\beta}} \tag{5}$$

where $\varepsilon^{\dot{\alpha}\dot{\beta}} = i(\sigma^2)^{\dot{\alpha}\dot{\beta}}$ and $\varepsilon_{\dot{\alpha}\dot{\beta}} = \varepsilon_{\alpha\beta}$.

(c) How does $\psi^\alpha\psi_\alpha$ transform under $SL(2,\mathbb{C})$?

(d) What happens if we assume that the components of ψ_α are commuting variables?

(e) Using the adjoint representation of $SL(2,\mathbb{C})$, one can show that the index structure of σ^μ is as follows

$$\sigma^\mu = \sigma^\mu_{\alpha\dot{\alpha}}, \tag{6}$$

i.e. σ^μ maps right-chiral spinor to the left-chiral spinor. Furthermore one can show that $(\bar{\sigma}^\mu)^{\dot{\alpha}\alpha} = \varepsilon^{\alpha\beta}\varepsilon^{\dot{\alpha}\dot{\beta}}\sigma^\mu_{\beta\dot{\beta}}$. Using these, show

$$\psi^\alpha\sigma^\mu_{\alpha\dot{\alpha}}\bar{\chi}^{\dot{\alpha}} = \psi\sigma^\mu\bar{\chi} = -\bar{\chi}\bar{\sigma}^\mu\psi. \tag{7}$$

3. **Dirac and Majorana spinors (for neutrinos)**
A *Dirac spinor* is a four-component spinor consisting of two Weyl spinors as follows

$$\Psi = \begin{pmatrix} \psi_\alpha \\ \bar{\chi}^{\dot{\alpha}} \end{pmatrix}. \tag{8}$$

Obviously Ψ transforms as $(1/2, 0) \oplus (0, 1/2)$ of the Lorentz group. In the Weyl (chiral) basis the γ-matrices have the following convenient form

$$\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix} \quad \text{and} \quad \gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \tag{9}$$

where $\sigma^\mu = (1, \vec{\sigma})$ and $\bar{\sigma}^\mu = (1, -\vec{\sigma})$. A *Majorana spinor* is a Dirac spinor Ψ with the following constraint

$$\Psi^c := C\bar{\Psi}^T = \Psi \tag{10}$$

where $C = i\gamma^2\gamma^0$ is the charge conjugation matrix.

(a) What does this constraint imply for ψ and $\bar{\chi}$ and what is the physical meaning of this condition?

(b) We have two projectors

$$P_L := \frac{1}{2}(1 - \gamma^5) \quad \text{and} \quad P_R := \frac{1}{2}(1 + \gamma^5). \tag{11}$$

Obviously P_L and P_R project Ψ to the left chiral (handed) part and to the right chiral part respectively. We write $\Psi_L := P_L\Psi$ and $\Psi_R := P_R\Psi$. Show that Ψ_L^c is right-handed. To that end, show $C^{-1}\gamma^5 C = (\gamma^5)^T$ first.

(c) The Lagrangian $\mathcal{L}_D$ for a Dirac spinor has the form

$$\mathcal{L}_D = i\bar{\Psi}\gamma^\mu\partial_\mu\Psi - m\bar{\Psi}\Psi \tag{12}$$

where $\bar{\Psi} = \Psi^\dagger\gamma^0$. The second term is called the *Dirac mass term*. Rewrite $\mathcal{L}_D$ in terms of ψ and $\bar{\chi}$

(d) Using the result of (a) rewrite the action $\mathcal{L}_M$ for a Majorana spinor in terms of ψ and $\bar{\chi}$

$$\mathcal{L}_M = i\bar{\Psi}\gamma^\mu\partial_\mu\Psi - \frac{m}{2}\bar{\Psi}\Psi. \tag{13}$$

The second term is called the *Majorana mass term*. Why is the factor 1/2 included in the mass term?