Exercises on Theoretical Astroparticle Physics

Prof. Dr. H.-P. Nilles – P.D. Dr. S. Förste

Let us calculate, how old we are.

1. A four dimensional homogeneous isotropic universe is described by the **Robertson–Walker metric**.

$$\mathrm{d}s^2 = \mathrm{d}t^2 - R(t)\left(\frac{\mathrm{d}r^2}{1-kr^2} + r^2\left(\mathrm{d}\theta^2 + \sin^2\theta \mathrm{d}\phi^2\right)\right)$$

A long but straight forward calculation gives the Ricci tensor:

$$R_{00} = -3\frac{\ddot{R}}{R}$$
$$R_{ij} = -\left[\frac{\ddot{R}}{R} + 2\frac{\dot{R}^2}{R^2} + \frac{2k}{R^2}\right] g_{ij}\,.$$

The energy momentum tensor of the fields in the universe respecting its symmetries is the one of a perfect fluid

$$T^{\mu}_{\ \nu} = \mathrm{diag}(\rho(t), -p(t), -p(t), -p(t))$$

with the energy density $\rho(t)$ and the pressure density $p(t)$.

(a) Compute the curvature scalar $\mathcal{R} = g^{\mu\nu}R_{\mu\nu}$. Write down the 00 and the ii components of the Einstein equations

$$R_{\mu\nu} - \frac{1}{2}\mathcal{R}g_{\mu\nu} = 8\pi G T_{\mu\nu}\,.$$

The 00 component is also called Friedmann equation.

(b) Derive the first law of thermodynamics

$$\frac{\mathrm{d}}{\mathrm{d}t}(\rho R^3) = -p\frac{\mathrm{d}}{\mathrm{d}t}R^3\,.$$

Now we have two independent equations but yet three independent functions, therefore we need another equation to get the chance of finding a solution. Fortunately there is a relation between the energy and the pressure density $p(t) = w\rho(t)$ depending on the dominating content in the universe. One finds

- $w = 0$ for static matter
- $w = 1/3$ for radiation (due to $T^{\mu}_{\ \mu} = 0$)

- $w = -1$ for vacuum energy (due to $T_{\mu\nu} \propto g_{\mu\nu}$) .

(c) How does the energy density change with the radius in these three cases?

(d) Define the **Hubble parameter** $H(t) := \frac{\dot{R}(t)}{R(t)}$,the critical density $\rho_C := \frac{3H^2}{8\pi G}$ and $\Omega := \frac{\rho}{\rho_C}$ and rewrite the Friedmann equation as

$$\frac{k}{H^2 R^2} = \Omega - 1\,.$$

In an expanding universe, how does the type of the universe (closed/flat/open) depend on the energy density?

(e) We introduce the notation with an index zero for the todays values for the quantities R, Ω, H ... . Rewrite the Friedmann equation again as

$$\left(\frac{\dot{R}}{R_0 H_0}\right)^2 = 1 - \Omega_0 + \Omega_0 \left(\frac{R_0}{R}\right)^a$$

with $a = 3w + 1$.

(f) We define the time t such that $R(t = 0) = 0$. This allows us to compute the age of the universe

$$t \equiv \int_0^{R_0} \frac{\mathrm{d}R}{\dot{R}}\,.$$

Show that

$$t = H_0^{-1} \int_0^1 \frac{\mathrm{d}x}{\sqrt{1 - \Omega_0 + \Omega_0 x^{-a}}}$$

and compute $t_0 = t(R_0)$ for a matter- and radiation dominated universe for the open, flat and closed case.