Physikalisches Institut
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Theoretische Physik

Exercise 1
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SS 10

Exercises on Group Theory

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–Home Exercises–

H 1.1 Isomorphism Theorems

The Isomorphism Theorems known from the lecture are:

1. Let $f : G \to H$ be a group homomorphism. Then we have the following properties:
   (i) The kernel $\ker f$ is a normal subgroup of G.
   (ii) The image $f(G)$ is a subgroup of H.
   (iii) The quotient $G/\ker f$ is isomorphic to $f(G)$ with the isomorphism given by

$$\begin{aligned} f :& G/\ker f \to f(G) \\ f :& g(\ker f) \to f(g) \end{aligned}$$

2. Let H be a subgroup and N be a normal subgroup of G. Then we have
   (i) The product HN is a subgroup of G. (The product is defined as $HN = \{hn | h \in H, n \in N\}$)
   (ii) The intersection $H \cap N$ is a normal subgroup of H.
   (iii) There is an isomorphism of quotient groups, $HN/N \cong H/(H \cap N)$.

3. Let H and N be normal subgroups of G, and let N be a subgroup of H. Then N is also normal in H, and

$$(G/N)/(H/N) \cong G/H \, .$$

Use the first one to prove the second and third.

H 1.2 Quotient and Product Groups

(a) Let $H \subset G$ be a subgroup. Show that for the quotient group G/H to be well-defined, H must be normal.

(b) The Euclidean group $E(n)$ has two obvious subgroups which are the group of pure rotations $\cong O(n)$ and the group of pure translations $\cong \mathbb{R}^n$. Which of them are normal? What is the corresponding quotient group? Find an example of two subgroups $H \subset G \subset E(N)$ such that G is normal in $E(n)$, H is normal in G but H is not normal in $E(n)$.

(c) Show that a G is a direct product of two subgroups H_1, H_2 if

- H_1 and H_2 are normal,
- $H_1 \cap H_2 = \{e\}$,
- they generate the group, $G = H_1 H_2$.

(d) Show that $U(n) \cong U(1) \times SU(n)$.

(e) Show further that $O(n) = \mathbb{Z}_2 \times SO(n)$ is only true for n odd.

A semidirect group product is defined as

$$\begin{aligned} G = N \ltimes H = \{(n,h) | n \in N\,, h \in H\} \\ (n,h)(n',h') = (n\theta(h)n', hh') \end{aligned}$$

with $\theta : H \to \mathrm{Aut}(N)$ being an action of H on N.

(f) Show that $O(2) \cong SO(2) \ltimes \mathbb{Z}_2$. Find θ.

(g) For the normal subgroup N of $E(n)$ you found above, show that $E(n) \cong N \ltimes E(n)/N$. What is the action of θ?