Physikalisches Institut
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Theoretische Physik

Exercise 2
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SS 10

Exercises on Group Theory

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–Home Exercises–

H 2.1 Group Actions

A group action is a homomorphism $G \to \mathrm{Sym}(X)$ from a group G to the group of bijections of a set X.

(a) Show that if a group action is *regular*, then there exists a bijection between the group G and X.

(b) Show that the subgroup $N = \{g \in G | gx = x \forall x \in X\}$ is a normal subgroup and the quotient group G/N acts *faithfully* on X.

(c) Show that for $x \in X$ the *little group* G_x is a subgroup of G and that G acts transitively on the *orbit* Gx.

(d) Show that being in an orbit is an equivalence relation which we denote by $\sim$.

(e) Prove the *orbit-stabilizer theorem* which states: Given $x \in X$, there is a bijection between the orbit Gx of x and the set of left cosets of the stabilizer G_x of x given by

$$gx \longmapsto gG_x \,.$$

(f) Consider the following group actions:

- The symmetric group S_n acting on an n-element set.
- The orthogonal group $O(n)$ acting on $\mathbb{R}^n$
- The orthogonal group $O(n)$ acting on the $(n-1)$-sphere S^{n-1}
- Any group G acting on itself by left-multiplication $g \to hg$
- Any group G acting on itself by conjugation $g \to hgh^{-1}$

Which of these actions is *faithful*, *transitive* or *free* and what are the group orbits?

(g) What is $\mathbb{R}^n / \sim$ for the $O(n)$ action?

H 2.2 More Isomorphism Theorem
Consider the following group homomorphisms:

- $G_1 \times G_2 \to G_1 \, , (g_1, g_2) \to g_1$
- $\mathbb{R}^n \to \mathbb{R}^r \, , (x_1, \dots, x_n) \to (x_1, \dots, x_r)$ with $r < n$
- $\det : GL(n) \to \mathbb{R}^*$
- $\mathbb{R} \to U(1) \, , x \to e^{\mathrm{i}x}$

Show that these maps are indeed homomorphisms. Use the Isomorphism theorem to find a normal subgroup given by the kernel. What is the corresponding isomorphism?

H 2.3 More on Groups

(a) Let $H \subset G$ be a subgroup. Show that the number of elements in each left coset is the same e.g. by constructing a bijection. Deduce from this that the order of H divides the order of G.

(b) Show that a group whose order is prime is necessarily cyclic.

(c) Consider a group G with $|G| = pq$ with p,q both prime. Show that every proper subgroup of G is cyclic.

(d) Let $g \in G$ with $|G| < \infty$. Show that $g^{|G|} = e$.

(e) List all the subgroups of any group G whose order is a prime number.

(f) Show that for p prime, the set $\mathbb{Z}_p^* = \mathbb{Z}_p - \{0\}$ is an Abelian group under multiplication.