Physikalisches Institut
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Theoretische Physik

Exercise 5
17. Mai 2010
SS 10

Exercises on Group Theory

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–Home Exercises–

H 5.1 Permutations

(a) Show that the signum defined by

$$\begin{aligned}\mathrm{sign}: S_n &\longrightarrow \{\pm 1\}\\ \sigma &\longmapsto \frac{P(x_{\sigma(1)},\dots,x_{\sigma(n)})}{P(x_1,\dots,x_n)}\,, \qquad \text{with } P(x_1,\dots,x_n) = \prod_{1\le i<j\le n}(x_i - x_j)\,,\end{aligned}$$

is a group homomorphism.

(b) Using that each permutation σ can be written as a composition of transpositions, $\sigma = \tau_1 \dots \tau_r$, deduce that $\mathrm{sign}(\sigma) = (-1)^r$. Deduce that although r is not well defined, we can always say if it is even or odd.

(c) Show that the alternating group, defined as

$$A_n = \{\sigma \in S_n | \mathrm{sign}(\sigma) = 1\}$$

is a normal subgroup of S_n. What is the order of A_n?

(d) Using the notation,

$$\sigma = \begin{pmatrix} 1 & \dots & n \\ \sigma(1) & \dots & \sigma(n)\end{pmatrix}\,.$$

show that

$$\begin{aligned}\sigma^{-1} &= \begin{pmatrix} \sigma(1) & \dots & \sigma(n) \\ 1 & \dots & n\end{pmatrix}\\ \begin{pmatrix} 1 & \dots & n \\ \sigma(1) & \dots & \sigma(n)\end{pmatrix} &= \begin{pmatrix} \pi(1) & \dots & \pi(n) \\ \sigma(\pi(1)) & \dots & \sigma(\pi(n))\end{pmatrix}\\ \sigma\pi\sigma^{-1} &= \begin{pmatrix} \sigma(1) & \dots & \sigma(n) \\ \sigma(\pi(1)) & \dots & \sigma(\pi(n))\end{pmatrix}\,.\end{aligned}$$

(e) What are the conjugacy classes of S_4 and A_4? How many elements do they have?

H 5.2 Cayley's Theorem

(a) Consider a finite group G and the map

$$\pi : G \longrightarrow S_n\,, \qquad n = |G|\,,$$
$$g \longmapsto \pi(g) = \begin{pmatrix} e & g_1 & \dots & g_{n-1} \\ g & gg_1 & \dots & gg_{n-1} \end{pmatrix}\,.$$

Show that π is an group homomorphism.

(b) Show that π is injective. This implies that G is isomorphic to $\pi(G)$ and thus can be considered as a subgroup of S_n.

(c) Show that the action of $\pi(G)$ on an n-element set is regular.

(d) Show that $\pi(g)$ consists of cycles of the length $\mathrm{ord}(g)$.

(e) Use this to show that all groups of prime order are cyclic.

H 5.3 $\mathbb{Z}_N$ irreps

(a) Find all irreducible representations of the cyclic group $\mathbb{Z}_N$.

Hints: What does the Abelianity of $\mathbb{Z}_N$ imply about the dimensions of the representation spaces? What does finiteness of $\mathbb{Z}_N$ imply? Use the formula

$$N = \sum_\mu n_\mu^2$$

where the sum runs over all irreducible representations μ and n_μ is the dimension.

(b) Use the orthogonality relations to deduce the formula

$$\frac{1}{N} \sum_{j=0}^{N-1} e^{2\pi \mathrm{i} n j/N} e^{-2\pi \mathrm{i} n' j/N} = \delta_{nn'}$$