Physikalisches Institut
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Theoretische Physik

Exercise 6
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SS 10

Exercises on Group Theory

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–Home Exercises–

H 6.1 S_3 characters

(a) Show that S_3 has three irreducible representations. What are their dimensions?

(b) Compute the character table of S_3. *Hint: Ex. H3.2(f)*

H 6.2 Representation on Polynomials

In Ex. 3.2 we have already seen how S_3 acts on a three-dimensional space. This space can be seen as the space of homogeneous polynomials in three variables of degree one, i.e. $P_1 = \left\{P \in \mathbb{K}[x_1, x_2, x_3] \middle| P(\lambda x_1, \lambda x_2, \lambda x_3) = \lambda P(x_1, x_2, x_3)\right\}$.

(a) Show that the space of homogeneous polynomial of degree n in k variables,

$$P_n = \left\{P \in \mathbb{K}[x_1, \ldots, x_k] \middle| P(\lambda x_1, \ldots, \lambda x_k) = \lambda^n P(x_1, \ldots, x_k)\right\}$$

is a vector space

(b) For now we fix $k = 3$. Show that the representation on P_1 induces a representation on P_n.

(c) A basis of P_n is given by the set of monomials, i.e. $\left\{x_{i_1} x_{i_2} \ldots x_{i_n} \middle| 1 \leq i_1 \leq i_2 \leq \ldots \leq i_n \leq 3\right\}$. What are the dimensions of P_1, P_2 and P_3?

(d) What is the dimension of P_n for general n?

(e) Use the bases of monomials to compute the characters of P_1, P_2 and P_3. *Hint: You don't need the whole representation matrices, just the diagonal elements.*

(f) Use the character table from H6.1 and the orthogonality relation for characters to show how often the irreducible representations are contained in the representation spaces P_n for $n = 1, 2, 3$.

(g) Find a basis for all one-dimensional irreducible subspaces.

(h) *(optional)* Find a basis of the remaining subspaces.

H 6.3 Characters

(a) Show that $\chi(g^{-1}) = \chi(g)^*$.

(b) Consider again the scalar product on the character space. Show that

$$\sum_a n_a \chi^{*a} \chi^a = |G|$$

if and only if the representation associated to χ is irreducible.

H 6.4 Inverse Conjugacy classes

Show that for each conjugacy class there is one conjugacy class containing all the invereses.