Physikalisches Institut
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Theoretische Physik

Exercise 7
7. June 2010
SS 10

Exercises on Group Theory

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–Home Exercises–

H 7.1 Character Products

(a) Show that for two representations D_1 and D_2 we have

$$\chi_{D_1 \otimes D_2} = \chi_{D_1} \cdot \chi_{D_2} \,.$$

(b) Use e.g. Ex. H6.3(b) to show that if D is an irreducible representation and D_1 is a one-dimensional representation, then $D \otimes D_1$ is also irreducible of the same dimension.

H 7.2 Character Table of S_4

We construct the S_4 character table without constructing the concrete representation matrices.

(a) Draw all Young diagrams with four boxes. Use the hook rule to determine the dimensions of the corresponding irreducible representations. Check the formula

$$|G| = \sum_\mu n_\mu^2 \,.$$

(b) Draw the blank character table. Fill in the $\chi(e)$ column. Use Ex. H5.1.(e) to fill the rows of the one-dimensional representations (the trivial T and alternating A).

(c) Deduce from H7.1 that for the two-dimensional representation $\mathbf{2}$ we have $\mathbf{2} \otimes A \cong \mathbf{2}$. What does this imply about the characters of the odd permutations?

(d) Use the orthonormality of the irreducible representations to complete the row $\chi_{\mathbf{2}}$.
Hint: The Characters of the symmetric group are always integers.

(e) Argue that for the three-dimensional representations $\mathbf{3}$ and $\mathbf{3}'$ we must have $\mathbf{3}' = \mathbf{3} \otimes A$. How does this relate their characters?

(f) Use again the orthonormality to complete the table.

(g) Use the table and orthogonality to decompose all products of two irreducible representations.

H 7.3 Decomposition of the Regular Representation

We take again S_3 and construct the vector space of linear combinations of group elements,

$$R := \left\{ \sum_{\sigma \in S_3} a_\sigma \sigma \middle| a_\sigma \in \mathbb{C} \right\}$$

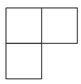
and the canonical group action

$$D(\sigma)\tau = \sigma\tau\,.$$

(a) Draw all Young diagrams with three boxes. Find all standard Young tableaux, i.e. all possibilities to write in the numbers $1, \ldots, 3$ such that in each row and column the numbers grow. You should find four possibilities.

(b) For each standard Young tableaux, we define the symmetrization operator known from the lecture. Show that these operators are projection operators.

(c) Now consider the Young diagram



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Show that the operators of the two standard Young tableaux of this diagram project on orthogonal spaces.

(d) Show that these two operators are related by

$$Y_2 = (23)Y_1(23)\,.$$

(e) Now choose the non-standard Young tableaux for this diagram

3	2
1	

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Show that the corresponding operator is not orthogonal to Y_1 and Y_2.