Physikalisches Institut
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Theoretische Physik

Exercise 8
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SS 10

Exercises on Group Theory

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–Home Exercises–

H 8.1 Normal discrete subgroups of Lie groups

Show that if a Lie group G has a discrete normal subgroup $N \lhd G$, then N lies in the center of G.

H 8.2 $SO(3)$ geometry

(a) Show that each $O \in SO(3)$ has an eigenvector with eigenvalue one. This allows us to parametrize $SO(3)$ with a unit vector $\hat{n}$ and a rotation angle α. Show that $SO(3)$ can be parametrized by a three-dimensional ball with opposite points identified on the boundary.

(b) Show that $SO(3)$ is not simply connected, i.e. there exists a closed cycle which is not contractable.

(c) Show that the Lie-algebra $\mathfrak{so}(3)$ is the vector space of antisymmetric matrices. Use the basis

$$(L_i)_{jk} = \epsilon_{ijk}\,, \qquad i = 1, \dots, 3\,,$$

to show the commutator

$$\big[L_i, L_j\big] = \epsilon_{ijk} L_k$$

H 8.3 Algebraic equivalence of $SO(3)$ and $SU(2)$

(a) Consider the set of Hermitean traceless 2×2 matrices.

$$A = \left\{ m \in \mathbb{C}^{2\times 2} \,\middle|\, \operatorname{tr} m = 0\,,\ m = m^\dagger \right\} \tag{1}$$

First show that the Pauli matrices form a basis of A. Thus for $m \in A$ we can write $m = m_i \sigma_i$ with,

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \sigma_2 = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix}, \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

(b) Show that for $U \in SU(2)$ and $m \in A$ we have $U^\dagger m U = n \in A$. This allows us to define a map

$$\begin{aligned} \omega : SU(2) &\longrightarrow \mathrm{Aut}(\mathbb{R}^3) \\ U &\longmapsto \omega(U) \end{aligned}$$

such that $n_i = \omega(U)_{ij} m_j$. Deduce the formula $\omega(U)_{ij} = \frac{1}{2}\,\mathrm{tr}\left(\sigma_i U^\dagger \sigma_j U\right)$.

(c) Show that ω is a homomorphism, i.e. $\omega(UV)_{ik} = \omega(U)_{ij}\omega(V)_{jk}$.

(d) Show that $\omega(U)_{ij} = \omega(U)^{-1}_{ji}$. This implies $\omega(U) \in O(3)$.

(e) Use the connectedness of $SU(2)$ to argue that $\det(\omega(U)) = +1$, i.e. $\omega(U) \in SO(3)$.

(f) Show that the Lie-algebra $\mathfrak{su}(2)$ is equal to A as a vector space.

(g) Show that $\mathfrak{su}(2)$ is isomorphic to $\mathfrak{so}(3)$ as Lie-algebras.

This already shows that there is a geometrical connection between $SO(3)$ and $SU(2)$. We will investigate this further on the next sheet.

H 8.4 Lie Bracket

(a) Show that a vector field $X = X^i(x)\partial_i$ is invariant under local coordinate transformations.

(b) Show that for two vector fields X and Y the product $X^i\partial_i Y^j \partial_j$ is not invariant under local coordinate transformations.

(c) Show that the Lie bracket $\mathcal{L}[X, Y]$ is invariant under local coordinate transformations.
Hint: Write the transformation matrix in terms of the old and new coordinates.