Physikalisches Institut
Universität Bonn
Theoretische Physik

Exercise 11
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SS 10

Exercises on Group Theory

Dr. Christoph Lüdeling

–Home Exercises–

H 11.1 Roots of $SU(4)$

The simple roots of $\mathfrak{su}(4)$ can be chosen as follows:

$$\alpha^1 = (1, 0, 0)\,, \qquad \alpha^2 = \left(-\frac{1}{2}, \frac{1}{\sqrt{2}}, -\frac{1}{2}\right)\,, \qquad \alpha^3 = (0, 0, 1)\,.$$

(a) Find the Cartan matrix $A^{ij} = \frac{2\alpha^i \cdot \alpha^j}{\alpha^i \cdot \alpha^i}$ and the Dynkin diagram.

(b) Consider the master formula

$$\frac{\alpha \cdot \beta}{\beta \cdot \beta} = -\frac{1}{2}(p - q)$$

where α and β are roots and $p(q)$ are the highest integers such that $\alpha + p\beta$ and $\alpha - q\beta$ are still roots. Let $\{\alpha^i\}$ be a set of simple roots and $\alpha = \sum a_i \alpha_i$ and $\beta = \sum b_j \alpha_j$. For α and β positive, what is q in terms of a_i and b_i? *Hint: Remember that for a positive root $\alpha = \sum a_i \alpha_i$, all a_i are nonnegative integers.*

(c) Use the master formula to determine which roots of $\mathfrak{su}(4)$ exist at level two and three, and show that there are none at level four.

(d) As we have seen, the generators of the non-simple roots can be constructed as $E_{\alpha+\beta} = e_{\alpha,\beta}[E_\alpha, E_\beta]$. Determine the normalization constants $e_{\alpha,\beta}$. *Hint: Exploit that E_β transforms in a definite way under the $\mathfrak{su}(2)$-subalgebra generated by $E_{\pm\alpha}$ and $\alpha_i H_i$.*

(e) Find the fundamental weights μ^i, i.e. the weights which satisfy $\frac{2\alpha^j \cdot \mu^i}{\alpha^j \cdot \alpha^j} = \delta^{ij}$.

H 11.2 Conjugate representations

Let ρ be a representation of $\mathfrak{g}$ with generators T_i.

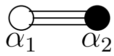
(a) Show that $\overline{\rho} := -\rho^*$ is also a representation, called the *complex conjugate representation.*

(b) What are the weights of $\overline{\rho}$?

(c) We call a representation *real* if $\overline{\rho} = U\rho U^{-1}$ for some automorphism U of the representation space. What does that mean for the weights? Based on this, argue that the adjoint representation is real.

H 11.3 The exceptional algebra G_2

The Dynkin diagram of the Lie algebra of the group G_2 is



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(a) Write down the Cartan matrix.

(b) Write down the simple roots *Hint: Choose w.l.o.g. the shorter root to be* $\alpha_2 = \left(0, \frac{1}{\sqrt{3}}\right)$.

(c) Determine all other positive roots using the master formula.

(d) What is the dimension of G_2? What is the Dynkin label of the highest root?

(e) Draw the roots in a coordinate system.

(f) Now consider the second fundamental weight. Work out the according representation. What is the dimension?

H 11.4 Highest weight procedure

(a) Write down the Dynkin diagram and the Cartan matrix of $\mathfrak{su}(3)$.

(b) Use the highest weight procedure to construct the irreducible representation associated to the Dynkin label $[2, 0]$. What is the dimension? Is the representation real?