Physikalisches Institut
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Theoretische Physik

Exercises 0
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Exercises on Theoretical Particle Physics II

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0.1 The massless Irrep. of Supersymmetry. (*0 credits*)

To explore which is the particle content of supersymmetric theories we will use the *Wigner Method*. For that purpose we will find a representation of a certain subgroup H of the SUSY group, for a given momemtum 4-vector in the rest frame q^μ, and search for H leaving this momemtum invariant, and for the representations of H on the $|q^\mu\rangle$ states. To obtain the general behaviour one should boost that frame to one with arbitrary momemtum. We will consider here the massless case, no central charges in the Algebra, and a theory with N supersymmetries.

(a) A massless particle in the rest frame can be described with the 4-momemtum $q_s^\mu = (m, 0, 0, m)$. Determine which generators of the SUSY Algebra leave q_s^μ invariant. Which are the commutation relations of the remaining generators from the Lorentz subgroup and which algebra do they expand?

(b) Use the Theorem: *Non trivial unitary representations of non compact groups are infinite dimensional* to argue that finding representations of H is equivalent to find representations of the generator $J = J_{12},\ \mathrm{i} \neq j$. This generator will give us the helicity of a given state.

(c) Show that the supersymmetry Algebra in the particular frame q_s^μ is given by

$$\{Q, Q\} = 0, \{\bar{Q}, \bar{Q}\} = 0, \tag{1}$$

$$\{Q^{1i}, \bar{Q}^{\dot{1}}_j\} = 0, \tag{2}$$

$$\{Q^{2i}, \bar{Q}^{\dot{2}}_j\} = 4m\delta^i_j \tag{3}$$

$$\left[Q^i_1, J\right] = -\frac{i}{2}Q^i_1 \tag{4}$$

$$\left[\bar{Q}_{\dot{1}}, J\right] = \frac{i}{2}\bar{Q}^i_{\dot{1}} \tag{5}$$

(d) Consider the states $Q^i_2|q^\mu_s\rangle$ and $\bar{Q}_{\dot{2}i}|q^\mu_s\rangle$, and impose positive norm on them, use Eq.(2) to show that the set $Q^i_2, \bar{Q}_{\dot{2}i}$ has zero action on the rest-frame.

(e) Note that Q^i_1 and $\bar{Q}^i_{\dot{1}}$ form a Clifford algebra, and they act as rising and lowering operators for J. Choose an state of a given helicity λ being the vacuum state for the operator Q^i_1

$$J|\lambda\rangle = i\lambda|\lambda\rangle \tag{6}$$

$$Q^i_1|\lambda\rangle = 0 \tag{7}$$

Describe the states that you will obtain by acting with the creation operators on the vacuum. Which helicity has a generic of these states and how many states are asociated with a given helicity? Which are the minimum and maximum possible helicities?

(f) To have a *CPT symmetric theory* one will need particles of both helicities, one should add then(if they are not present) the representations with helicities $-\lambda$ to $-\lambda + N/2$. Consider helicities with maximal $\lambda_{max} = 2$, describe the spectra of the theories with $N = 1$ and $N = 4$. Which is the maximal extended supergravity theory?

0.2 The massive Irrep. of Supersymmetry. *(0 credits)*

We will consider now the massive irreducible representations of SUSY with central charges trivially realized.

(a) Take as a rest-frame one with momentum $q_s^\mu = (m, 0, 0, 0)$. Which are the generators conforming a subset H, which leave q_s^μ invariant?

(b) Compute the supersymmetry algebra acting on the rest-frame states to be:

$$
\begin{aligned}
\{Q^{Ai}, \bar{Q}^{\dot{B}j}\} &= 2\delta^{A\dot{B}}\delta^i_j m && (8)\\
\{Q, Q\} &= 0, \{\bar{Q}, \bar{Q}\} = 0 && (9)\\
[J_m, J_n] &= \epsilon_{mnr} J_r && (10)\\
\left[Q^{Ai}, J_m\right] &= i(\sigma_m)^A_B Q^{Bi} && (11)\\
\left[\bar{Q}^{\dot{A}i}, J_m\right] &= i(\sigma_m)^{\dot{A}}_{\dot{B}} \bar{Q}^{\dot{B}i} && (12)
\end{aligned}
$$

(c) Consider now the Clifford vacuum $Q^i_A|q^\mu_s\rangle = 0 \quad \forall_{A,i}$ construct the states by appliying systematically the creation operators on the vacuum. Which is the number of states you have for a given N?

(d) Consider $N = 1$, given a vacuum with spins $s = 0, \frac{1}{2}, 1, \frac{3}{2}, 2$, construct the described representations specifiying their spin.

(e) Define the hermitian generators given by

$$
\Gamma^i_{2A-1} = \frac{1}{2m}(Q^{Ai} + \bar{Q}^{\dot{A}i})
$$
$$
\Gamma^i_{2A} = \frac{i}{2m}(Q^{Ai} - \bar{Q}^{\dot{A}i}),
$$

Compute the new Clifford algebra, define the parity operator $\Gamma_{4N+1} = \prod_{p=1}^4 \prod_{i=1}^N \Gamma^i_p$, prove

$$
\Gamma^2_{4N+1} = 1, \{\Gamma_{4N+1}, \Gamma^i_p\} = 0 \qquad (13)
$$

Analize is eigenvalues in the constructed states.

(f) The 4N elements of the Clifford algebra carry the group $SO(4N)$, which is an invariance group of the whole algebra. This group contains $SU(2) \times USp(2N)$. Show that the $SU(2)$ rotations generators in $SO(4N)$ is represented by

$$
s_k = -\frac{i}{4m}(\sigma_k)^A_B [Q^{jB}, (Q^{jA})^*].
$$

(g) The states of a given spin can be classified by that subgroup of $SO(4N)$ commuting with the appropiate $SU(2)$ rotation subgroup. Consider the generators

$$\begin{aligned} \Lambda^i_j &= \frac{i}{2m}\left[Q^{Ai}, (Q^{Aj})^*\right], \\ k^{ij} &= \frac{i}{2m}\left[Q^{Ai}, Q^j_A\right], k^{ij} = (k_{ij})^\dagger. \end{aligned} \tag{14}$$

Do the change of variables to $Q^a_A = Q^i_A \delta^a_i$ for $a = 1...N$, $Q^a_A = \epsilon_{AB}(Q^i_B)^* \delta^a_i$ for $a = N+1...2N$, define the new generators

$$s^{ab} = \frac{i}{2m}\left[Q^{Aa}, Q^b_A\right], \tag{15}$$

show that they span the algebra of $USp(2N)$.