Physikalisches Institut
Universität Bonn
Theoretische Physik

Exercises 2
26 April 2010
SS 2010

Exercises on Theoretical Particle Physics II

Priv.Doz.Dr. S.Förste

Due 03/05/2010

2.1 The SUSY Algebra and the Chiral Representation (*16 credits*)

The SUSY Algebra relates in a non-trivial way the Poincaré group with generators $P_\mu, M_{\mu\nu}$, with and Internal symmetry group with generators T_r by the inclusion of anticommuting generators Q^i_α, the demand of a Z_2 graded structure for it, and the fullfillment of the Coleman-Mandula Theorem for the subset $\{P_\mu, M_{\mu\nu}, Q^i_\alpha\}$. The $Q_\alpha, \bar{Q}_{\dot\beta} = (Q_\beta)^*$ transform in the Lorentz group representations $(0, \frac{1}{2})$ and $(\frac{1}{2}, 0)$ respectively. The Algebra reads

$$\begin{aligned} \{Q_\alpha, \bar{Q}_{\dot\beta}\} &= 2(\sigma^\mu)_{\alpha\dot\beta} P_\mu, \quad [Q_\alpha, P_\mu] = 0, \\ [M_{\mu\nu}, Q_\alpha] &= i(\sigma_{\mu\nu})_\alpha{}^\beta Q_\beta, \quad \left[M_{\mu\nu}, \bar{Q}^{\dot\alpha}\right] = i(\bar\sigma_{\mu\nu})^{\dot\alpha}{}_{\dot\beta} \bar{Q}^{\dot\beta}. \end{aligned} \tag{1}$$

(a) The SUSY algebra can be viewed as a Lie Algebra by introducing Grassmann variables $\theta_\alpha, \bar\theta_{\dot\alpha}$. Check that this new algebra is given by the commutators

$$\left[(\theta Q), (\bar{Q}\bar\theta)\right] = 2(\theta\sigma^\mu\bar\theta)P_\mu, \quad [P_\mu, (\theta Q)] = \left[P_\mu, (\bar{Q}\bar\theta)\right] = 0. \tag{2}$$

(*1 credit*)

(b) Define the corresponding group element associated to the Lie Algebra (2) as

$$S(a^\mu, \alpha, \bar\alpha) := \exp\left[\alpha Q + \bar{Q}\bar\alpha - i a^\mu P_\mu\right]. \tag{3}$$

Show that $S(a^\mu, \alpha, \bar\alpha) S(b^\mu, \beta, \bar\beta)$ is again a group element. (*2 credits*)

(c) Multiplication of group elements induces a motion in the parameter space, called the **superspace**, with coordinates $(x^\mu, \theta, \bar\theta)$. This serves to define a representation of the SUSY group on **superfields** $\Phi(x^\mu, \theta, \bar\theta)$ as

$$\begin{aligned} s(a^\mu, \alpha, \bar\alpha) : (x^\mu, \theta, \bar\theta) &\rightarrow (x^\mu + a^\mu - i\alpha\sigma^\mu\bar\theta + i\theta\sigma^\mu\bar\alpha, \theta + \alpha, \bar\theta + \bar\alpha), \\ S(a^\mu, \alpha, \bar\alpha)\Phi(x^\mu, \theta, \bar\theta) &= \Phi(x^\mu + a^\mu - i\alpha\sigma^\mu\bar\theta + i\theta\sigma^\mu\bar\alpha, \theta + \alpha, \bar\theta + \bar\alpha). \end{aligned} \tag{4}$$

Use an infinitesimal transformation to show that the SUSY algebra on superfields $\Phi(x^\mu, \theta, \bar\theta)$ is realised by

$$P_\mu = i\partial_\mu, \quad Q_\alpha = \partial_\alpha - i(\sigma^\mu)_{\alpha\dot\beta}\bar\theta^{\dot\beta}\partial_\mu, \quad \bar{Q}_{\dot\beta} = -\bar\partial_{\dot\beta} + i\theta^\alpha(\sigma^\mu)_{\alpha\dot\beta}\partial_\mu \tag{5}$$

(*2 credits*)

(d) Check that (4) is fulfilled for linear combinations and products of superfields. In addition show that the defined operators (5) form a representation of the SUSY algebra by explicitly verifiying the (anti-) commutation relations (1). This tell us that (4) defines a linear representation of (1), i.e. on a vectorspace. (*2 credits*)

(e) Define a (SUSY) covariant derivative D_α by

$$D_\alpha(\delta_{(\epsilon,\bar{\epsilon})}\Phi) = \delta_{(\epsilon,\bar{\epsilon})}(D_\alpha\Phi), \tag{6}$$

where the infinitesimal transformation $\delta = \epsilon Q + \bar{Q}\bar{\epsilon}$ comes from the $S(0,\epsilon,\bar{\epsilon})$ expansion. This definition implies that $D\Phi$ transforms as a superfield too. Show that the following derivatives are covariant:

$$\begin{aligned} D_\alpha &= \partial_\alpha + i(\sigma^\mu)_{\alpha\dot{\beta}}\bar{\theta}^{\dot{\beta}}\partial_\mu, \\ D_{\dot{\beta}} &= -\bar{\partial}_{\dot{\beta}} - i\theta^\alpha(\sigma^\mu)_{\alpha\dot{\beta}}\partial_\mu. \end{aligned} \tag{7}$$

(*2 credits*)

(f) Next define **left** and **right chiral representations** by

$$\begin{aligned} S_L(a^\mu,\alpha,\bar{\alpha}) &:= \exp\left[\alpha Q - ia^\mu P_\mu\right]\exp\left[\bar{Q}\bar{\alpha}\right], \\ S_R(a^\mu,\alpha,\bar{\alpha}) &:= \exp\left[\bar{Q}\bar{\alpha} - ia^\mu P_\mu\right]\exp\left[\alpha Q\right]. \end{aligned} \tag{8}$$

Consider the left representation S_L, obtain its relation with the representation S in (3). Check that $S_L(a^\mu,\alpha,\bar{\alpha})S_L(b^\mu,\beta,\bar{\beta})$ is a group element. (*2 credits*)

(g) A superfield in the left-chiral respresentation is defined as

$$S_L(a^\mu,\alpha,\bar{\alpha})\left[\phi_L(x^\mu,\theta,\bar{\theta})\right] = \phi_L(x^\mu + a^\mu + 2i\theta\sigma^\mu\bar{\alpha},\theta+\alpha,\bar{\theta}+\bar{\alpha}). \tag{9}$$

Determine the representations of the SUSY generators Q_L and $\bar{Q}_L$. (*2 credits*)

(h) Check that the following operators define covariant derivatives

$$\begin{aligned} D_{L\alpha} &= \partial_\alpha + 2i(\sigma^\mu)_{\alpha\dot{\beta}}\bar{\theta}^{\dot{\beta}}\partial_\mu, \\ \bar{D}_{L\dot{\beta}} &= -\bar{\partial}_{\dot{\beta}}, \end{aligned} \tag{10}$$

by computing the commutator with the SUSY transformation S_L. (*1 credit*)

(i) Define now **chiral superfields** by the constrains

$$\begin{aligned} \bar{D}\Phi(x,\theta,\bar{\theta}) &= 0, \quad \text{for } \textbf{left-chiral}, \\ D\Phi(x,\theta,\bar{\theta}) &= 0, \quad \text{for } \textbf{right-chiral}. \end{aligned} \tag{11}$$

This definition is independent of the representation. Work in the representation S of (3) to show that the component fields are not constrain by differential equations in x. Choose the left-chiral representation $\bar{D}\Phi = \bar{D}_L\phi_L$, to deduce the general form of a left chiral superfield. (*2 credits*)

Hint: Make a Taylor expansion in θ, to define the component fields of Φ.

(j) Consider the infinitesimal SUSY transformation $\delta_{(\epsilon,\bar{\epsilon})}$ of a left-chiral superfield ϕ_L. How do the component fields of ϕ_L transform? (*2 credits*)

Hint: Use the left-chiral representation of the SUSY generators Q_L and $\bar{Q}_L$ and assume that the transfromation is small: $\epsilon\sigma^\mu\bar{\epsilon} \approx 0$.