Physikalisches Institut
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Theoretische Physik

Exercises 4
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# Exercises on Theoretical Particle Physics II

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## 4.1 Super Yang-Mills theory and coupling to matter (*19 credits*)

The way matter couplings are realized in supersymmetric actions will guide us to the **non-Abelian** generalization of the gauge invariant action we already studied. Consider a chiral superfields $\Phi$ transforming under a global symmetry

$$\Phi \mapsto \Phi' = e^{-\mathrm{i}\lambda^a \rho(T_a)}\Phi, \qquad \lambda^a \in \mathbb{R}, \quad a = 1, \dots, \dim(\mathfrak{g}), \tag{1}$$

in a representation[1] $\rho$ of a Lie algebra $\mathfrak{g}$ with generators $T_a$. In order to gauge this symmetry consistently the transformed superfield $\Phi'$ has to remain chiral.

1. Check that (1) respects the chirality of $\Phi$ for $\lambda \in \mathbb{R}$ constant and for $\lambda \equiv \Lambda(x, \theta)$ a complete chiral superfield. Although $W(\Phi)$ can be arranged to be gauge invariant, $\Phi^\dagger\Phi$ cannot. Determine its transformation behaviour. (*2 credits*)

2. In order for this to be gauge invariant introduce a minimal coupling of the vectorsuperfield to the matter contained in the chiral superfield of the form

$$\mathcal{L}_{\text{matter}} \supset \Phi^\dagger e^V \Phi\Big|_{\theta^2\bar{\theta}^2}, \qquad V = V^a T_a, \quad a = 1, \dots, \dim(\mathfrak{g}). \tag{2}$$

   Determine the right transformation property of $e^V$ for gauge invariance. What is the first order transformation of $V$? Can you still perform the WZ-gauge? (*3 credits*)

3. Rewrite (2) in the left-chiral represenation by shifting $x^\mu$. This yields $e^{V-2\mathrm{i}\theta\sigma^\mu\bar{\theta}\partial_\mu}$. Why do you expect the covariant derivative to appear? Calculate the D-term ($\theta^2\bar{\theta}^2$-term) of (2) in the WZ-gauge in the left-chiral representation using $(V_{\mathrm{WZ}})^n = 0$ for $n \geq 3$. Identify the covariant derivatives. (*3 credits*)

4. Turning to the kinetic term of the non-Abelian gauge sector we have to generalize it further. The non-Abelian field strength is defined by

$$W_\alpha = -\tfrac{1}{4}\bar{D}\bar{D}\left(e^{-V} D_\alpha e^V\right), \qquad \bar{W}_{\dot{\alpha}} = \tfrac{1}{4}DD\left(e^V \bar{D}_{\dot{\alpha}} e^{-V}\right). \tag{3}$$

   How does $W_\alpha$ transform under a gauge transformation of $e^V$? Insert $e^{V_{\mathrm{WZ}}} = 1 + V_{\mathrm{WZ}} + \frac{1}{2}V_{\mathrm{WZ}}^2$ to deduce

$$W_\alpha = -\tfrac{1}{4}\bar{D}\bar{D}D_\alpha V + \tfrac{1}{8}\bar{D}\bar{D}\left[V, D_\alpha V\right].$$

   Compare this to the Abelian case. Calculate $W_\alpha$ explicitly. You obtain the same result as on sheet 2, replacing ordinary derivatives by covariant ones. (*3 credits*)

[1] In the following we will omit the letter $\rho$ for convenience.

5. Scale the superfield by $V \mapsto 2gV$, where $g$ denotes the **gauge coupling constant**. Next, introduce a complex coupling constant $\tau = \frac{\Theta}{2\pi} + \frac{4\pi \mathrm{i}}{g^2}$ containing the **theta-angle** $\Theta$ and determine the action of the gauge sector given by

$$\mathcal{L}_{\text{gauge}} = \frac{1}{32\pi}\mathrm{Im}\left(\tau \int d^2\theta \mathrm{Tr} W^\alpha W_\alpha\right). \tag{4}$$

   *Hint:* $\mathrm{Tr}W^\alpha W_\alpha$ *is identical to the expanded Lagrangian for a* $U(1)$ *gauge theory with covariant derivatives instead of ordinary ones. Then, multiply this by* $\tau$ *and determine the imaginary part .* *(4 credits)*

6. Combine the matter and gauge sector of the action. Integrate out the auxiliary fields to determine the full scalar potential. *(4 credits)*

The result reads

$$\begin{aligned}
\mathcal{L} &= \mathcal{L}_{\text{gauge}} + \mathcal{L}_{\text{matter}} \\
&= \tfrac{1}{32\pi}\mathrm{Im}(\tau \int d^2\theta \mathrm{Tr} W^\alpha W_\alpha) + \int d^2\theta d^2\bar\theta^2 \Phi^\dagger e^{2gV}\Phi + \left(\int d^2\theta W(\Phi) + \text{h.c.}\right) \\
&= \mathrm{Tr}\left(-\tfrac{1}{4}F^{\mu\nu}F_{\mu\nu} - \mathrm{i}\lambda\sigma^\mu D_\mu\bar\lambda\right) + \tfrac{\Theta}{32\pi^2}g^2 \mathrm{Tr} F_{\mu\nu}\tilde F^{\mu\nu} \\
&\quad + (D_\mu\varphi)^\dagger D^\mu\varphi - \mathrm{i}\psi\sigma^\mu D_\mu\bar\psi + \mathrm{i}\sqrt{2}g\varphi^\dagger\lambda\psi - \mathrm{i}\sqrt{2}g\bar\psi\bar\lambda\varphi \\
&\quad - \tfrac{1}{2}\frac{\partial^2 W}{\partial\varphi^i\varphi^j}\psi^i\psi^j - \tfrac{1}{2}\frac{\partial^2 \bar W}{\partial\bar\varphi^i\bar\varphi^j}\bar\psi^i\bar\psi^j - V(\varphi^\dagger,\varphi) + \text{total derivatives},
\end{aligned} \tag{5}$$

with $\tilde F^{\mu\nu} = \frac{1}{2}\epsilon^{\mu\nu\rho\sigma}F_{\rho\sigma}$, the scalar potential

$$V(\varphi^\dagger,\varphi) = F^\dagger F + \tfrac{1}{2}D^2 = \sum_i \left|\frac{\partial W}{\partial\varphi^i}\right|^2 + \frac{g^2}{2}\sum_a \left|\varphi^\dagger T_a\varphi\right|^2 \tag{6}$$

and covariant derivatives

$$\begin{aligned}
D_\mu\lambda &= \partial_\mu\lambda - \mathrm{i}g\left[V_\mu^b, \lambda\right], & D_\mu\varphi &= \partial_\mu\varphi - \mathrm{i}gV_\mu^a T_a\varphi, \\
D_\mu\psi &= \partial_\mu\psi - \mathrm{i}gV_\mu^a T_a\psi, & F_{\mu\nu} &= \partial_\mu V_\nu - \partial_\nu V_\mu - \mathrm{i}g\left[V_\mu, V_\nu\right].
\end{aligned} \tag{7}$$