Physikalisches Institut
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Theoretische Physik

Exercises 11
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Exercises on Theoretical Particle Physics II

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11.1 Tensor product of Matrices (*4 credits*)

For two matrices A and B of dimension $M \times N$ and $P \times Q$, the Kronecker product $A \otimes B$ is a $MP \times NQ$ matrix with elements

$$(A \otimes B)_{im,jn} = A_{ij} B_{mn} \,.$$

The indices run over $im = 11, 12, ..., NQ$ etc. For example, if A is a 2×2 matrix, the Kronecker product with B is

$$A \otimes B = \begin{pmatrix} a_{11}B & a_{12}B \\ a_{21}B & a_{22}B \end{pmatrix} .$$

Clearly, it is linear and associative. Check the following properties of the Kronecker product:

(a) Transpose and complex conjugation distribute over the Kronecker product, i.e.

$$(A \otimes B)^T = A^T \otimes B^T \,, \qquad (A \otimes B)^* = A^* \otimes B^* \,.$$

(*1 credit*)

(b) If dimensions match, matrix multiplication factorises:

$$(A \otimes B)(C \otimes D) = (AC \otimes BD)$$

(*1 credit*)

(c) If A and B are square matrices of dimensions $M \times M$ and $N \times N$, we have for the trace and determinant

$$\det(A \otimes B) = (\det A)^N \cdot (\det B)^M \,, \qquad \operatorname{tr}(A \otimes B) = \operatorname{tr} A \operatorname{tr} B \,.$$

(*2 credits*)

11.2 Gamma Matrices in various Dimensions (*8 credits*)

We can recursively define a particular representation of the Γ matrices for all dimensions in the following way: First we consider even dimensions. In $D = 2$, start with

$$\Gamma^0 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} , \qquad \Gamma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} .$$

Now assume the Γ matrices in $D-2$ dimensions to be γ^μ . In D dimensions, define

$$\Gamma^\mu = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \otimes \gamma^\mu \,, \qquad \Gamma^{D-2} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \otimes \mathbb{1} \,, \qquad \Gamma^{D-1} = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix} \otimes \mathbb{1} \,.$$

For odd dimensions, $D = 2k + 1$, we take the $2k$-dimensional Γ matrices and add Γ^* (see sheet 9).

(a) Show that this procedure defines a set of Γ matrices in any dimension. (*4 credits*)

(b) Show that, in any dimensions, Γ^0 and all odd Γ^i for $i \geq 3$ are antisymmetric, while Γ^1 and the even Γ^i are symmetric! Conclude that Γ^3 , Γ^5 ,..., Γ^9 are imaginary and the other ones are real. (*4 credits*)

11.3 Majorana Spinors in various Dimensions (*9 credits*)

Consider the Majorana condition, i.e. a reality condition on a spinor of the form $\psi^* = B\psi$. This requires

$$B\Sigma^{\mu\nu}B^{-1} = \Sigma^{\mu\nu *} \,, \qquad \text{and} \qquad BB^* = 1\,.$$

(a) Consider even dimensions first. Use the Γ matrix representation defined in the previous problem. Let

$$B = \Gamma^3\Gamma^5 \dots \Gamma^{D-1} \,, \qquad B' = \Gamma^* B\,.$$

Show that

$$B\Gamma^\mu B^{-1} = -(-)^{\frac{D}{2}}\Gamma^{\mu *} \,, \qquad B'\Gamma^\mu B'^{-1} = (-)^{\frac{D}{2}}\Gamma^{\mu *} \,,$$

so both B and B' satisfy the first condition above. (*2 credits*)

(b) For which D do they also satisfy the second one? (*3 credits*)

(c) Under which condition is the Majorana condition compatible with a chirality condition $\Gamma^*\psi = \pm\psi$? (*2 credits*)

(d) The definitions of B and B' also extend in $D+1$ dimensions. Do they both generate consistent Majorana conditions? In which dimensions? (*2 credits*)