Physikalisches Institut
Universität Bonn
Theoretische Physik

Exercise 01
17 April 2012
SS 12

Exercises on String Theory II

Prof. Dr. H.P. Nilles, Priv. Doz. Dr. S. Förste

–Home Exercises–
Due 24. April 2012

On this exercise sheet, we investigate the twisted and untwisted massless matter content of the $\mathbb{C}^3/\mathbb{Z}_3$ orbifold. In order to calculate the irreducible representations of the gauge group under which the matter states transform, group theoretical properties have to be exploited. We do this for the simplest case possible, i.e. for the standard embedding. It is intriguing that already the simplest choice of the gauge embedding gives rise to the correct chiral representations of a Standard Model GUT group.

In the second exercise, we examine the Hodge diamond of the orbifold and calculate the Euler number in two different ways.

Exercise 1.1: Spectrum of the $\mathbb{Z}_3$ orbifold in Standard Embedding ***(13 credits)***

Remember that the mass m_L of a left–mover is given by

$$\frac{m_L^2}{4} = \frac{p^2}{2} + N - 1\,, \qquad (1)$$

where $N = \sum N^i$ is the total number of oscillators N^i in the i^{th} torus, $p \in \Lambda_{\text{E}_8\times\text{E}_8}$ is the $\text{E}_8 \times \text{E}_8$ lattice–valued left–moving momentum, and the -1 arises as a normal ordering constant. For the rest of the exercise, we will deal with massless states which have $m_L = 0$.

(a) Use (1) to argue that only the roots of $\text{E}_8 \times \text{E}_8$ contribute to the massless spectrum. *(1 credit)*

On the last exercise sheet, you already calculated the roots of $\text{E}_8 \times \text{E}_8$ that correspond to the gauge fields. The rest of the roots correspond to weights of (chiral) matter states. Hence we only have to look at the $\text{E}_6 \times \text{SU}(3)$.

(b) Use the simple roots $\vec{\alpha}^{\,i}$ of $\text{E}_6\times\text{SU}(3)$ calculated on the last exercise sheet to calculate the Cartan matrices $M^{ij} = 2\frac{\vec{\alpha}^i\cdot\vec{\alpha}^j}{\vec{\alpha}^i\cdot\vec{\alpha}^i}$ of both groups. *(2 credits)*

(c) Perform the highest weight procedure for the fundamental representations of E_6 and SU(3). *(4 credits)*
Hint: Remember that the fundamental is represented by $\boxed{1\ 0\ \dots\ 0}$ *in the Dynkin basis.*

(d) Use the results of (a) and (c) to give the massless spectrum of the untwisted sector in terms of the representations of the gauge group. (*2 credits*)

This completes the spectrum analysis of the untwisted spectrum. Next, we proceed with the twisted spectrum. Here, the momenta as well as the zero point energy (i.e. the normal ordering constant) are shifted due to the orbifold action. The mass equation for the twisted sector reads

$$\frac{m_L^2}{4} = \frac{(p+t)^2}{2} + N - 1 + \delta c\,, \tag{2}$$

where

$$t = \frac{1}{3}(1,1,-2,0,0,0,0,0)(0,0,0,0,0,0,0,0) \qquad \text{and} \qquad \delta c = \frac{1}{3}\,. \tag{3}$$

It is convenient to define the shifted momenta $p_{\text{sh}} := p + t$. Furthermore, note that the N^i are quantized in multiples of $\frac{1}{3}$ now.

(e) For which combinations of p_{sh} and N can $m_L^2 = 0$ now be satisfied? Under which representations of the gauge group do the states transform? (*4 credits*)
Hint: To find the representation, calculate the Dynkin labels of the p_{sh}. *Compare them with the results from (c). Note that the spectrum is the same at all 27 fixed points.*

Exercise 1.2: Hodge diamond and Euler number **(*7 credits*)**

In order to characterize topological spaces, it is often useful to look at the Hodge numbers of the manifold. Let us describe $\mathbb{C}^3$ with the coordinates z^i, $i = 1, 2, 3$ plus their complex conjugates $\overline{z}^i$. All forms on the orbifold can be obtained from those forms of the torus which are left invariant by the orbifold action $\theta:\ (z^1, z^2, z^3) \mapsto (e^{2\pi i/3}z^1, e^{2\pi i/3}z^2, e^{4\pi i/3}z^3)$ and correspondingly for $\overline{z}^i$.

(a) Calculate all (p, q) forms that are left invariant by the orbifold action. The number of independent ones is denoted by the Hodge number $h^{(p,q)}$. (*3 credits*)

(b) In addition to the Hodge numbers found above, each fixed point contributes 1 to $h^{(1,1)}$. Calculate the Euler number $\chi = \sum_{p,q}(-1)^{p+q}h^{(p,q)}$. (*1 credit*)

(c) The Euler number can also be calculated as follows: Start with the Euler number $\chi(X)$ of $X = (T^2)^3$. Cut out the 27 fixed points P and calculate $\chi(X - P)$ (note that $X - P$ is not compact anymore, which will be repaired later). Next, divide out the orbifold action $\mathbb{Z}_3$ (which acts without fixed points on $X - P$) and calculate $\chi(\frac{X-P}{\mathbb{Z}_3})$. Now, make the space compact again by gluing in a copy of $\mathbb{CP}^2$ with $\chi(\mathbb{CP}^2) = 3$ at each fixed point. Calculate the Euler number of the final space $\chi(\widetilde{X}) = \chi(\frac{X-P}{\mathbb{Z}_3}) + 27\chi(\mathbb{CP}^2)$. Compare with the previous result. (*3 credits*)
Hint: You may use $\chi(A + B) = \chi(A) + \chi(B)$ *and* $\chi(\frac{A}{G}) = \chi(A)/|G|$ *with* $|G|$ *the cardinality of the finite group* G.