Physikalisches Institut
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Theoretische Physik

Exercise 02
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SS 12

Exercises on String Theory II

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To be discussed on 3 May 2012

In the exercise sheet we continue discussing the geometrical properties of the compactification spaces and their physical implications. In the first exercise we attempt to unambiguosly determine the chirality of the states we found for the $\mathbb{Z}_3$ orbifold. Exercise 2 is an exploration of the interplay between the number of chiral families and the presence of invariant forms in the manifold.

Exercise 2.1: Chiral States of the $\mathbb{Z}_3$ Orbifold (*12 credits*)

Whereas the left moving part of the heterotic string is responsible for the gauge group, the right moving part accounts for the target space supersymmetries. In the computation of the right moving massless spectrum only the fermions contribute. The effect of the NS and R sectors can be seen more intuitively after bosonization: The massless states are then given by weights in the vector or spinor lattices of the transverse SO(8)

$$\text{Vector:} \quad (n_1, n_2, n_3, n_4)\,, \qquad \sum_{i=1}^{4} n_i \text{ odd}$$

$$\text{Spinor:} \quad \left(n_1 + \frac{1}{2}, n_2 + \frac{1}{2}, n_3 + \frac{1}{2}, n_4 + \frac{1}{2}\right), \qquad \sum_{i=1}^{4} n_i \text{ even}$$

Recall that SO(8) is the little group for massless states. Thus, these weights allow for a simple interpetation. We can take the first entry to be the eigenvalue under the 4D helicity operator and the remaining three components to run as weights of the internal SO(6). Taking $v = (0, \frac{1}{3}, \frac{1}{3}, -\frac{2}{3})$ to be the orbifold twist, a right moving state $|q\rangle$ (with q being a weight) transforms as

$$|q\rangle \xrightarrow{\theta} e^{-2\pi i q\cdot v}\,|q\rangle\,. \tag{1}$$

For the untwisted sector the mass equation reads

$$\frac{m_R^2}{4} = \frac{q^2}{2} - \frac{1}{2} \tag{2}$$

(a) Find all massless states. Use (1) to find the invariant states. Why do they give rise to the 4D vector multiplet? (*2 credits*)

(b) For which vector lattice solutions q_v is

$$q_s = q_v + (-\tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}) \tag{3}$$

also a massless state? (*1.5 credits*)

(c) Show that for the remaining q_v you found in (a), the spinors

$$q_s = q_v + (\tfrac{1}{2}, -\tfrac{1}{2}, -\tfrac{1}{2}, -\tfrac{1}{2}) \tag{4}$$

are also massless states. Note that the prescriptions (3) and (4) serve to build up chiral multiplets. Note also that the states in (4) are the CPT conjugates of (3). (*1.5 credits*)

(d) Use the weights you found in (b) and tensor them with the left moving pieces you obtained in exercise 1.1 (d). Which tensor products are invariant under the point group action? We can take them to be the left chiral untwisted fields. (*3 credits*)

(e) Now specialize to the states of the twisted sector. The normal ordering constant as well as the weights get shifted and the mass equation reads

$$\frac{m_R^2}{4} = \frac{q_{\text{sh}}^2}{2} - \frac{1}{2} + \delta_C \tag{5}$$

where $q_{\text{sh}} = q + v$. Find the massless states of this twisted sector. According to the convention we established in (d), are these states left or right chiral? Where are their CPT conjugates? (*2 credits*)

(f) Use the previous result and your findings from exercise 1.1 (e) to write down the left chiral twisted states. (*3 credits*)

Exercise 2.2: Hodge Numbers and Chiral Families (***8 credits***)

It can be shown that the only free Hodge numbers of a Calabi-Yau threefold are $h^{1,1}$ and $h^{2,1}$.

(a) Use the results from exercise 1.2 to write the Euler characteristic of an arbitrary Calabi-Yau threefold in terms of $h^{1,1}$ and $h^{2,1}$. (*2 credits*)

(b) For the particular case of the standard embedding, the SU(3) factor in the gauge decomposition is in close relation with the decompostion induced on the spinor of SO(8)

$$\mathbf{8}_s \to \mathbf{1}_{1/2} \oplus \mathbf{1}_{-1/2} \oplus \mathbf{3}_{1/2} \oplus \bar{\mathbf{3}}_{-1/2}\,, \tag{6}$$

where the index labels the chirality. Similarly the beaking of the adjoint of E_8 is given by

$$\mathbf{248} \to (\mathbf{78}, \mathbf{1}) \oplus (\mathbf{1}, \mathbf{8}) \oplus (\mathbf{27}, \mathbf{3}) \oplus (\overline{\mathbf{27}}, \bar{\mathbf{3}})\,. \tag{7}$$

The left chiral states we found in the previous exercise transform then in the $\bar{\mathbf{3}}$ of the holonomy group and in the $(\mathbf{27}, \mathbf{3})$ of $E_6 \times SU(3)$. Forget about the E_6 indices carried by these states and concentrate on the SU(3) indices. Can you relate these states to a certain class of forms? Does the multiplicity of left chiral $\mathbf{27}$'s coincide with the Hodge numbers for this class? (*3 credits*)

(c) The number of (2,0) forms is always equal to zero. This is however not the reason for the absence of $\overline{\mathbf{27}}$'s in the spectrum. Argue why they can be related to the Hodge number $h^{2,1}$. (*3 credits*)