Physikalisches Institut
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Theoretische Physik

Exercise 03
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SS 12

Exercises on String Theory II

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To be discussed on 15 May 2012

On this exercise sheet we examine the description of Calabi-Yau manifolds via projective spaces. To this end, we discuss basic properties of $\mathbb{CP}^N$ and derive a condition that CY manifolds in these spaces have to fulfill. Then we look at some simple realizations in various dimensions.

Exercise 3.1: Basics on projective spaces (*8 credits*)

The first thing we need in our discussion are Chern classes. For a general vector bundle $\mathcal{V}$ of rank r (i.e. the fiber is an r–dimensional vector space), the total Chern class is defined in terms of the curvature 2–form $\mathcal{F}$ of the bundle $\mathcal{V}$ as

$$c(\mathcal{V}) := \det(1 + \frac{1}{2\pi}\mathcal{F}) := c_0(\mathcal{V}) + c_1(\mathcal{V}) + \ldots := 1 + \frac{1}{2\pi}\mathrm{tr}(\mathcal{F}) + \ldots \tag{1}$$

where the k^{th} Chern class c_k is a closed $2k$–form. The Chern classes are useful as they are topological invariants.

(a) For a given vector bundle of rank r over an N dimensional manifold, what is the highest non–zero Chern class? (*1 credit*)

(b) Take the special case where the vector bundle $\mathcal{V}$ is the tangent bundle of a Ricci-flat Kähler manifold X to argue that then $c_1(TX) = 0$. (*1 credit*)

Let us now specialize to the case where the Calabi–Yau (CY) X is given as the zero set of an equation in complex projective space $\mathbb{CP}^N$. $\mathbb{CP}^N$ is a space of complex dimension N with coordinates $(z_0, z_1, \ldots, z_N) \neq (0, 0, \ldots, 0)$ and an equivalence relation

$$(z_0, z_1, \ldots, z_N) \sim (\lambda z_0, \lambda z_1, \ldots, \lambda z_n)\,, \qquad \lambda \in \mathbb{C}^* := \mathbb{C}\backslash\{0\}\,. \tag{2}$$

(c) The space $\mathbb{CP}^N$ can be covered with open patches $U_\alpha := \{z_\alpha \neq 0\}$. In these patches, one uses the affine coordinates $x_{i,\alpha} := z_i/z_\alpha$. Give the transition function on the overlap of $U_\alpha \cap U_\beta$. (*1 credit*)

(d) Argue that in $\mathbb{CP}^N$ only homogeneous polynomials are well–defined. (*1 credit*)

Obviously, the equivalence relation (2) defines a line in $\mathbb{C}^{N+1}$ through the origin and the point $(z_0, z_1, \ldots, z_N)$. By fibering this line over every point in $\mathbb{CP}^N$, one obtains a line bundle, the so–called *tautological line bundle*. We denote this line bundle by $\mathcal{O}(-1)$. The dual of this line bundle, the so–called *hyperplane bundle*, defines a linear polynomial in the homogenoeus coordinates and is denoted by $\mathcal{O}(1)$. Polynomials of degree d are obtained from tensoring $\mathcal{O}(1)$ d times with itself, $\mathcal{O}(1)^d := \mathcal{O}(d)$.

(e) Using (a), we can write $c(\mathcal{O}(1)) = 1 + H$ for some closed 2–form H. Use (1) and $\mathcal{F}_{V\otimes W} = \mathcal{F}_V \oplus \mathcal{F}_W$ for line bundles to calculate $c(\mathcal{O}(d))$. (*1 credit*)

Assume now that X is given as the solution to a polynomial equation $S(z_i) = 0$ where S is homogeneous of degree d. The total Chern class of X is given in terms of the total Chern class of $\mathbb{CP}^N$ and of the total Chern class of the bundle NS normal to the hypersurface defined by S via the adjunction formula

$$c(X) = c(\mathbb{CP}^N)/c(NS)\,, \tag{3}$$

where $c(\mathbb{CP}^N) = \prod_{i=0}^{N}(1 + H)$.

(f) Use (3) to read off the first Chern class $c_1(X)$. Show that $c_1(X)$ is trivial for $d = N+1$. *Hint: Express S in terms of H and Taylor expand the denominator.* (*3 credits*)

Exercise 3.2: Calabi–Yaus as hypersurfaces in projective spaces (*12 credits*)

Let us now apply the above results to describe CYs as hypersurfaces in projective spaces. We will investigate examples of Calabi–Yau manifolds in 1,2, and 3 complex dimensions.

(a) Let us start with the CY 1–fold, i.e. a CY manifold in $d = 1$ (complex) dimensions. Using the results from Exercise 3.1, it is given as a cubic equation in $\mathbb{CP}^2$. Write down the most general cubic equation. (*1 credit*)

CYs given in this way always have one Kähler parameter, $h^{1,1} = 1$. The the number of complex structure parameters $h^{d-1,1}$ is given by the number of independent coefficients in the defining equation up to $GL(d)$ transformations.

(b) How many independent complex structure parameters are there? (*1 credit*)

(c) Use your results to calculate the Euler number. Which manifold is a CY 1–fold topologically equivalent to? (*1 credit*)

Another way to calculate the Euler number is via the integral of the highest-dimensional Chern class, $\chi = \int_X c_d$.

(d) Use (3) to calculate the top Chern class. Calculate the integral using $\int_X c_d = c_d \cdot (N+1)H$ with $H^N \equiv 1$. (*1 credit*)

(e) Next we repeat the analysis for the quartic in $\mathbb{CP}^3$, which gives a CY 2–fold, a so–called K3 manifold. Calculate the number of complex structure coefficients and subsequently the Euler number by summing the $h^{p,q}$ (note that both the Kähler and the complex structure contribute to $h^{1,1}$ here). Compare your result with the calculation of the Euler number via integration of the top Chern class. (*4 credits*)

(f) Repeat the same analysis for the quintic in $\mathbb{CP}^4$ to obtain a CY 3–fold. (*4 credits*)