Physikalisches Institut
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Theoretische Physik

Exercise 04
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SS 12

Exercises on String Theory II

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–Home Exercises–
To be discussed on 22 May 2012

In the exercise we construct the conformal group and analyze its corresponding algebra.

Exercise 4.1: The Conformal Group in $d > 2$ Dimensions ***(20 credits)***

The conformal group is defined to be the subgroup of all coordinate transformations $x \mapsto x'$ which leave the metric invariant up to an overall factor $\Omega(x)$

$$\eta_{\mu\nu}(x) \mapsto \eta_{\mu\nu}(x') = \Omega(x)\eta_{\mu\nu}(x) \tag{1}$$

(a) Consider an infinitesimal transformation $x^\mu \mapsto x^\mu + \epsilon^\mu$. Show that it belongs to the conformal group if the so called *conformal Killing vector equation* is satified:

$$\partial_\mu \epsilon_\nu - \partial_\nu \epsilon_\mu = \frac{2}{d}(\partial \cdot \epsilon)\eta_{\mu\nu}\,. \tag{2}$$

(*3 credits*)

(b) In order to find all infinitesimal conformal transfomations one has to find the most general solution to eq. (2). Useful equations can be found by acting on it with $\partial^\rho\partial^\sigma$. Take $\rho = \mu$, $\sigma = \nu$, and show that this leads to

$$\Box(\partial \cdot \epsilon) = 0\,. \tag{3}$$

(*2 credits*)

(c) Now choose only $\sigma = \nu$ to prove

$$\partial_\mu \partial_\rho (\partial \cdot \epsilon) = 0\,. \tag{4}$$

Show that this implies

$$\partial \cdot \epsilon = d(\lambda - 2b_\alpha x^\alpha)\,, \tag{5}$$

where λ and b_α are constants chosen for later convenience. (*2 credits*)

(d) Differentiate (2) with respect to x^α. Show that this conduces to the following result

$$\partial_\alpha \epsilon_\nu - \partial_\nu \epsilon_\alpha = 4(x_\alpha b_\nu - x_\nu b_\alpha) - 2w_{\alpha\nu} \tag{6}$$

where $w_{\alpha\nu}$ is a constant antisymmetric tensor. (*2 credits*)

(e) Use the previous equation together with (2) to find *(2 credits)*

$$\epsilon_\mu = a_\mu + \lambda x_m u + w_{\mu\nu} x_\nu + b_\mu x^2 - 2(b \cdot x) x_\mu \tag{7}$$

(f) One can associate a generator of the conformal algebra to each of the generators (7). The momentum operator P_μ is responsible for the translations a_μ. Similarly the generators $M_{\mu\nu}$ of the Lorentz group $SO(1, d-1)$ induce the infinitesimal boosts and rotations related to $w_{\mu\nu}$. These generators together lead to the Poincaré algebra. The remaining parameters are associated to the dilatation operator D and the generators K_μ of the special conformal tranformations

$$x \mapsto x' = \frac{x + bx^2}{1 + 2b \cdot x + b^2 x^2} \,. \tag{8}$$

Let us consider a translation $e^{-\mathrm{i}a^\mu P_\mu}$. Expanding this term and comparing it to (7), one can show that $P_\mu = \mathrm{i}\partial_\mu$ as expected. In order to account for the antisymmetry of $M_{\mu\nu}$, Lorentz transformations are written according to the convention $\Lambda = \exp\{-\frac{\mathrm{i}}{2} w_{\mu\nu} M^{\mu\nu}\}$. One can then choose

$$M_{\mu\nu} = \mathrm{i}(x_\mu \partial_\nu - x_\nu \partial_\mu) \,. \tag{9}$$

Apply similiar arguments to show that the remaining generators are of the form

$$D = \mathrm{i} x^\mu \partial_\mu \,, \quad K_\mu = \mathrm{i}\left[x^2 \partial_\mu - 2 x_\mu x^\nu \partial_\nu\right] \,. \tag{10}$$

(1 credit)

(g) Use the explicit form of K_μ and consider a special conformal transformation $e^{-\mathrm{i}b^\mu K_\mu}$ acting on a given coordinate x^σ. Show that this exponential agrees with the expansion of the denominator in (8) to first order. *(3 credits)*

(h) The commutation relations between the generators of the Poincaré group are already known

$$\begin{aligned} [M_{\mu\nu}, M_{\rho\sigma}] &= -\mathrm{i}(\eta_{\mu\rho} M_{\nu\sigma} - \eta_{\mu\sigma} M_{\nu\rho} - \eta_{\nu\rho} M_{\mu\sigma} + \eta_{\nu\sigma} M_{\mu\rho}) \,, \\ [M_{\mu\nu}, P_\rho] &= -\mathrm{i}(\eta_{\mu\rho} P_\nu - \eta_{\nu\rho} P_\mu) \,. \end{aligned} \tag{11}$$

Prove the following commutation relations: *(3 credits)*

$$[D, P_\mu] = -\mathrm{i} P_\mu \,, \qquad [D, K_\mu] = \mathrm{i} K_\mu \,, \tag{12a}$$

$$[D, M_{\mu\nu}] = 0 \,, \qquad [P_\mu, K_\nu] = 2\mathrm{i} M_{\mu\nu} - 2\mathrm{i}\eta_{\mu\nu} D \,, \tag{12b}$$

$$[M_{\mu\nu}, K_\rho] = -\mathrm{i}(\eta_{\mu\rho} K_\nu - \eta_{\nu\rho} K_\mu) \,. \tag{12c}$$

(i) Use the previous results to find an isomorphism between the conformal algebra and that of $SO(2, d)$. The fact that the conformal group in d dimensions is isomorph to the symmetry group of AdS_{d+1}, is in fact one of the key ingredients fro the AdS/CFT correspondence to work. *(2 credits)*