Physikalisches Institut
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Theoretische Physik

Exercise 09
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SS 12

Exercises on String Theory II

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–Home Exercises–
To be discussed on 05 July 2012

Exercise 9.1: Properties of the θ-function (*10 credits*)

The complete string partition function has to be invariant under modular transformations. To this end, it is built from functions such that the combined expression has this property. In this exercise we will deal with the θ-function. It is defined as

$$\theta\left[\begin{array}{c}\alpha\\ \beta\end{array}\right](z,\tau)=\sum_{n\in\mathbb{Z}} q^{\frac{1}{2}(n-\alpha)^2}e^{2\pi i(z-\beta)(n-\alpha)}\,,\quad q=e^{2\pi i\tau}\,. \tag{1}$$

Here, $z\in\mathbb{C}/\Lambda$ is the coordinate on the worldsheet torus lattice Λ spanned by 1 and τ. Furthermore, $\alpha,\beta\in\mathbb{R}$ are called the characteristics of the θ-function. This form of the θ-function is the sum representation.
Equivalently, one can express the θ-function in the product representation as

$$\theta\left[\begin{array}{c}\alpha\\ \beta\end{array}\right](z,\tau)=e^{-2\pi i\alpha(z-\beta)}q^{\frac{1}{2}\alpha^2}\prod_{n\geq 1}\left[(1-q^n)\prod_{s=\pm1}\left(1+e^{2\pi i s(z-\beta)}q^{n-\frac{1}{2}-s\alpha}\right)\right]\,. \tag{2}$$

(a) We want to show that these functions indeed agree by checking their periodicities. Show that (*5 credits*)

$$\theta\left[\begin{array}{c}\alpha\\ \beta\end{array}\right](z+1,\tau)=e^{-2\pi i\alpha}\theta\left[\begin{array}{c}\alpha\\ \beta\end{array}\right](z,\tau)\,, \tag{3}$$

$$\theta\left[\begin{array}{c}\alpha\\ \beta\end{array}\right](z+\tau,\tau)=e^{2\pi i(\beta-z-\frac{1}{2}\tau)}\theta\left[\begin{array}{c}\alpha\\ \beta\end{array}\right](z,\tau)\,. \tag{4}$$

Hint: To show (4) in the product representation, you may want to use at some point that $1+e^{-2\pi i(z-\beta)}q^{-\frac{1}{2}+\alpha}=q^{\alpha}e^{2\pi i(\beta-z-\frac{1}{2}\tau)}(1+e^{2\pi i(z-\beta)}q^{1-\frac{1}{2}-\alpha})\,.$

(b) The θ-function is also periodic in its characteristics. It has the properties

$$\theta\left[\begin{array}{c}\alpha\\ \beta\end{array}\right](z,\tau+1)=e^{-i\pi\alpha(\alpha+1)}\theta\left[\begin{array}{c}\alpha\\ \beta+\alpha+\frac{1}{2}\end{array}\right](z,\tau) \tag{5}$$

Show this using the product representation. (*2 credits*)

The behavior of the θ- function under $\tau \to -\frac{1}{\tau}$ requires more work. One finds that

$$\theta \left[\begin{array}{c} \alpha \\ \beta \end{array} \right] (\frac{z}{\tau}, -\frac{1}{\tau}) = \sqrt{-i\tau}\, e^{2\pi i (\frac{z^2}{2\tau} + \alpha\beta)} \theta \left[\begin{array}{c} \beta \\ -\alpha \end{array} \right] (z, \tau) \,. \tag{6}$$

(c) To show this, first argue that the θ-function can be written in integral form as

$$\theta \left[\begin{array}{c} \alpha \\ \beta \end{array} \right] (z, \tau) = \int_{-\infty}^{\infty} dx\, e^{2\pi i [\frac{\tau}{2}(x-\alpha)^2 + (z-\beta)(x-\alpha)]} \sum_{n\in\mathbb{Z}} \delta(x-n) \,. \tag{7}$$

Then use Poisson resummation $\sum_{n\in\mathbb{Z}} \delta(x-y-n) = \sum_{p\in\mathbb{Z}} e^{-2\pi i p(x-y)}$ and complete the square in the integral. After completing the square, use the standard Gaussian integral $\int_{-\infty}^{\infty} dx\, e^{-a(x-y)^2} = \sqrt{\frac{\pi}{a}}$ to prove (7). *(3 credits)*

Exercise 9.2: Exam review questions – not to be handed in ***(0 credits)***

Orbifolds

- What are untwisted and twisted strings?
- Wht is the shift vector and twist vector? What is shifted by the shift vector?
- Where do the contributions in the mass formula for strings on orbifolds come from?
- Why does the gauge group arise from the untwisted sector? How is the $E_8 \times E_8$ broken? Why can it result in a chiral theory?

Calabi-Yau manifolds

- What is the defining property of a Calabi-Yau manifold?
- What is the structure of the Hodge diamond? What is the meaning of the non-vanishing Hodge numbers?
- Give simple examples of Calabi-Yau manifolds written as complete intersections in d=1,2,3 complex dimensions.

CFT

- What are conformal transformations?
- What is special about the conformal group in 2 dimensions?
- What is the Virasoro algebra?
- What are conformal primary fields?
- What are conformal weights? What are the conformal weights of a free boson and a free fermion?
- What is an operator product expansion? What is the central charge?