Exercises on Theoretical Particle Astrophysics

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–Home Exercises–

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2.1 The Hierarchy Problem

20 points

Consider the Yukawa sector in the Standard Model (SM) Lagrangian

$$\mathcal{L}_{\text{SM}} \supset -y^{(e)}_{ij}\overline{L}^i H e^j_R - y^{(d)}_{ij}\overline{Q}^i H d^j_R - y^{(u)}_{ij}\overline{Q}^i \tilde{H}\overline{u}^j_R \; + \; h.c. \tag{1}$$

We have used i, j as family indices and introduced the doublet $\tilde{H} = (\mathrm{i}\sigma^2)H^*$ to construct a gauge invariant Yukawa coupling for the up-type quarks.

(a) Show that $\tilde{H}$ and H transform identically under $SU(2)_L$ and carry opposite $U(1)_Y$ charges. (*1 point*)

(b) Take the Higgs VEV to be of the form $\langle H\rangle = \frac{1}{\sqrt{2}}(0, v)^T$. This allows for the following redefinition of the excitation modes in the Higgs field

$$H = \exp\left\{\frac{\mathrm{i}}{v}\xi^a(x)T^a\right\}\begin{pmatrix} 0 \\ \frac{1}{\sqrt{2}}(v + h(x))\end{pmatrix} ,$$

in which $\xi^a(x)$ and $h(x)$ are real fields. Argue why this reparametrization is possible. What is the role of the fields $\xi^a(x)$ after spontaneous symmetry breaking (SSB)? From your answer to the previous questions you can deduce that there is the possibility to gauge away the contribution of the ξ^a's by means of an $SU(2)_L$ transformation. Use this so-called *unitary gauge* and eq. (1) to find the couplings of the fermions to the Higgs boson $h(x)$ as well as their corresponding mass terms. (*1.5 points*)

(c) The mass terms previously found are not in a diagonal form. In other words, flavor eigenstates are not mass eigenstates. The mass matrices are brought to a diagonal form at the cost of introducing the CKM matrix for the quark sector (and the PMNS matrix for the neutrino sector). Why can we simultaneously diagonalize mass matrices and Yukawa couplings $(h\overline{f}_i f_j)$? This implies that the SM Higgs is not expected to introduce any flavor changing neutral currents at tree level.

(d) Use experimental values to show that the Yukawa coupling for the top quark is very close to one. Compare it to those for the bottom quark and the electron. (*1 point*)

From the theoretical point of view this is a very puzzling result: The mass spectrum of the standard model expands over six orders of magnitude! In the light of SSB, the masses are all set by the electroweak (EW) scale ($v = 246$ GeV) and thus, the hierarchy for the masses is blamed to the pattern of Yukawa couplings. This hierarchy can be better understood by considering running of the coefficients. The renormalization group equations serve to explain the hierarchy between the electron and bottom quark Yukawas: If one assumes that these are of the same order of magnitude at a much higher energy scale, the contributions from the QCD coupling g_3 drive them to the observed values at the EW scale. Unfortunately this argument does not apply for the top quark, since the high value for its Yukawa (~ 1) cancels against g_3 making its running negligible. The top-Yukawa is then an *IR fixed point* in the standard model, but why that is the case still remains as an open question.

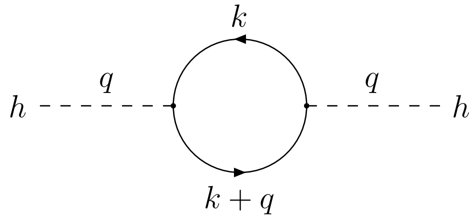

Figure 1: Fermion loop correction to the Higgs propagator

(e) Next we want to investigate the loop corrections to the Higgs propagator. Inspired by your result from (b), consider a generic Dirac fermion f with mass m and Yukawa coupling y_f. Show that the matrix element for the fermion loop depicted in fig. 1 is given by (*2 points*)

$$\mathrm{i}\Pi(q) = -\frac{y_f^2}{2}\int \frac{\mathrm{d}^4k}{(2\pi)^4}\,\mathrm{tr}\left[\frac{\not{k}+m}{(k^2-m^2+\mathrm{i}\epsilon)}\frac{\not{k}+\not{q}+m}{((k+q)^2-m^2+\mathrm{i}\epsilon)}\right]. \quad (2)$$

(Hint: Some useful Feynman rules are given at the end of the sheet.)

(d) To simplify the computations take the zero momentum limit ($q \to 0$). Recall that $k^2 = (k^0)^2 - (k^i)^2$. Therefore, the integral of eqn. (2) is over Minkowski space. It is much more convenient to perform such integrals in 4-dim Euclidean space. To do so, one has to perform a *Wick rotation*: (*3 points*)

(i) View k^0 as a complex variable. Draw the complex k^0-plane. The integration is along the real axis. Mark the position of the poles of the integrand in eqn. (2).

(ii) Use Cauchy's integral theorem to argue that the integral from $-\infty$ to $+\infty$ is equal to the integral from $-\mathrm{i}\infty$ to $+\mathrm{i}\infty$.

(iii) Define new (Euclidean) coordinates: $k^0 = \mathrm{i}n^0$ and $k^i = n^i$ and rewrite the integral in terms of k^μ. At the end, rename n^μ to k^μ.

(iv) Now we can set $\epsilon \to 0$, because there is no divergence on the path of integration.

(v) Rewrite your integral in spherical coordinates by using $d^4k = 2\pi^2 |k|^3 d|k|$. The result should read:

$$\mathrm{i}\Pi(0) = \frac{\mathrm{i}y_f^2}{4\pi^2} \int_0^{+\infty} \frac{(|k|^2 - m^2)|k|^3 d|k|}{(|k|^2 + m^2)^2}, \tag{3}$$

(e) Using the boundaries from 0 to $+\infty$, we see that the integrand diverges. We regularize the integral by an energy cutoff, i.e. we integrate from 0 to Λ. Show that the resulting matrix element takes the form

$$\Pi(0) = \frac{y_f^2}{8\pi^2} \left[\Lambda^2 - 6m^2 \log\left(\frac{\Lambda}{m}\right) + \cdots \right]. \tag{4}$$

where the dots denote finite terms. (*2 points*)

(f) Show that the first order correction to the Higgs propagator in the zero momentum limit is given by

$$-\frac{\mathrm{i}}{m_h^2}\left[1 + \frac{\Pi(0)}{m_h^2}\right].$$

Now calculate the correction to all orders (several one-loop diagrams one after another). Use the geometric series

$$\frac{1}{1-x} = 1 + x + x^2 + ...$$

to show that all fermion loops shift the Higgs mass by $\Delta m_h^2 = -\Pi(0)$. (*1 point*)

The previous result is the reason for the so called *Hierarchy problem*. On one hand, the Higgs mass parameterizes the electroweak (EW) scale, but as we could observe, this quantity receives huge quantum corrections from the particles which couple (directly or indirectly) to the Higgs field. In particular, we saw that such corrections scale quadratically with the cutoff scale Λ, so that one expects the m_h^2 to be of the order of Λ^2. On the other hand, Λ is interpreted as the energy scale at which new physics kicks in. If we take Λ to be the Planck scale, we see that the Higgs mass measured recently at the LHC ($m_h = 126$ GeV) is unnaturally small (17 orders of magnitude smaller than the expected quantum corrections).

Among all particles of the standard model, the Higgs boson is the only field which experiences this traumatic effect. All other particles receive corrections which are logarithmic in the cutoff. However, the quarks, leptons and the $W^\pm$ and Z bosons obtain their masses from $\langle H \rangle \sim m_h$, so that the entire mass spectrum is directly or indirectly sensitive to the unnatural cutoff effects.

(g) Many solutions have been proposed to stabilize the EW scale against radiative corrections and here we illustrate the supersymmetric one. Consider a complex scalar S with the following mass and interaction terms

$$\mathcal{L} \supset m_S^2 |S|^2 - \sqrt{2} y_S^{(1)} m_S h |S|^2 - \frac{1}{2} y_S^{(2)} h^2 |S|^2$$

Draw the loop contributions of S to the Higgs propagator and write down the corresponding matrix elements (at zero momentum) *(2.5 points)*

$$\mathrm{i}\Pi_S(0) = \mathrm{i}\Pi_S^{(1)}(0) + \mathrm{i}\Pi_S^{(2)}(0)\,.$$

(h) Use the techniques developed in (d) to arrive at the following result *(3.5 points)*

$$\Pi_S(0) = -\frac{1}{16\pi^2}\left[y_S^{(2)}\Lambda^2 - m_S^2(2y_S^{(2)} + 4(y_S^{(1)})^2)\log\left(\frac{\Lambda}{m_S}\right) + \cdots\right]. \tag{5}$$

(i) Note the very similar structure of eqs. (4) and (5). From these it follows that a certain systematic cancelation is possible, provided a certain relation between y_f and $y_S^{(2)}$. However, if this cancelation is left at the level of the parameters, one faces the problem of fine tuning. A more convincing solution is to propose a symmetry which protects this relation. The symmetry which suits for this purpose is known as supersymmetry (SUSY), it postulates that for each Weyl fermion in the spectrum there is a complex scalar which carries the same quantum numbers. Take two scalars S_L and S_R to be the superpartners of the fermion f introduced in (e). Why does one need two scalars? Assume that the scalars have the same masses and coupling strengths. SUSY predicts the following relations

$$y_S^{(2)} = (y_f)^2 = (y_S^{(2)})^2\,, \tag{6}$$

$$m_S^2 = m^2\,, \tag{7}$$

use them to show that Δm_h^2 now scales logarithmically with Λ. *(1 point)*

(j) At the energy scales probed by colliders no superpartners have been observed so far, this means that if SUSY plays any role in nature it must be broken at a higher scale. When SUSY is *softly* broken eq. (6) still holds, ensuring that the quadratic divergence is avoided. Nevertheless, the masses of the scalar superpartners are expected to be larger than those of the fermions: $m_S^2 = m^2 + \Delta m_{SUSY}^2$. Write down the radiative correction to the Higgs mass in the soft SUSY breaking scenario. Can the SUSY breaking scale Δm_{SUSY}^2 reintroduce the hierarchy problem? *(1.5 points)*

Feynman Propagator of Fermions with Momentum q	$\mathrm{i}\frac{\not{q}+m}{q^2-m^2+\mathrm{i}\epsilon}$
Feynman Propagator of Bosons with Momentum q	$\mathrm{i}\frac{1}{q^2-m^2+\mathrm{i}\epsilon}$
Loop momentum k	$\int\frac{\mathrm{d}^4k}{(2\pi)^4}$
Fermion loop	$\cdot(-1)$
Boson loop	$\cdot(+1)$

Table 1: Some Feynman rules relevant for this exercise