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Theoretische Physik

Exercise 3
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SS 13

Exercises on Theoretical Particle Astrophysics

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Due 3rd May

In this exercise we will continue the study of Lie algebras and Lie groups, which we have started in the first sheet. For this purpose we start with an explicit example (the Lie algebra of $SU(3)$) and then we introduce the machinery required to study the representations of $\mathfrak{su}(N)$ in a general fashion.

H 3.1 The $\mathfrak{su}(3)$ Algebra *10 points*

Consider the Gell-Mann matrices

$$T^1 = \frac{1}{2}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\,, \quad T^2 = \frac{1}{2}\begin{pmatrix} 0 & -\mathrm{i} & 0 \\ \mathrm{i} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\,, \quad T^3 = \frac{1}{2}\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}\,, \quad T^4 = \frac{1}{2}\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}\,,$$

$$T^5 = \frac{1}{2}\begin{pmatrix} 0 & 0 & -\mathrm{i} \\ 0 & 0 & 0 \\ \mathrm{i} & 0 & 0 \end{pmatrix}\,, \quad T^6 = \frac{1}{2}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}\,, \quad T^7 = \frac{1}{2}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -\mathrm{i} \\ 0 & \mathrm{i} & 0 \end{pmatrix}\,, \quad T^8 = \frac{1}{2\sqrt{3}}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}\,.$$

(a) Show that they serve as a basis for the $\mathfrak{su}(3)$ algebra. Evaluate the commutators of these matrices to determine the structure constants f^{abc}. Show that, with the normalization used here, f^{abc} is totally antisymmetric. *Hint: You may wish to check only a representative sample of the commutators.* (*4 points*)

(b) Why is $\{T^3, T^8\}$ a good choice for the Cartan subalgebra? Show that the (complex) basis transformation

$$T_\pm = T^1 \pm \mathrm{i}T^2\,, \qquad U_\pm = T^4 \pm \mathrm{i}T^5\,, \qquad V_\pm = T^6 \pm \mathrm{i}T^7\,,$$

diagonalizes both f^{3ab} and f^{8ab}. (*3 points*)

(c) Take the eigenvalues of a given element with the Cartan-generators and write them as two component vectors. Draw these vectors in a coordinate system. Name a physical example where you know this pattern from? (*2 points*)

(d) Use the previous diagram to identify the positive and simple roots and show that they give you the expected Cartan matrix. *1 points*

H 3.2 Representations of $\mathfrak{su}(N)$ *10 points*

In this exercise we will now have a closer look at representations of SU(N) groups.

(a) Recall the definition of the adjoint $\text{ad}\, a(b) := [a, b]$.
Show that the adjoint is a representation of the Lie algebra

$$\text{ad}\big([a,b]\big) = [\text{ad}\, a,\, \text{ad}\, b]\ , \quad \text{for } a, b \in \mathfrak{g}\, .$$

(*1 point*)

PLEASE NOTE!

♣ The bracket $[\cdot\,,\cdot]$ on the left-hand side denotes the abstract Lie-bracket, but on the right-hand side it denotes the commutator.

♣ The adjoint representation ad of a Lie algebra $\mathfrak{g}$ on a vector space V is a linear mapping ad : $\mathfrak{g} \to \text{End}(V)$, where V is equal to the Lie algebra itself, i .e. $V = \mathfrak{g}$ This means that when we computed the Dynkin diagram of SU(N), we implicitly used the adjoint representation of SU(N):

$$\text{ad}\, h(e_{ab}) = [h, e_{ab}]\, . \tag{1}$$

Furthermore, we had the eigenvalue equation

$$\text{ad}\, h(e_{ab}) = \alpha_{e_{ab}}(h)\, e_{ab}\, , \tag{2}$$

which defined the roots $\alpha_{e_{ab}}$.
This eigenvalue equation can now be generalized to non-adjoint representations ρ on some vector space V. Let ϕ^i be a basis of V. We denote the representations of the elements of the Cartan subalgebra $h \in H$ by $\rho(h)$ and the representations of the step operators e_α by $\rho(e_\alpha)$. Then eq. (2) reads: $\rho(h)\,\phi^i = M^i(h)\,\phi^i$. Since the linear functions M^i act on elements $h \in H$ and give (real) numbers, they are elements of the dual space H^*. They are called **weights**. The corresponding vectors ϕ^i are called **weight vectors**. Note that roots are the weights of the adjoint representation!
You may have already gotten that simple roots α_j span H^*, so it is possible to reexpress the weights by simple roots $M^i = \sum_j c_{ij}\alpha_j$, where the coefficients c_{ij} are in general non-integers. A weights M^i is called **positive**, if the first non-zero coefficients is positive. We write $M^i > M^j$, if $M^i - M^j > 0$.
A weight is called the **highest weight**, denoted by Λ, if $\Lambda > M^i\ \forall M^i \neq \Lambda$

(b) Suppose that ϕ^i is a weight vector with weight M^i. Show that $\rho(e_\alpha)\phi^i$ is a weight vector with weight $M^i + \alpha$ unless $\rho(e_\alpha)\phi^i = 0$.
Hint Use eqs. (1) and (2) and the fact that ρ is a representation. Thus it makes sense to think of the $\rho(e_\alpha)$ as raising operators and the $\rho(e_{-\alpha})$ as lowering operators. (*1 point*)

(c) Consider now a representation ρ of SU(N). We denote the generators $\rho(t_a)$. For elements of the Cartan subalgebra, we may also write $\rho(h)$. Follow from

$$[\rho(t_a), \rho(t_b)] = \mathrm{i}\, f_{abc}\, \rho(t_c)\,,$$

that $-\rho(t_a)^*$ forms a representation, called the *complex conjugate* of ρ. We denote it by $\overline{\rho}$. ρ is said to be a real representation if it is equivalent to its complex conjugate. (*1 point*)

(d) Show that if M^i is a weight in ρ, $-M^i$ is a weight in $\overline{\rho}$.
Hint: Use the fact that Cartan generators are hermitean (*1 point*)

Now we are well equipped to construct the representations. For a finite dimensional representation we will find a state with highest weight Λ, which is annihilated by all positive root operators. Then we can get all states by acting with the lowering operators on it. In order to do this, we present the weights by the Dynkin labels

$$m_i := \frac{2\langle M, \alpha_i\rangle}{\langle \alpha_i, \alpha_i\rangle}\,.$$

where M denotes a weight. The dynkin labels always consist of integer numbers which for a highest weight state are non-negative. It is easy to see that acting with $E_{-\alpha_i}$ corresponds to substracting the ^{i}th row of the Cartan matrix from the Dynkin label. Now you can construct all irreducible representations via the following procedure:

- ♢ start with the Dynkin label m with non-negative entries, representing the highest weight state
- ♢ if the ^{i}th entry of the Dynkin label m_i is positive, you can get m_i new states by substracting m_i times the ^{i}th row of the Cartan matrix
- ♢ repeat the last step for all new steps, for $i = 1 \ldots r$
- ♢ at the end you should arrive at the lowest weight state with only non-positive entries in the Dynkin label.

Lets now get more concrete and turn to an example:

(e) Construct the **5** and the **10** of $\mathfrak{su}(5)$ with the highest Dynkin labels $(1, 0, 0, 0)$ and $(0, 1, 0, 0)$. What are the higest Dynkin labels of the $\overline{\mathbf{5}}$ and the $\overline{\mathbf{10}}$? Also, construct the adjoint, the **24**, from the Dynkin label $(1, 0, 0, 1)$. How can you see that it is real? (*6 points*)

References

[1] J. Fuchs and C. Schweigert., *Symmetries, Lie Algebras and Representations.* Cambridge University Press, 2003.

[2] H. Georgi., *Lie Algebras in Particle Physics: From Isospin to Unified Theories*, Westview Press, 1999.

[3] C. Luedeling., *Group Theory for Physicists*, Lecture Notes, Bonn, SS 2010. `http://www.th.physik.uni-bonn.de/nilles/people/luedeling/grouptheory/data/grouptheorynotes.pdf`