Physikalisches Institut
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Theoretische Physik

Exercise 4
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SS 13

Exercises on Theoretical Particle Astrophysics

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Due 10th May

4.1 Majorana Fermions *5 points*

We write a four component Dirac spinor in the chiral representation as a composition of two Weyl spinors

$$\Psi = \begin{pmatrix} \psi_\mathrm{L} \\ \psi_\mathrm{R} \end{pmatrix} .$$

A *Majorana spinor* is a Dirac spinor Ψ with the following constraint

$$\Psi^c := C\overline{\Psi}^T = \Psi \,, \tag{1}$$

where $C = \mathrm{i}\gamma^2\gamma^0$ is the charge conjugation operator.

(a) Show that $(\Psi^c)^c = \Psi$. What does eq. (1) imply for ψ_L and ψ_R and what is the physical meaning of this condition? (*1.5 points*)

(b) The Lagrangian $\mathscr{L}_\mathrm{D}$ for a Dirac spinor has the form

$$\mathscr{L}_\mathrm{D} = \overline{\Psi}\,(\mathrm{i}\gamma^\mu\partial_\mu)\,\Psi - m\overline{\Psi}\Psi \,,$$

where the second term is called the *Dirac mass term*. Rewrite $\mathscr{L}_\mathrm{D}$ in ψ_L and ψ_R. (*1 point*)

(c) Using the result of (b) rewrite the action $\mathscr{L}_\mathrm{M}$ for a Majorana spinor in terms of $\psi_{\mathrm{L/R}}$

$$\mathscr{L}_\mathrm{M} = \overline{\Psi}\,(\mathrm{i}\gamma^\mu\partial_\mu)\,\Psi - \frac{m}{2}\overline{\Psi}\Psi \,.$$

The second term is called the *Majorana mass term*. Why is the factor 1/2 included in the mass term? (*1 point*)

(d) Remember the projectors $P_{\mathrm{L/R}} = 1/2(\mathbb{1} \mp \gamma^5)$. As you know $P_{\mathrm{L/R}}$ project Ψ onto the left/right handed part, respectively. We denote $\Psi_{\mathrm{L/R}} := P_{\mathrm{L/R}}\Psi$. Show that

$$\left(\Psi_{\mathrm{L/R}}\right)^c = (\Psi^c)_{\mathrm{R/L}}$$
$$\overline{\left(\Psi_{\mathrm{L/R}}\right)^c}\left(\Psi_{\mathrm{R/L}}\right)^c = \overline{\Psi_{\mathrm{R/L}}}\Psi_{\mathrm{L/R}}$$

(1.5 points)

4.2 $SO(10)$ GUTs: $U(1)_{\mathrm{B-L}}$ and the See-Saw Mechanism *15 points*

Here we revise some of the most outstanding features of $SO(10)$ grand unified theories.

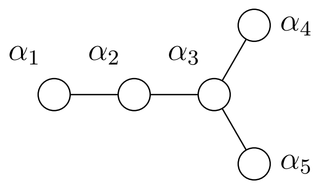

Figure 1: The Dynkin diagram of $\mathfrak{so}(10)$.

(a) Use the Dynkin diagram of $SO(10)$ to infer its corresponding Cartan matrix A_{ij}. The highest weight of the fundamental (**10**) and the spinor (**16**) representation are $(1,0,0,0,0)$ and $(0,0,0,0,1)$. Apply the highest weight procedure to construct these representations explicitly. Are they real? If not, give the highest weight of the complex conjugate representation. *(1,5 points)*

To study the breakdown of $SO(10)$, let us introduce the **Dynkin symmetry breaking** procedure: To each simple root one assigns an integer number, called the **Kač label** a_i. They are given as the coefficients of the decompositions of the highest root in the basis of simple roots. Deleting any node with Kač label $a_i = 1$ from the Dynkin diagram gives a maximal regular subalgebra times a $U(1)$ factor.

(b) The Dynkin label for the highest root is $(0,1,0,0,0)$. Write the highest root as a combination of simple roots. Use this to give all possible alternatives for the Dynkin symmetry breaking of $SO(10)$. *(1 point)*

(c) We specialize to the breaking $SO(10) \to SU(5) \times U(1)_X$, and hence we delete the node α_4 from the Dynkin diagram. The generator of $U(1)_X$ must anihilate all roots of $\mathfrak{su}(5)$. Show that

$$\alpha_x = 2\alpha_1 + 4\alpha_2 + 6\alpha_3 + 5\alpha_4 + 3\alpha_5\,,$$

fulfills these conditions. Note that α_i $i = 1,2,3,5$ together with α_X form a Cartan subalgebra for $\mathfrak{so}(10)$, hence any weight M is an eigenvector of α_X, and its eigenvalue is the $U(1)_X$ charge. How does α_X act on M? *(2 points)*

(d) Both the **10** and the **16** are reducible representations of $SU(5)$. To identify their corresponding branching rules we can use the highest weight constructions obtained in (a). Erase all arrows involving α_4 and identify the disconnected pieces. To read out the representation for each of the pieces, transform each Dynkin label m of $\mathfrak{so}(10)$ into an $\mathfrak{su}(5)$ one by deleting the fourth entry

$$m = (m_1, m_2, m_3, m_4, m_5) \mapsto (m_1, m_2, m_3, m_5)\,.$$

Use your results from the previous exercise sheet, together with the generator α_X to arrive at the following branchings

$$\begin{aligned} \mathbf{10} &= \mathbf{5}_2 \oplus \overline{\mathbf{5}}_{-2}\,, \\ \mathbf{16} &= \mathbf{1}_{-5} \oplus \overline{\mathbf{5}}_3 \oplus \mathbf{10}_{-1}\,, \end{aligned} \tag{2}$$

where the subindices label the charges under $U(1)_X$. *(1.5 points)*

As you already know, $SU(5)$ unifies all gauge interactions into a single, semi-simple Lie group. The irreducible representations $\mathbf{10} \oplus \overline{\mathbf{5}}$ furnish a complete family as one can see from the branchings

$$\begin{aligned} \mathbf{10} &\to \overbrace{(\mathbf{3},\mathbf{2})_{1/6}}^{Q_\mathrm{L}} \oplus \overbrace{(\overline{\mathbf{3}},\mathbf{1})_{-2/3}}^{\overline{u_\mathrm{R}}} \oplus \overbrace{(\mathbf{1},\mathbf{1})_{1}}^{\overline{e_\mathrm{R}}}\,, \\ \overline{\mathbf{5}} &\to \underbrace{(\overline{\mathbf{3}},\mathbf{1})_{1/3}}_{\overline{d_\mathrm{R}}} \oplus \underbrace{(\mathbf{1},\mathbf{2})_{-1/2}}_{L_\mathrm{L}}\,, \end{aligned} \tag{3}$$

in which the hypercharges have been put as subindices. As you can see from eq. (2), we can combine these representations, together with a right handed neutrino $\overline{\nu_\mathrm{R}}$ ($\mathbf{1}_{-5}$) to form the **16**-plet of $SO(10)$.

(e) Use the branchings (2) and (3). Find $U(1)_{\mathrm{B-L}}$ as a linear combination of $U(1)_X$ and $U(1)_Y$. *(Hint: All quarks have* $\mathrm{B-L}$ *charge* $1/3$*, whereas leptons have charge* -1*.)* *(1.5 points)*

(f) Physically speaking the breakdown of $SO(10)$ proceeds similarly as the standard Higgs mechanism. So we need a scalar field S to develop a VEV. We ensure that $SU(5) \subset SO(10)$ remains intact by assigning the VEV to an $SU(5)$ singlet of S. Note that the **16** contains a singlet of $SU(5)$, the same happens for the **45** and **126**

$$\begin{aligned} \mathbf{45} &= \mathbf{1}_0 \oplus \mathbf{10}_{-4} \oplus \overline{\mathbf{10}}_4 \oplus \mathbf{24}_0\,, \\ \mathbf{126} &= \mathbf{1}_{-10} \oplus \overline{\mathbf{5}}_{-2} \oplus \mathbf{10}_{-6} \oplus \overline{\mathbf{15}}_6 \oplus \mathbf{45}_2 \oplus \overline{\mathbf{50}}_{-2}\,. \end{aligned} \tag{4}$$

Under which of these representations can S transform so that its VEV does not break $U(1)_{\mathrm{B-L}}$? *(0.5 points)*

(g) Show that by taking S to transform as the **126**, $U(1)_{\mathrm{B-L}}$ gets broken to a $\mathbb{Z}_2$ discrete subgroup. *(1.5 points)*

Finally we consider this breaking scheme to study neutrino masses. We denote by s the $SU(5)$ singlet of the **126** and assume the standard model Higgs scalar to descend from the **10**-plet of $SO(10)$

$$H = (\mathbf{1}, \mathbf{2})_{1/2} \subset \mathbf{5}_2 \subset \mathbf{10}\,.$$

(h) What is the $U(1)_X$ charge for H? Verify that the couplings

$$\overline{L_\mathrm{L}}\, \tilde{H}\, \nu_\mathrm{R} \subset \overline{\mathbf{16}} \otimes \mathbf{10} \otimes \overline{\mathbf{16}} \quad \text{and} \quad s^* \overline{(\nu_\mathrm{R})^c}\, \nu_\mathrm{R} \subset \overline{\mathbf{126}} \otimes \mathbf{16} \otimes \mathbf{16}$$

are gauge invariant (recall that $\tilde{H} = \mathrm{i}\sigma^2 H^*$). As these couplings are allowed in the lagrangian, denote their coupling strengths by y_ν and y_s, respectively. *(0.5 points)*

(i) Give the corresponding VEVs to H and s and show that the previous couplings lead to the following *Dirac-Majorana mass term for the neutrinos*

$$\mathscr{L}_\mathrm{DM} = -\frac{1}{2}\left[m_\mathrm{D}(\overline{\nu_\mathrm{L}}\, \nu_\mathrm{R} + \overline{\nu_\mathrm{R}}\, \nu_\mathrm{L}) + m_\mathrm{L}\overline{\nu_\mathrm{L}}\, (\nu_\mathrm{L})^c + m_\mathrm{R}\overline{(\nu_\mathrm{R})^c}\, \nu_\mathrm{R}\right] \tag{5}$$

Write m_L, m_R and m_D in terms of y_ν, y_s, $\langle s \rangle$ and v (the Higgs VEV). Use these results to explain why m_L must be zero, m_D is of the order of the electroweak symmetry breaking scale $M_W \sim 100\,\mathrm{GeV}$ and $m_\mathrm{R} \sim M_\mathrm{GUT} \sim 10^{16}\,\mathrm{GeV}$. *(1.5 points)*

(j) Show that eq. (5) can be written in matrix form as

$$\mathscr{L}_\mathrm{DM} = -\frac{1}{2}\begin{pmatrix}\overline{\nu_\mathrm{L}} & \overline{(\nu_\mathrm{R})^c}\end{pmatrix} \mathcal{M} \begin{pmatrix}(\nu_\mathrm{L})^c \\ \nu_\mathrm{R}\end{pmatrix}$$

with

$$\mathcal{M} = \begin{pmatrix} m_\mathrm{L} & m_\mathrm{D} \\ m_\mathrm{D} & m_\mathrm{R}\end{pmatrix}$$

being the *neutrino mass matrix. Hint: Use your results from exercise 4.1* *(1 point)*

(k) In this setup, diagonalize $\mathcal{M}$ using an orthogonal matrix A

$$A^T \mathcal{M} A = \mathrm{diag}(m_1, m_2).$$

Show that to the first non-vanishing order in the (small) parameter $\rho := m_\mathrm{D}/m_\mathrm{R}$ that the eigenvalues are $m_1 = -m_\mathrm{D}^2/m_\mathrm{R}$ and $m_2 = m_\mathrm{R}$. Find the rotation matrix A to the first order in ρ for the diagonalization. What does $\rho \ll 1$ imply for the mass eigenstates? Insert the estimations done in (i) and compare the mass of the light neutrino to actual experimental bounds. *(2.5 points)*

We see that by making one mass heavy the other one becomes very light. For this reason setups of this kind are generically referred to as *See-Saw mechanism.*