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Theoretische Physik | SS 13

Exercises on Theoretical Particle Astrophysics

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Due 17th April

5.1 Chiral Anomalies *à la* Fujikawa *12 points*

Consider the standard QCD lagrangian for a massless quark

$$\mathcal{L} = -\frac{1}{2}\mathrm{tr}[F_{\mu\nu}F^{\mu\nu}] + \overline{q}\mathrm{i}\not{D}q\,, \tag{1}$$

with $\not{D} = \gamma^\mu D_\mu = \gamma^\mu(\partial_\mu + \mathrm{i}gA_\mu)$, $A_\mu = A^a_\mu T^a$ and $F_{\mu\nu} = F^a_{\mu\nu}T^a$, where

$$F^a_{\mu\nu} = (\partial_\mu A^a_\nu - \partial_\nu A^a_\mu) - g f^{abc} A^b_\mu A^c_\nu\,, \tag{2}$$

and T^a being the generators of $SU(3)$, for which the Lie-bracket reads $[T^a, T^b] = \mathrm{i}f^{abc}T^c$.

(a) Show that a chiral transformation $q \to \exp\{\mathrm{i}\alpha\gamma^5/2\}q$ leaves the lagrangian (1) unchanged. (*1 point*)

(b) Now let us consider the more general case in which the chiral transformation is local, i.e. $\alpha = \alpha(x^\mu)$. Show that in this case (*1 point*)

$$\mathcal{L} \to \mathcal{L} + \alpha(x^\mu)\partial_\mu j^\mu_5\,, \quad \text{with} \quad j^\mu_5 = \overline{q}\gamma^\mu\gamma_5 q\,.$$

Now we consider the path integral

$$\mathcal{Z} = \int \mathcal{D}A^a_\mu\, \mathcal{D}q\, \mathcal{D}\overline{q}\, \exp\left\{\mathrm{i}\int \mathrm{d}^4x\mathcal{L}\right\}\,.$$

(c) Argue that the path integral measure $\mathcal{D}q\,\mathcal{D}\overline{q}$ is not invariant under chiral transformations. (*1 point*)

To study this in more detail we expand q and $\overline{q}$ in the basis of eigenstates of the Dirac operator

$$q = \sum_n a_n\phi_n(x^\mu)\,, \quad \overline{q} = \sum_n \hat{a}_n\hat{\phi}_n(x^\mu)\,,$$

with

$$(\mathrm{i}\not{D})\phi_m = \lambda_m \phi_m \,, \quad \hat{\phi}_m(\mathrm{i}\not{D}) = \lambda_m \hat{\phi}_m$$

and $a_n, \hat{a}_n$ being Grassmann coefficients. With this we can write the path integral measure as

$$\mathcal{D}q\,\mathcal{D}\bar{q} = \prod_m \mathrm{d}a_m \mathrm{d}\hat{a}_m \,. \tag{3}$$

(d) Show that chiral transformations act on the Grassmann coefficients according to (*1 point*)

$$a_m \to \sum_n \left(\delta_{n,m} + \frac{1}{2} \underbrace{\int \mathrm{d}^4 x i\alpha(x)\hat{\phi}_m \gamma^5 \phi_n}_{C_{mn}} + \cdots \right) a_n \tag{4}$$

Hint: Use the orthogonality relations $\int \mathrm{d}^4 x \hat{\phi}_n \phi_m = \delta_{n,m}$.

(e) Use the previous result together with (3) to show that for an *infintesimal* chiral trafo one has (*1 point*)

$$\mathcal{D}q\,\mathcal{D}\bar{q} \to \exp\{-\mathrm{tr}C\}\mathcal{D}q\,\mathcal{D}\bar{q} \,. \tag{5}$$

Hint: $\det A = e^{\mathrm{tr}(\log A)}$.

Let us examine our result in more detail, the trace term contains a γ_5 which might suggest that it vanishes, however we are taking an infinite sum. To properly study its behavior we introduce a regulator

$$\sum_n \bar{\phi}_n \gamma^5 \phi_n = \lim_{M\to\infty} \sum_n \bar{\phi}_n \gamma^5 \phi_n e^{\lambda_n^2/M^2} . \tag{6}$$

(f) Prove that

$$\lambda_m^2 = (i\not{D})^2 = -D^2 + \frac{g}{2}\sigma^{\mu\nu}F_{\mu\nu} \,,$$

with $\sigma^{\mu\nu} = \frac{\mathrm{i}}{2}[\gamma^\mu, \gamma^\nu]$. (*2 points*)

Since we are taking the limit $M \to \infty$ only small wavelengths contributes, this justifies an expansion of the exponential in terms of $\sigma \cdot F$. Tracing over Dirac indices and ignoring the background A_μ we obtain

$$\sum_n \bar{\phi}_n \gamma^5 \phi_n = \lim_{M\to\infty} \mathrm{tr}\left[\gamma^5 \frac{1}{2}\left(\frac{g}{2M^2}\sigma^{\mu\nu}F_{\mu\nu}\right)^2\right] \langle x| e^{-\frac{\partial^2}{M^2}} |x\rangle \,.$$

at leading order.

(g) Why does the linear term in $\sigma \cdot F$ vanish? (*0,5 points*)

(h) Evaluate the expression $\langle x| e^{-\frac{\partial^2}{M^2}} |x\rangle$ as an integral in momentum space. (*2 points*)

(i) Take the trace to finally show that a chiral transformation in the path integral can be reabsorbed in the exponential i.e.

$$\mathcal{Z} \to \int \mathcal{D}A^a_\mu \, \mathcal{D}q \, \mathcal{D}\overline{q} \, \exp\left\{ \mathrm{i} \int \mathrm{d}^4x \left[\mathcal{L} + \alpha(x^\mu) \left(\partial_\mu j^\mu_5 + \frac{g^2}{32\pi^2} \mathrm{tr}[F_{\mu\nu}\tilde{F}^{\mu\nu}] \right) \right] \right\} \tag{7}$$

where $\tilde{F}^{\mu\nu} = \epsilon^{\mu\nu\rho\sigma} F_{\rho\sigma}$. *(2.5 points)*

This functional derivation is due to Fujikawa, and leads straighforwardly to the result

$$\partial_\mu j^\mu_5 = -\frac{g^2}{32\pi^2} \mathrm{tr}[F_{\mu\nu}\tilde{F}^{\mu\nu}] \,, \tag{8}$$

The behavior we have just studied is characteristic of an anomaly. Such anomalies have many important consequences in physics. For our purposes, this machinery has been developed to deal with a very special aspect of the axial anomaly: the appearance of an additional parameter in the Standard Model and its associated *strong CP problem.*

5.2 The Strong CP Problem *8 points*

It is perfectly valid to add the following term to the Lagrangian (1)

$$\mathcal{L}_\theta = \frac{g^2\theta}{32\pi^2} \mathrm{tr}[F_{\mu\nu}\tilde{F}^{\mu\nu}] \,, \tag{9}$$

(a) Show that this term is P and CP odd. *(1 point)*

(b) Show that a *global* chiral transformation can be used to gauge $\mathcal{L}_\theta$ away from $\mathcal{Z}$. *(1 point)*

(c) From the previous computation we see that the θ-term is non physical. Now we want to show that this is no longer true for the case of massive quarks. To see that we introduce a mass term for q

$$\mathcal{L}_m = -\frac{m}{2} \, \overline{q}(1+\gamma_5)q - \frac{m^*}{2} \, \overline{q}(1-\gamma_5)q \,,$$

This might look strange but it is, in fact, the most general mass term one can write down for a Dirac fermion. For which values of m is this term P and CP invariant? *(2 points)*

(d) Use a chiral transformation to show that the mass can be made real at the cost of shifting the θ parameter: *(1 point)*

$$\theta \to \overline{\theta} = \theta - \mathrm{Arg}(m) \,.$$

The new parameter $\overline{\theta}$ is then the physical measure of CP violation in QCD. As a C conserving operator it contributes to electric dipole moment of the neutron. Surprisingly, current experiments require $\overline{\theta} \lesssim 10^{-9}$. The smallness of theta is usually referred to as *the strong CP problem.*

(e) There is still a puzzle to consider. To see this show that $\mathcal{L}_\theta$ can be written as a total derivative, i.e.

$$\mathrm{tr}[F_{\mu\nu}\tilde{F}^{\mu\nu}] = \partial^\mu K_\mu\,, \quad K_\mu = -\epsilon^{\mu\nu\rho\sigma}\left(A^a_\nu F^a_{\rho\sigma} - \frac{2}{3} g f^{abc} A^a_\nu A^b_\rho A^c_\sigma\right).$$

Hint: Use $\mathrm{tr}(T^a T^b) = -\delta^{ab}/2$ (*1 point*)

From the previous observations one could naively think that $\mathcal{L}_\theta$ can be integrated out as a boundary term. However 't Hooft has shown that the resulting surface term can not be neglected. This is because QCD allows for *instantons* i.e. solutions to the equations of motion which have a finite action and give a non vanishing surface integral.

Now we want to explore the *axion* solution to the strong CP problem

(f) Instead of working with the total divergence term. Shift $\overline{\theta}$ to the mass sector to find the following sort of "potential" for $\overline{\theta}$ (*1 point*)

$$V(\overline{\theta}) = |m|\left[\overline{q}q\,\cos\overline{\theta} + \mathrm{i}\,\overline{q}\gamma_5 q\,\sin\overline{\theta}\right]$$

(g) We can take this potential as a perturbation of the vacuum energy. Remember that in QCD the quark condensate has a non zero vacuum expectation value, in terms of effective pion operators we can write

$$\overline{q}q = \langle\overline{q}q\rangle\cos(\pi^0/f_\pi)\,, \quad \overline{q}\gamma_5 q = \mathrm{i}\langle\overline{q}q\rangle\sin(\pi^0/f_\pi)\,.$$

with f_π being the Pion decay constant. Use these results to evaluate $\langle V(\overline{\theta})\rangle$ and find the minima of this potential. *Hint:* $\langle\overline{q}q\rangle = -250$ *MeV.* (*1 point*)

The previous result shows that the vacuum energy gets minimized at a point where CP is conserved, as expected, since points of enhanced symmetry are always stationary points. This observation, however is not very useful since $\overline{\theta}$ is not a dynamical parameter. One can introduce a new particle, namely a Goldstone boson a of a $U(1)_{\mathrm{PQ}}$ global symmetry (which is broken at a higher scale M). Then one can prove that a has a coupling of the form

$$\mathcal{L}_a = \frac{g^2}{32\pi^2}\frac{a}{M}\mathrm{tr}[F_{\mu\nu}\tilde{F}^{\mu\nu}]\,, \tag{10}$$

so that a shift in the axion $a \to a - M\overline{\theta}$ leaves us with no theta term in the lagrangian.