Physikalisches Institut
Universität Bonn
Theoretische Physik

Exercise 9
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SS 13

Exercises on Theoretical Particle Astrophysics

Prof. Dr. Hans Peter Nilles

http://www.th.physik.uni-bonn.de/nilles/

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Due 21th June

9.1 Dark Matter in Galaxies *8 points*

The observation of the galactic rotation curve[1] 1 yields a deficit of mass in the galaxy. Under the assumption of spherical symmetry of a rotating galaxy one can calculate the mass inside a sphere of a given radius from the circular velocity of the stars at its surface and compare it to an estimation from the visible stars.

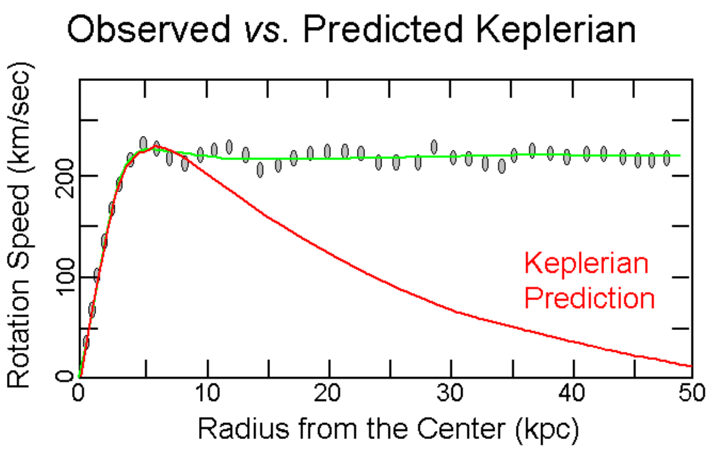

Figure 1: rotation curve of a galaxy

(a) Give a formula which expresses the circular velocity in terms of the enclosed mass and the distance to the galactic center. Verify the virial theorem for gravitationally bound systems $< T >= - < V > /2$. (*2 points*)

(b) Assume the simplest case of a constant mass density ρ_0 inside a radius r_0. How does the rotation curve look like? (*1 point*)

[1]Image taken from $http://www.astronomy.ohio-state.edu/~thompson/162/Lecture40.html$

(c) A more realistic distribution is of the form

$$\rho(r) = \frac{\rho_0 r_0^2}{r^2\,(1 + r/r_0)^{\alpha}}\,.$$

Derive the rotation curve $v(r)$. Which value of α gives a flat rotation curve at $r \gg r_0$ as shown in the measurements? *(3 points)*

(d) At $r = 10^5$ light years the measurement yields $v_{calc} = 15\text{km/s}$ and $v_{meas} = 225\text{km/s}$. Calculate the visible as well as the true galaxy mass. What is the percentage of dark matter in the galaxy? How high is the average dark matter mass density?
Hint: $G = 6.67 \times 10^{-11}\ m^3\ kg^{-1}\ s^{-2}$ *(2 points)*

9.2 The Boltzmann equation *12 points*

Consider a stable particle ψ. In a comoving volume, we know that the number of ψ and $\overline{\psi}$ changes only through annihilation and inverse annihilation processes (with χ we indicate all the possible final states):

$$\psi\overline{\psi} \leftrightarrow \chi\overline{\chi}\,.$$

Under certain simplifying assumptions, the Boltzmann equation that rules the evolution of the number density for ψ and can be written as:

$$\frac{dn_\psi}{dt} + 3Hn_\psi = -\langle\sigma_A|v|\rangle(n_\psi^2 - (n_\psi^{\text{EQ}})^2)\,, \tag{1}$$

where $\sigma_A|v|$ is the total annihilation cross section, and n_ψ^{EQ} is the particle number density at thermal equilibrium. Let us take a system in which the assumptions that lead to the previous formula are fulfilled, and consider the following questions:

(a) Take a particle ψ and use the following quantity

$$Y = \frac{n_\psi}{s}\,,$$

where s is the entropy density. Using the conservation of entropy per comoving volume ($sa^3 = \text{constant}$), show that *(1 point)*

$$\dot{n}_\psi + 3Hn_\psi = s\dot{Y}\,. \tag{2}$$

(b) Let m be the mass of the particle ψ. Now introduce the quantity

$$x \equiv \frac{m}{T}\,. \tag{3}$$

During the radiation dominated era, define also $H(m) \simeq 1.67(g*)^{\frac{1}{2}} m^2/m_{Pl}$, and $H(x) = H(m)x^{-2}$. Show that the Boltzmann equation becomes *(4 points)*

$$\frac{dY}{dx} = \frac{-x - \langle\sigma_A|v|\rangle s}{H(m)}(Y^2 - Y_{\text{E}}^2)\,. \tag{4}$$

(c) Write the expression for $Y_{\mathrm{EQ}}(x)$ (notice, as a function of x), in the case $x \gg 3$ (that is, the non-relativistic limit), and in the case $3 \gg x$ (the relativistic limit). Suppose that the freezed out occurs at $x \equiv x_f$ while still in the relativistic case. What is the value of $Y_{\mathrm{EQ}}(x)$ at x_f ? *(3 points)*

(d) We have derived the x-dependent Boltzmann equation

$$\frac{dY}{dx} = \frac{\lambda}{x^2}(Y^2 - Y_{\mathrm{E}}^2)\,, \tag{5}$$

where λ is parametrized by

$$\lambda = \frac{m^3 \langle \sigma_A |v| \rangle}{H(m)} \tag{6}$$

and can be considered as constant in this exercise. At late times, i.e. well after freeze-out, Y will be much larger than Y_{EQ} and the relation

$$\frac{dY}{dx} \backsimeq \frac{\lambda Y^2}{x^2}(x \ll 1) \tag{7}$$

holds. Integrate equation (7) analytically in order to derive the approximation

$$Y_\infty \backsimeq \frac{x_f}{\lambda}\,. \tag{8}$$

Typically one can consider Y_f being significantly larger than Y_∞. *(4 points)*