Exercises on Theoretical Particle Physics II

Prof. Dr. H.P. Nilles

Due 14.4.2014

1. Weyl spinors and Grassmann variables (*10 credits*)

In this exercise, we want to illustrate the connection between spinors and Grassmann variables and get used to the spinor index conventions.
Let θ_α, $\alpha = 1, 2$ be anti-commuting complex Grassmann variables

$$\{\theta_\alpha, \theta_\beta\} = 0. \tag{1}$$

As left-chiral Weyl spinors, they transform in the $(1/2, 0)$ representation of the Lorentz group

$$\theta \mapsto (D_L)\theta = \exp\left[-\frac{\mathrm{i}}{2}\omega_{\mu\nu}\sigma^{\mu\nu}\right]\theta \tag{2a}$$

$$\bar{\theta} \mapsto (D_R)\bar{\theta} = \exp\left[-\frac{\mathrm{i}}{2}\omega_{\mu\nu}\bar{\sigma}^{\mu\nu}\right]\bar{\theta} \tag{2b}$$

with the Pauli matrices $\sigma^\mu := (\mathbb{1}, \sigma^i)$, $\bar{\sigma}^\mu := (\mathbb{1}, -\sigma^i)$ and the spin generators $\sigma^{\mu\nu} := \frac{\mathrm{i}}{4}(\sigma^\mu\bar{\sigma}^\nu - \sigma^\nu\bar{\sigma}^\mu)$, $\bar{\sigma}^{\mu\nu} := \frac{\mathrm{i}}{4}(\bar{\sigma}^\mu\sigma^\nu - \bar{\sigma}^\nu\sigma^\mu)$.

(a) Let $\epsilon_{\alpha\beta} = \epsilon^{\alpha\beta}$ be the totally antisymmetric 2×2 tensor, i.e. $\epsilon_{\alpha\beta} = -\epsilon_{\beta\alpha}$ with normalization $\epsilon_{12} = 1$. Show that: $\epsilon_{\alpha\beta}\epsilon^{\beta\gamma} = -\delta_\alpha^\gamma$. (*1 credit*)

(b) On last year's exercise you proved

$$\sigma_2 = (D_L)^T\sigma_2 D_L. \tag{3}$$

Why does this mean that σ_2 is a spinor metric? Being a metric, it can be used to raise and lower spinor indices. (*1 credit*)

(c) Show that $\epsilon = \mathrm{i}\sigma_2$ is an equivalent choice for the metric. From now on, we use ϵ to raise and lower spinor indices:

$$\theta^\alpha := -\epsilon^{\alpha\beta}\theta_\beta. \tag{4}$$

Give the inverse of this relation. (*2 credits*)

(d) We define the conjugate Grassmann variable $\bar{\theta}^{\dot{\alpha}}$ as $\bar{\theta}^{\dot{\alpha}} := (\theta^\alpha)^*$. Verify (2b), i.e. show that it transforms in the $(0, 1/2)$ representation of the Lorentz group (as a right-chiral Weyl spinor).
Hint: You showed last year that $\sigma_2 D_L \sigma_2 = D_R^*$. Use this to calculate the transformation of $\epsilon^{\alpha\beta}\theta_\beta$. (*2 credits*)

(e) The conventions for contracting spinor indices are:

$$\xi\psi := \xi^\alpha\chi_\alpha \quad \text{and} \quad \bar{\xi}\bar{\chi} := \bar{\xi}_{\dot{\alpha}}\bar{\chi}^{\dot{\alpha}} \tag{5}$$

Verify the following identities: (*2 credits*)

(i)	$\xi^\alpha\chi_\alpha = -\xi_\alpha\chi^\alpha$	and	$\bar{\xi}_{\dot{\alpha}}\bar{\chi}^{\dot{\alpha}} = -\bar{\xi}^{\dot{\alpha}}\bar{\chi}_{\dot{\alpha}}$
(ii)	$\xi\chi = \chi\xi$	and	$\bar{\xi}\bar{\chi} = \bar{\chi}\bar{\xi}$

(f) Prove furthermore: (*2 credits*)

$$\begin{array}{llllll} \text{(i)} & \theta^\alpha\theta^\beta = \frac{1}{2}\epsilon^{\alpha\beta}\theta\theta & \text{and} & \theta_\alpha\theta_\beta = \frac{1}{2}\epsilon_{\alpha\beta}\theta\theta \\ \text{(ii)} & \bar\theta^{\dot\alpha}\bar\theta^{\dot\beta} = -\frac{1}{2}\epsilon^{\dot\alpha\dot\beta}\bar\theta\bar\theta & \text{and} & \bar\theta_{\dot\alpha}\bar\theta_{\dot\beta} = -\frac{1}{2}\epsilon_{\dot\alpha\dot\beta}\bar\theta\bar\theta \end{array}$$

In summary, the components of spinors are Grassmann variables. They anti-commute and transform in the correct Lorentz representations.

2. Weyl spinors and Pauli matrices (*7 credits*)

This exercise is intended to further establish the relation between the Pauli matrices, the spinor metric, and spinors as Grassmann variables.

(a) Use the Lorentz transformations (2) to deduce the spinor index structure for the Pauli matrices σ^μ, $\bar\sigma^\mu$ and the spin generators $\sigma^{\mu\nu}$, $\bar\sigma^{\mu\nu}$ in terms of the conventions introduced in (5). (*2 credits*)

(b) Check the following identites: (*2 credits*)

$$\begin{array}{lll} \text{(i)} & (\bar\sigma^\mu)^T & = -\epsilon\sigma^\mu\epsilon \\ \text{(ii)} & (\sigma^\mu)^{\alpha\dot\beta} & = (\bar\sigma^\mu)^{\dot\beta\alpha} \end{array}$$

(c) Verify furthermore: (*3 credits*)

$$\begin{array}{lll} \text{(i)} & \bar\xi\bar\sigma^\mu\chi & = -\chi\sigma^\mu\bar\xi \\ \text{(ii)} & \chi_\alpha\bar\xi_{\dot\beta} & = \frac{1}{2}(\sigma^\mu)_{\alpha\dot\beta}(\chi\sigma_\mu\bar\xi) \\ \text{(iii)} & (\theta\sigma^\mu\bar\theta)(\theta\sigma^\nu\bar\theta) & = \frac{1}{2}\eta^{\mu\nu}(\theta\theta)(\bar\theta\bar\theta) \end{array}$$

Hint: $\eta_{\mu\nu}\sigma^\mu_{\alpha\dot\alpha}\sigma^\nu_{\beta\dot\beta} = 2\epsilon_{\alpha\beta}\epsilon_{\dot\alpha\dot\beta}$ *(Prove it)*

3. Grassmann variable calculus (*3 credits*)

This exercise is intended to introduce differentiation of Grassmann variables and to investigate the consequences.

The Grassmann differentiation is defined as

$$\partial_\alpha := \frac{\partial}{\partial\theta^\alpha} \text{ and } \bar\partial^{\dot\alpha} := \frac{\partial}{\partial\bar\theta_{\dot\alpha}}, \tag{6}$$

with the usual relation $\partial_\alpha\theta^\beta = \delta_\alpha^\beta$ and $\bar\partial^{\dot\alpha}\bar\theta_{\dot\beta} = \delta^{\dot\alpha}_{\dot\beta}$. However, the product rule must be defined with a minus sign:

$$\partial_\alpha(\theta^\beta\theta^\gamma) = \delta^\beta_\alpha\theta^\gamma - \theta^\beta\delta^\gamma_\alpha.$$

Show that:

$$\begin{array}{lll} \text{(i)} & \partial^\alpha & = \epsilon^{\alpha\beta}\partial_\beta \\ \text{(ii)} & \partial^\alpha\partial_\alpha(\theta\theta) & = \bar\partial_{\dot\alpha}\bar\partial^{\dot\alpha}(\bar\theta\bar\theta) = 4 \end{array}$$