

Exercises on Theoretical Particle Physics II

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DUE 22.4.2014

4. The SUSY Algebra and the Chiral Representation (20 credits)

The SUSY algebra relates in a non-trivial way the Poincaré group with generators $P_\mu, M_{\mu\nu}$, with the anticommuting generators Q_α^i . The $Q_\alpha, \bar{Q}_{\dot{\beta}} = (Q_\beta)^*$ transform in the Lorentz group representations $(1/2, 0)$ and $(0, 1/2)$ respectively. The algebra reads

$$\begin{aligned} \{Q_\alpha, \bar{Q}_{\dot{\beta}}\} &= 2(\sigma^\mu)_{\alpha\dot{\beta}} P_\mu, & [Q_\alpha, P_\mu] &= 0, \\ [M_{\mu\nu}, Q_\alpha] &= i(\sigma_{\mu\nu})_\alpha{}^\beta Q_\beta, & [M_{\mu\nu}, \bar{Q}_{\dot{\alpha}}] &= i(\bar{\sigma}_{\mu\nu})^{\dot{\alpha}}{}_{\dot{\beta}} \bar{Q}_{\dot{\beta}}. \end{aligned} \quad (1)$$

- (a) The SUSY algebra can be viewed as a Lie algebra by introducing Grassmann variables $\theta_\alpha, \bar{\theta}_{\dot{\alpha}}$. Check that this new algebra is given by the commutators

$$[(\theta Q), (\bar{Q}\bar{\theta})] = 2(\theta\sigma^\mu\bar{\theta})P_\mu, \quad [P_\mu, (\theta Q)] = [P_\mu, (\bar{Q}\bar{\theta})] = 0. \quad (2)$$

(2 credits)

- (b) Define the corresponding group element associated to the Lie Algebra (2) as

$$S(a^\mu, \alpha, \bar{\alpha}) := \exp [\alpha Q + \bar{Q}\bar{\alpha} - ia^\mu P_\mu]. \quad (3)$$

Show that $S(a^\mu, \alpha, \bar{\alpha})S(b^\mu, \beta, \bar{\beta})$ is again a group element. (2 credits)

- (c) Multiplication of group elements induces a motion in the parameter space, called the **superspace**, with coordinates $(x^\mu, \theta, \bar{\theta})$. This serves to define a representation of the SUSY group on **superfields** $\Phi(x^\mu, \theta, \bar{\theta})$ as

$$\begin{aligned} S(a^\mu, \alpha, \bar{\alpha}) : (x^\mu, \theta, \bar{\theta}) &\mapsto (x^\mu + a^\mu - i\alpha\sigma^\mu\bar{\theta} + i\theta\sigma^\mu\bar{\alpha}, \theta + \alpha, \bar{\theta} + \bar{\alpha}), \\ S(a^\mu, \alpha, \bar{\alpha})\Phi(x^\mu, \theta, \bar{\theta}) &= \Phi(x^\mu + a^\mu - i\alpha\sigma^\mu\bar{\theta} + i\theta\sigma^\mu\bar{\alpha}, \theta + \alpha, \bar{\theta} + \bar{\alpha}). \end{aligned} \quad (4)$$

Use an infinitesimal transformation to show that the SUSY algebra on superfields $\Phi(x^\mu, \theta, \bar{\theta})$ is realised by

$$P_\mu = i\partial_\mu, \quad Q_\alpha = \partial_\alpha - i(\sigma^\mu)_{\alpha\dot{\beta}}\bar{\theta}^{\dot{\beta}}\partial_\mu, \quad \bar{Q}_{\dot{\beta}} = -\bar{\partial}_{\dot{\beta}} + i\theta^\alpha(\sigma^\mu)_{\alpha\dot{\beta}}\partial_\mu. \quad (5)$$

(2 credits)

- (d) Show that the operators (5) form a representation of the SUSY algebra by explicitly verifying the (anti-) commutation relations (1). (2 credits)

(e) Define a (SUSY) covariant derivative D_α by

$$D_\alpha(\delta_{(\epsilon, \bar{\epsilon})}\Phi) = \delta_{(\epsilon, \bar{\epsilon})}(D_\alpha\Phi), \quad (6)$$

where the infinitesimal transformation $\delta_{(\epsilon, \bar{\epsilon})} = \epsilon Q + \bar{Q}\bar{\epsilon}$ comes from the $S(0, \epsilon, \bar{\epsilon})$ expansion. This definition implies that $D_\alpha\Phi$ transforms as a superfield, too. Show that the following derivatives are covariant:

$$\begin{aligned} D_\alpha &= \partial_\alpha + i(\sigma^\mu)_{\alpha\dot{\beta}}\bar{\theta}^{\dot{\beta}}\partial_\mu, \\ D_{\dot{\beta}} &= -\bar{\partial}_{\dot{\beta}} - i\theta^\alpha(\sigma^\mu)_{\alpha\dot{\beta}}\partial_\mu. \end{aligned} \quad (7)$$

(2 credits)

(f) Next define **left** and **right chiral representations** by

$$\begin{aligned} S_L(a^\mu, \alpha, \bar{\alpha}) &:= \exp[\alpha Q - ia^\mu P_\mu] \exp[\bar{Q}\bar{\alpha}], \\ S_R(a^\mu, \alpha, \bar{\alpha}) &:= \exp[\bar{Q}\bar{\alpha} - ia^\mu P_\mu] \exp[\alpha Q]. \end{aligned} \quad (8)$$

Consider the left representation S_L , obtain its relation with the representation S in (3). Check that $S_L(a^\mu, \alpha, \bar{\alpha})S_L(b^\mu, \beta, \bar{\beta})$ is a group element. (2 credits)

Hint: Use the result of part (g) from this exercise.

(g) Show that

$$S(a^\mu, \alpha, \bar{\alpha}) = S_L(a^\mu - i\alpha\sigma^\mu\bar{\alpha}, \alpha, \bar{\alpha}) = S_R(a^\mu + i\alpha\sigma^\mu\bar{\alpha}, \alpha, \bar{\alpha}). \quad (2 \text{ credits})$$

(h) A superfield in the left-chiral representation is defined by

$$S_L(a^\mu, \alpha, \bar{\alpha})[\phi_L(x^\mu, \theta, \bar{\theta})] = \phi_L(x^\mu + a^\mu + 2i\theta\sigma^\mu\bar{\alpha}, \theta + \alpha, \bar{\theta} + \bar{\alpha}). \quad (9)$$

Determine the representations of the SUSY generators Q_L and \bar{Q}_L . (2 credits)

(i) Check that the following operators define covariant derivatives

$$\begin{aligned} D_{L\alpha} &= \partial_\alpha + 2i(\sigma^\mu)_{\alpha\dot{\beta}}\bar{\theta}^{\dot{\beta}}\partial_\mu, \\ \bar{D}_{L\dot{\beta}} &= -\bar{\partial}_{\dot{\beta}}, \end{aligned} \quad (10)$$

by computing the commutator with the SUSY transformation S_L . (2 credits)

(j) Define now **chiral superfields** by the constraints

$$\begin{aligned} \bar{D}_L\Phi(x, \theta, \bar{\theta}) &= 0, \quad \text{for **left-chiral**}, \\ D_L\Phi(x, \theta, \bar{\theta}) &= 0, \quad \text{for **right-chiral**}. \end{aligned} \quad (11)$$

This definition is independent of the representation. Use the representation S of (3) to work out the constraints on the component fields. Choose the left-chiral representation $\bar{D}_L\Phi_L = 0$, to deduce the general form of a left chiral superfield. (2 credits)

Hint: Make a Taylor expansion in θ to define the component fields of Φ .

(k) Consider the infinitesimal SUSY transformation $\delta_{(\epsilon, \bar{\epsilon})}$ of a left-chiral superfield ϕ_L . How do the component fields of ϕ_L transform? (2 credits)

Hint: Use the left-chiral representation of the SUSY generators Q_L and \bar{Q}_L and assume that the transformation is small: $\epsilon\sigma^\mu\bar{\epsilon} \approx 0$.