An Introduction to
Superstring Theory

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1 Introduction

In spite of all its beautiful features the Bosonic String Theory left us with some problems like the absence of fermions and the tachyon in the spectrum. Because particles can be devided into bosons and fermions and we want to have a theory which describes them all we must change the action so that there are also fermions in the spectrum.

Fermions obey anticommutation relations so the most obvious step will be to introduce anticommuting coordinates ψ^μ into the action, fermionic partners for the bosonic coordinates X^μ which are already there.

The tachyon in the Bosonic String Theory, a state with a negative mass square ($m^2 < 0$), is caused by the normal ordering constant one had to introduce. It is at least an alarming fact and a hint that the vacuum is incorrectly identified. If the action is supersymmetric, i.e. invariant under a transformation which mixes bosonic and fermionic coordinates, there is good hope that the tachyon will be eliminated from the spectrum. One has to be carefully in distinguishing between world-sheet- and space-time-supersymmetry. In this article we discuss only two-dimensional world-sheet supersymmetry.

In the Bosonic String Theory it was found that at the quantum level the theory makes sense only in 26 dimensions. Similarly in the Superstring Theory we will find the critical dimension to be 10.

2 The Action

In this section we search for a suitable action that determines the motion of the string, the one dimensional object in a d dimensional space-time (called *target space*), for which we built our theory. As an action for higher dimensional objects one chooses the generalization of the point particle action which is the length of its world-line. Thus for the string (1-dim.) we take the surface that it sweeps out in the target space. This 2-dimensional *world-sheet* of the string is parametrized by the two parameters τ, σ, where $\tau \in [\tau_0, \tau_1]$ and $\sigma \in [0, 2\pi]$ for the closed, $\sigma \in [0, \pi]$ for the open string. In the Bosonic String Theory the action was found to be

$$S = -\frac{1}{4\pi\alpha'} \int \mathrm{d}\tau \mathrm{d}\sigma \partial_\alpha X^\mu \partial^\alpha X_\mu,$$

where $\alpha \in \{0, 1\}$, $\eta^{\alpha\beta} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$ and the $X^\mu(\tau, \sigma)$ are free bosonic fields. One can show that this form of the action is equivalent to the two-dimensional volume of the worldsheet. The bosonic string action thus is the action of a two-dimensional free field theory. To include fermions into the theory we now add a second term which is built of fermion fields Ψ. Because we want to have supersymmetry (invariance under a transformation that mixes the bosonic and the fermionic coordinates) as a symmetry of the action, we must choose the Ψ as superpartners of the X^μ. For this reason they have to be two-component spinors $\Psi_A(\tau, \sigma), A \in \{-, +\}$, i.e.

$$\Psi = \begin{pmatrix} \Psi_- \\ \Psi_+ \end{pmatrix}$$

There are surprisingly few choices that lead to interesting theories, one that does is to choose them to have a second index μ which labels the target space coordinates: $\Psi^\mu_A, \quad \mu \in \{0, ..., d-1\}$. But this μ is a vector index, meaning that instead of the Ψ^μ_A being two-component spinors they transform in the vector representation of the target space Lorentz group $SO(d-1, 1)$. We finally take the action to be:

$$S = -\frac{1}{4\pi\alpha'} \int \mathrm{d}\tau \mathrm{d}\sigma \left(\partial_\alpha X^\mu \partial^\alpha X_\mu + i\overline{\Psi}^\mu \varrho^\alpha \partial_\alpha \Psi_\mu \right). \tag{1}$$

Here the ϱ^α are chosen to be

$$\varrho^0 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad \varrho^1 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \tag{2}$$

thus they are two-dimensional Dirac-matrices (two-dimensional Dirac-matrices are in general denoted by ϱ, four-dimensional by γ and d-dimensional by Γ) and it can easily be verified by matrix multiplication that they satisfy the two-dimensional Clifford-Algebra:

$$\{\varrho^\alpha, \varrho^\beta\} = -2\eta^{\alpha\beta}. \tag{3}$$

Because the ϱ^α are purely imaginary the Dirac-operator $i\varrho^\alpha\partial_\alpha$ is real and it makes sense to choose the two-component spinor Ψ_A real, as a *Majorana-spinor*. Due to the reality of the spinor components $\overline{\Psi} = \Psi^t\varrho^0 = i(\Psi_+, -\Psi_-)$.
Thus we have build an action including d bosonic fields X^μ and d fermionic fields Ψ_A^μ. The reader should not be disturbed by the fact that the Ψ_A^μ transform as a vector of $SO(d-1,1)$, the Lorentz group in d dimensions. Actually there is no contradiction to the *Spin-Statistics Theorem*. We are considering a two-dimensional field theory and the fermion fields Ψ_A^μ transform as spinors under transformations of the 2-dim worldsheet as the theorem demands. It says nothing about the features under transformations of the target space, which can be seen as inner symmetries from the two-dimensional point of view.

2.1 Light-Cone Gauge

In the previous chapter we found that the superstring-action showed invariance under reparametrizations of the world-sheet. This feature now will be used to go to light-cone gauge. The reason for this gauge will become obvious very soon: In light-cone gauge it will be possible to get rid of all unphysical degrees of freedom and keep only the physical ones. For that we perform the transformation

$$\begin{pmatrix} \tau \\ \sigma \end{pmatrix} \longrightarrow \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} \tau \\ \sigma \end{pmatrix} \tag{4}$$

and call the new coordinates $\sigma^\pm := \tau \pm \sigma$ *light-cone coordinates*. The derivatives transform according to

$$\partial_\pm = \frac{1}{2}(\partial_\tau \pm \partial_\sigma).$$

Therefore the metric in light-cone coordinates is calculated from the worldsheet metric $(h^{\alpha\beta})^{\alpha\beta} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$ to be:

$$\eta^{++} = \frac{\partial\sigma^+}{\partial\sigma^\mu}\frac{\partial\sigma^+}{\partial\sigma^\nu}\eta^{\mu\nu} = -\frac{\partial\sigma^+}{\partial\sigma^0}\frac{\partial\sigma^+}{\partial\sigma^0} + \frac{\partial\sigma^+}{\partial\sigma^1}\frac{\partial\sigma^+}{\partial\sigma^1} = -1+1 = 0$$

and in the same way

$$\eta^{--} = 0 \qquad \eta^{-+} = \eta^{+-} = -\frac{1}{2}.$$

The light-cone indizes $+,-$ are raised and lowered with that metric.

Now the fields X^μ, Ψ^μ are redefined according to

$$X^\pm := \frac{1}{\sqrt{2}}(X^0 \pm X^1) \qquad \Psi^\pm := \frac{1}{\sqrt{2}}(\Psi^0 \pm \Psi^1), \tag{5}$$

and the transversal fields $X^i, \Psi^i \quad i = 2, ..., d-2$. The reason of this will be seen soon.

2.2 Symmetries

We will now use the symmetries of the action to reduce the degrees of freedom, to delete all fields which depend in a trivial way on the world-sheet parameters or can be expressed by other fields.

First there is a reparametrization invariance, i.e. the action (1) is invariant under $\tau \to \tau'$, $\sigma \to \sigma'$. In the following we will see that for reducing the number of free fields a very useful gauge is the choice of light-cone coordinates $\sigma^{\pm} = \tau \pm \sigma$.

With the knowledge of the previous chapter we are prepared to transform the action into light-cone coordinates. By using that the determinant of the transformation matrix, defined in (4) is given by $\det(A) = 2$ the calculation runs as follows

$$
\begin{aligned}
S = -\frac{1}{4\pi\alpha'} \int & \mathrm{d}\tau\mathrm{d}\sigma\, (\partial_\alpha X^\mu \partial^\alpha X_\mu + i\overline{\Psi}^\mu \varrho^\alpha \partial_\alpha \Psi_\mu) \\
= \frac{1}{4\pi\alpha'} \int & \mathrm{d}\tau\mathrm{d}\sigma\, (\partial_\tau X^\mu \partial_\tau X_\mu - \partial_\sigma X^\mu \partial_\sigma X_\mu - i\overline{\Psi}^\mu (\varrho^0 \partial_0 + \varrho^1 \partial_1)\Psi_\mu) \\
= \frac{1}{2\pi\alpha'} \int & \mathrm{d}\tau\mathrm{d}\sigma\, (2 \cdot \frac{1}{4}(\partial_\tau - \partial_\sigma) X^\mu (\partial_\tau + \partial_\sigma) X_\mu \\
& -2 \cdot \frac{i}{2} \cdot \frac{1}{2} \begin{pmatrix} \Psi^\mu_+, & -\Psi^\mu_- \end{pmatrix} \begin{pmatrix} (\partial_\tau - \partial_\sigma)\delta\Psi_{+\mu} \\ -(\partial_\tau + \partial_\sigma)\delta\Psi_{-\mu} \end{pmatrix} \\
= \frac{1}{2\pi\alpha'} \int & \mathrm{d}\sigma^+ \mathrm{d}\sigma^- (\partial_- X^\mu \partial_+ X_\mu + \frac{i}{2}(\Psi^\mu_+ \partial_- \Psi_{+\mu} + \Psi^\mu_- \partial_+ \Psi_{-\mu})).
\end{aligned}
$$

First we consider the following two symmetries:

Poincaré-invariance: The Poincaré-invariance in 2 - and d dimensions of the action is obvious from the index structure of the appearing terms (scalar products only).

Supersymmetry: The action is also invariant under the following two-dimensional worldsheet supersymmetry, a transformation which mixes the bosonic and the fermionic degrees of freedom:

$$
\begin{aligned}
\delta X^\mu &= i(\epsilon_+ \Psi^\mu_- - \epsilon_- \Psi^\mu_+) && (6) \\
\delta \Psi^\mu &= -i\varrho^\alpha \partial_\alpha X^\mu \epsilon, && (7)
\end{aligned}
$$

where ϵ is a two-component non-chiral Majorana-spinor with infinitesimal components. In components the second condition reads (by using the explicit form of the Dirac-matrices)

$$
\begin{aligned}
\delta \Psi^\mu_- &= -2\epsilon_+ \partial_- X^\mu && (8) \\
\delta \Psi^\mu_+ &= 2\epsilon_- \partial_+ X^\mu. && (9)
\end{aligned}
$$

For checking the invariance under this supersymmetry we use the anticommutation relation for the spinor components $\epsilon_+\Psi_- = -\Psi_-\epsilon_+$. We first calculate the additional terms (only those which are linear in ϵ because we are considering an infinitesimal transformation) which arrive after applying $X^\mu \rightarrow X^\mu + \delta X^\mu$ of the bosonic term (we just write the integrand but it is naturally to be understood that we consider $S(X^\mu, \Psi^\mu) \rightarrow S(X^\mu + \delta X^\mu, \Psi^\mu + \delta\Psi^\mu)$:

$$\partial_-(X^\mu + \delta X^\mu)\partial_+(X_\mu + \delta X_\mu) = \partial_- X^\mu \partial_+ X_\mu + \partial_- X^\mu \partial_+(\delta X_\mu) + \partial_-(\delta X^\mu)\partial_+ X_\mu$$

$$= \partial_- X^\mu \partial_+ X_\mu + i\partial_- X^\mu \partial_+(\epsilon_+\Psi_{-\mu} - \epsilon_-\Psi_{+\mu}) + \partial_-(\epsilon_+\Psi_{-\mu} - \epsilon_-\Psi_{+\mu})\partial_+ X^\mu$$

$$= \partial_- X^\mu \partial_+ X_\mu + i(\epsilon_+(\partial_- X^\mu \partial_+ \Psi_{-\mu} + \partial_- \Psi_-^\mu \partial_+ X^\mu) - \epsilon_-(\partial_- X^\mu \partial_+ \Psi_{+\mu} + \partial_- \Psi_+^\mu \partial_+ X_\mu)).$$

Then we perform the transformation $\Psi^\mu \rightarrow \Psi^\mu + \delta\Psi^\mu$ to the fermionic term

$$\frac{i}{2}((\Psi_+^\mu + \delta\Psi_+^\mu)\partial_-(\Psi_{+\mu} + \delta\Psi_{+\mu}) + (\Psi_-^\mu + \delta\Psi_-^\mu)\partial_+(\Psi_{-\mu} + \delta\Psi_{-\mu})) =$$

$$\frac{i}{2}(\Psi_+^\mu \partial_- \Psi_{+\mu} + \Psi_-^\mu \partial_+ \Psi_{-\mu} + \delta\Psi_+^\mu \partial_- \Psi_{+\mu} + \Psi_+^\mu \partial_-(\delta\Psi_{+\mu}) + \delta\Psi_-^\mu \partial_+ \Psi_{-\mu} + \Psi_-^\mu \partial_+(\delta\Psi_{-\mu}))$$

$$= \frac{i}{2}(\Psi_+^\mu \partial_- \Psi_{+\mu} + \Psi_-^\mu \partial_+ \Psi_{-\mu}) + i(\epsilon_- \partial_+ X^\mu \partial_- \Psi_{+\mu} + \Psi_+^\mu \partial_-(\epsilon_- \partial_+ X_\mu))$$
$$-\epsilon_+ \partial_- X^\mu \partial_+ \Psi_{-\mu} - \Psi_-^\mu \partial_+(\epsilon_+ \partial_- X_\mu))$$

$$= \frac{i}{2}(\Psi_+^\mu \partial_- \Psi_{+\mu} + \Psi_-^\mu \partial_+ \Psi_{-\mu}) + i(-\epsilon_+(\partial_- X^\mu \partial_+ \Psi_{-\mu} + \Psi_-^\mu \partial_+ \partial_- X_\mu) +$$
$$\epsilon_-(\partial_+ X^\mu \partial_- \Psi_{+\mu} - \Psi_+^\mu \partial_- \partial_+ X_\mu))$$

and obviously the first term of the bosonic part cancels with the first term of the fermionic and the fourth of the bosonic with the third of the fermionic part. Now we integrate the two remaining terms of the fermionic part by parts (they are actually still written in an integral although this is not obvious in the above formula). This is exemplarily done for the first one(it is first transformed to (τ, σ), in order to discuss the boundary terms) :

$$2\,i\int d\tau\, d\sigma\, \Psi_-^\mu(\partial_\tau - \partial_\sigma)(\partial_\tau + \partial_\sigma)X_\mu = 2\,i\int d\sigma(\Psi_-^\mu(\partial_\tau + \partial_\sigma)X_\mu)|_{\tau=\tau_0}^{\tau_1}$$
$$-2\,i\int d\tau(\Psi_-^\mu(\partial_\tau + \partial_\sigma)X_\mu)|_{\sigma=0}^{2\pi} - i\int d\sigma^+ d\sigma^- \partial_- \Psi_-^\mu \partial_+ X_\mu$$

Both boundary terms vanish, the first because of the chosen boundary conditions and the last cancels with the second term of the bosonic part. In analogy one can show that also the last remaining term is cancelled. Thus we have showed that under the supersymmetry transformation $\delta S = 0$, that it really is a symmetry of our action. One can further show that the action is still invariant if the transformation parameter ϵ is taken of the form

$$\epsilon_- = \epsilon_-(\sigma^-) \qquad \epsilon_+ = \epsilon_+(\sigma^+)$$

what defines partially local symmetries.

3 Equations of Motion

The equations of motion of the Superstring are derived from the action by a variational principle. By performing the variation $X^\mu \to X^\mu + \delta X^\mu$ with the boundary conditions $\delta X^\mu(\tau\!=\!\tau_0) = \delta X^\mu(\tau\!=\!\tau_1) = 0$ and

- $\delta X^\mu(\sigma \in \{0, \pi\})$ arbitrarily chosen, but $X'^\mu = 0$ for open strings (*Neumann boundary conditions*)
- $\delta X^\mu(\sigma + \pi) = \delta X^\mu(\sigma)$ for closed strings (*Dirac boundary conditions*)

The calculation yields as in the bosonic case (see Appendix A) the equations of motion:

$$\partial_+ \partial_- X^\mu = 0 \tag{10}$$

with the corresponding boundary terms in each case.

The variation of the fermionic part of the action is slightly more difficult. By performing the variation $\Psi^\mu \to \Psi^\mu + \delta\Psi^\mu$ with the boundary conditions $\Psi^\mu(\tau \in \{\tau_0, \tau_1\}) = 0$ the calculation is the following:

$$\begin{aligned}
\delta S &= -\frac{i}{4\pi\alpha'} \int \mathrm{d}\tau \mathrm{d}\sigma (\overline{\Psi}^\mu \varrho^\alpha \delta(\partial_\alpha \Psi_\mu)) \\
&= \frac{i}{4\pi\alpha'} \int \mathrm{d}\tau \mathrm{d}\sigma (\partial_\alpha \overline{\Psi}^\mu \varrho^\alpha \delta(\Psi_\mu)) - \frac{i}{4\pi\alpha'} \int \mathrm{d}\tau (\left(\overline{\Psi^\mu_+}, \ -\overline{\Psi^\mu_-} \right) \varrho^1 \begin{pmatrix} \delta\Psi^\mu_- \\ \delta\Psi^\mu_+ \end{pmatrix})|_{\sigma \in \partial S}
\end{aligned}$$

Because we are considering closed strings only $\sigma \in [0, \pi]$ and the boundary term becomes $(-\Psi^\mu_+ \delta\Psi_{+\mu} + \Psi^\mu_- \delta\Psi_{-\mu})|^\pi_{\sigma=0}$. The equation of motion for the fermion fields $\partial_\alpha(\overline{\Psi^\mu})\varrho^\alpha = 0$ is the Dirac conjugate of a massless two-dimensional Dirac equation

$$\varrho^\alpha \partial_\alpha \Psi^\mu = 0. \tag{11}$$

With the special matrix representation of the ϱ^α in mind one finds the equivalent form of the equations $(\partial_\tau + \partial_\sigma)\Psi^\mu_- = 0, (\partial_\sigma - \partial_\tau)\Psi^\mu_+ = 0$ and finally

$$\partial_+ \Psi^\mu_- = \partial_- \Psi^\mu_+ = 0. \tag{12}$$

which are supplemented by the boundary conditions

$$(-\Psi_{+\mu} \delta\Psi^\mu_+ + \Psi_{-\mu} \delta\Psi^\mu_-)|^{\sigma=\pi}_{\sigma=0} = 0. \tag{13}$$

which will be discussed shortly.

Thus Ψ_-^μ and $\partial_- X^\mu$ are both functions of σ^- only, $\Psi_+^\mu, \partial_+ X^\mu$ of σ^+, a fact that makes it more transparent why the action is invariant under supersymmetry transformations, which mixes Ψ_-^μ with $\partial_- X^\mu$ and Ψ_+^μ with $\partial_+ X^\mu$. The equations of motion will have solutions of negative norm, due to the metric-components h_{01} and h_{10} a fact that will directly lead to negative norm states after quantization.

3.1 Solutions and Boundary Conditions

Now we call our attention to the boundary conditions $(-\Psi_{+\mu}\delta\Psi_+^\mu + \Psi_{-\mu}\delta\Psi_-^\mu)|_{\sigma=0}^{\pi} = 0$ which we received while varying the action. For closed strings the variation of Ψ_+^μ has to be taken independently of the one of Ψ_-^μ, because we do not want the boundary conditions to break part of the supersymmetry $\delta\Psi_- = -2\epsilon_+\partial_- X^\mu$ and $\delta\Psi_+ = 2\epsilon_-\partial_+ X^\mu$, with $\epsilon_\pm$ independent of each other. It is obvious that there are two possible choices of boundary conditions, either periodicity or anti-periodicity of each of the two spinor components under shifts of σ by π.[1] The possibility for the two choices defines two sectors in which the solutions can be:

- The **Ramond Sector**: $\Psi_\pm^\mu(\tau, \sigma + \pi) = \Psi_\pm^\mu(\tau, \sigma)$
- The **Neveu-Schwarz Sector**: $\Psi_\pm^\mu(\tau, \sigma + \pi) = -\Psi_\pm^\mu(\tau, \sigma)$

The general solution of the equations of motion can be written in terms of the following mode expansion, for the *Ramond Sector*(R-sector):

$$\Psi_-^\mu = \sum_{n\in\mathbb{Z}} d_n^\mu e^{-2in(\tau-\sigma)} \tag{14}$$

$$\Psi_+^\mu = \sum_{n\in\mathbb{Z}} \tilde{d}_n^\mu e^{-2in(\tau+\sigma)} \tag{15}$$

and for the *Neveu-Schwarz Sector*(NS-sector):

$$\Psi_-^\mu = \sum_{n\in\mathbb{Z}+\frac{1}{2}} b_n^\mu e^{-2in(\tau-\sigma)} \tag{16}$$

$$\Psi_+^\mu = \sum_{n\in\mathbb{Z}+\frac{1}{2}} \tilde{b}_n^\mu e^{-2in(\tau+\sigma)}, \tag{17}$$

where the sum runs over half-integer numbers $(..., -\frac{1}{2}, \frac{1}{2}, \frac{3}{2}, ...)$ in order to satisfy the appropriate boundary conditions.
The formula for the X^μ is the same as in the Bosonic String Theory (see Appendix A).

[1] From here on we will choose $\sigma_{max} = \pi$, although we are considering closed strings.

3.2 Constraint Equations

The above mentioned negative norm states (called ghosts) will be a problem for the probabilistic interpretation of the states and therefore we must try to remove them. This can be done in light-cone gauge if we use certain constraint equations, which reduce the number of degrees of freedom. The first demand is, in analogy to the bosonic case that the equation of motion for the metric holds. Thus the energy-momentum tensor must vanish. But to calculate the Euler-Lagrange equations for the metric, we must take care of the spinor term. It is necessary to insert a vielbein e^a_α which here is a zweibein and its superpartner, the gravitino χ_α. Then one can generalize our action to a form (still supersymmetric) that can be varied due to the zweibein and gives the energy-momentum tensor:

$$T_{\alpha\beta} = -\frac{2\pi}{e}\frac{\delta S}{\delta e^\beta_a} e_{\alpha a}. \tag{18}$$

We skip this calculation for the sake of briefness and just present the components of the (two-dimensional) energy-momentum tensor in light-cone gauge:

$$\begin{aligned}
T_{++} &= \partial_+ X^\mu \partial_+ X_\mu + \frac{i}{2}\Psi^\mu_+ \partial_+ \Psi_{\mu +} && (19)\\
T_{--} &= \partial_- X^\mu \partial_- X_- + \frac{i}{2}\Psi^\mu_- \partial_- \psi_{\mu -} && (20)\\
T_{+-} &= 0 && (21)\\
T_{-+} &= 0. && (22)
\end{aligned}$$

Hence it is easily to be seen that the trace of the energy-momentum tensor ($h^{\alpha\beta}T_{\alpha\beta}$) vanishes:

$$h^{\alpha\beta}T_{\alpha\beta} = -\frac{1}{2}T_{+-} - \frac{1}{2}T_{-+} = 0$$

In the Bosonic String Theory the two coordinates $X^\pm$ were eliminated by imposing $T_{++} = T_{--} = 0$ as *constraint equations* on the solutions of the equations of motion (see Appendix A). This is the same in the Superstring Theory but here also the first two components of Ψ^μ have negative norm and must be removed. Therefore we need more constraint equations and we consider the Noether *supercurrents* J_α which are yielded by applying Noether's method to the supersymmetry transformation:

$$\begin{aligned}
J_+ &= \Psi^\mu_+ \partial_+ X_\mu && (23)\\
J_- &= \Psi^\mu_- \partial_- X_\mu. && (24)
\end{aligned}$$

The components of the energy-momentum tensor and the supercurrent satisfy the algebra:

$$\{J_\pm(\sigma), J_\pm(\sigma')\}_{P.B.} = \pi\delta(\sigma - \sigma')T_{\pm\pm}(\sigma) \quad \{J_+(\sigma), J_-(\sigma')\} = 0. \tag{25}$$

Therefore it does not seem consistent to set only $T_{\alpha\beta} = 0$ and we demand for all X^μ, Ψ^μ_A the constraint equations

$$T_{++} = T_{--} = J_+ = J_- = 0. \tag{26}$$

It is to be mentioned that the supercurrent can also be derived from the above mentioned action by varying it with respect to the gravitino χ^α.

The above mentioned reparametrization invariance is not yet completely fixed (by going to light-cone coordinates). Conformal coordinate transformations $\sigma^\pm \to \tilde{\sigma}^\pm(\sigma^\pm)$ are still allowed. Therefore we take the choice $\tau \to \frac{1}{2}(\tilde{\sigma}^+(\sigma^+) + \tilde{\sigma}^-(\sigma^-)$ and it is evident that

$$\partial_+\partial_-\tau = 0. \tag{27}$$

Because the equations of motion for the bosonical degrees of freedom are the same as in the Bosonic String Theory (see Appendix A), explicitly $\partial_+\partial_- X^\mu = 0$, they meet the same equation as τ and therefore we can choose $X^+ \sim \tau$. We decide to take the $+$-component because then the constraint equations can be solved, otherwise a squareroot will arise as a problem. Thus the first component of X^μ is fixed:

$$X^+ = x^+ + p^+\tau \tag{28}$$

with x^+, p^+ constant numbers. In addition to the fixing of X^+ we also choose

$$\begin{pmatrix} \Psi_- \\ \Psi_+ \end{pmatrix}^{\mu=+} = 0 \tag{29}$$

what will turn out to be a good choice because it enables us to solve the constraints, as described in the following.

Now we can use the constraint equations to solve X^- and Ψ^- in terms of the transversal coordinates X^i and Ψ^i, $\quad (i = 2, ..., d-1)$. We write them explicitly:

$$\begin{aligned}
-\frac{p^+}{2}\Psi^-_+ + \Psi^i_+\partial_+ X^i &= 0 \\
-\frac{p^+}{2}\Psi^-_- + \Psi^i_-\partial_- X^i &= 0 \\
-p^+\partial_+ X^- + (\partial_+ X^i)^2 - \frac{i}{2}\Psi^i_+\partial_+\Psi^i_+ &= 0 \\
-p^+\partial_- X^- + (\partial_- X^i)^2 - \frac{i}{2}\Psi^i_-\partial_-\Psi^i_- &= 0
\end{aligned}$$

and find

$$(\Psi_\pm)^- = \frac{2}{p^+}\Psi^i_\pm\partial_\pm X^i \tag{30}$$

$$\partial_\pm X^- = \frac{1}{p^+}((\partial_\pm X^i)^2 + \frac{i}{2}\Psi^i_\pm\partial_\pm\Psi^i_\pm). \tag{31}$$

Hence the imposing of the constraints reduced the number of physical degrees of freedom and in the end we are left with only $d-2$ bosonic and $d-2$ fermion fields.

4 Quantization of the Supersymmetric String

Up to this point we treated the Superstring as a classical theory but now it is time to quantize it. For the quantization process we calculate the Poisson Brackets of the classical general coordinates and their canonically conjugated momenta and then perform the replacement

$$[\quad,\quad]_{P.B.} \longrightarrow \frac{1}{i}[\quad,\quad] \tag{32}$$

for the bosonic and

$$[\quad,\quad]_{P.B.} \longrightarrow \frac{1}{i}\{\quad,\quad\} \tag{33}$$

for the fermionic coordinates. The fields X^μ, Ψ^μ will be seen as operators and we will find that they can be interpreted as creation and annihilation operators. Thus we will achieve a second quantization where all the states of the spectrum can be constructed by acting with them on a vacuum state.

For the bosonic fields the results are the same as in the Bosonic String Theory because the corresponding part of the action is not changed.

But for the fermionic coordinates the situation is a bit more complicated because there are certain constraints, called *second class constraints*. The classical conjugated momenta are given by

$$\Pi_A^\mu := \frac{\partial L}{\partial(\partial_\tau \Psi_{A\mu})} = -\frac{i}{4\pi\alpha'}\Psi_A^\mu. \tag{34}$$

From this follows the above mentioned second class constraint $\Phi_A^\mu = \Pi_A^\mu + \frac{i}{4\pi\alpha'}\Psi_A^\mu = 0$, second class because $[\Phi_A^\mu(\tau,\sigma), \Phi_B^\nu(\tau,\sigma')]_{P.B.} = -\frac{i}{2\pi\alpha'}\delta(\sigma-\sigma')\delta_{AB}\eta^{\mu\nu}$. Therefore the Poisson Brackets of the classical solutions and their conjugated momenta have to be replaced by Dirac Brackets (see Appendix B) and the calculation yields:

$$[\Psi_+^\mu(\sigma), \Psi_+^\nu(\sigma')]_{D.B.} = [\Psi_-^\mu(\sigma), \Psi_-^\nu(\sigma')]_{D.B.} = i\pi\eta^{\mu\nu}\delta(\sigma-\sigma') \tag{35}$$

$$[\Psi_+^\mu(\sigma), \Psi_-^\nu(\sigma')]_{D.B.} = 0. \tag{36}$$

Now all we have to do to quantize the string is to regard the modes α, b, d as operators and replace the Dirac Brackets by anticommutators $[\quad,\quad]_{D.B.} = \frac{1}{i}\{\quad,\quad\}$. So we find the anticommutation relations

$$\{\Psi_+^\mu(\sigma), \Psi_+^\nu(\sigma')\} = \{\Psi_-^\mu(\sigma), \Psi_-^\nu(\sigma')\} = \pi\eta^{\mu\nu}\delta(\sigma-\sigma') \tag{37}$$

$$\{\Psi_+^\mu(\sigma), \Psi_-^\nu(\sigma')\} = 0. \tag{38}$$

By inserting the formula for $\Psi_\pm$ we find the anticommutators for the Fourier coefficients

$$\pi\eta^{\mu\nu}\delta(\sigma-\sigma') = \{\Psi_+^\mu(\sigma), \Psi_+^\nu(\sigma')\} = \{\sum_{m\in\mathbb{Z}} \tilde{d}_m^\mu e^{-2im\sigma^+}, \sum_{n\in\mathbb{Z}} \tilde{d}_n^\mu e^{-2in\sigma^+}\}$$

$$= \sum_{n,m\in\mathbb{Z}} \{\tilde{d}^{\mu}_{m}, \tilde{d}^{\nu}_{m}\} e^{-2i(m\sigma^{+}+n\sigma^{+})}.$$

By using $\delta(\sigma) = \frac{1}{\pi}\sum_{k=-\infty}^{\infty} e^{2ik\sigma}$ for the δ-function we get

$$\sum_{n\in\mathbb{Z}} \left(\eta^{\mu\nu} e^{2in(\sigma^{+}-\sigma'^{+})} - \sum_{m\in\mathbb{Z}} \{\tilde{d}^{\mu}_{m}, \tilde{d}^{\nu}_{m}\} e^{-2in(\sigma'^{+}+\frac{m}{n}\sigma^{+})} \right) = 0$$

which is only satisfied if

$$\{\tilde{d}^{\mu}_{m}, \tilde{d}^{\nu}_{m}\} = \eta^{\mu\nu}\delta_{n+m,0}.$$

In analogy we calculate all other commutators:

$$\{b^{\mu}_{r}, b^{\nu}_{s}\} = \{\tilde{b}^{\mu}_{r}, \tilde{b}^{\nu}_{s}\} = \eta^{\mu\nu}\delta_{r+s,0} \qquad \text{NS} \tag{39}$$

$$\{d^{\mu}_{r}, d^{\nu}_{s}\} = \{\tilde{d}^{\mu}_{r}, \tilde{d}^{\nu}_{s}\} = \eta^{\mu\nu}\delta_{r+s,0} \qquad \text{R} \tag{40}$$

otherwise the above relations won‘t be true. From the reality of the Majorana spinors the condition $(b^{\mu}_{r})^{\dagger} = b^{\mu}_{-r}$ arises for $r > 0$ and the same for the d‘s. Then the operators meets the usual harmonic oszillator anticommutation relations $\{b^{\mu\,\dagger}_{r}, b^{\nu}_{s}\} = \eta^{\mu\nu}\delta_{r,s}$ and the negative frequency modes can be seen as raising operators, the positive frequency modes as lowering operators. But what about zero-Fourier index? We will discuss this topic in the next section. For this reason we can construct each state by acting with raising operators on a vacuum state (*second quantization*). The vacuum (of the right-moving NS-sector for example) is defined by

$$\alpha^{\mu}_{m}|k\rangle = b^{\mu}_{r}|k\rangle = 0 \quad \forall \alpha, b > 0 \tag{41}$$

A general state then is the tensor product of the right- and left-moving modes. The number operator is given by the formula

$$N = N^{(a)} + N^{(b)} = \sum_{m=1}^{\infty} \alpha_{-m}\cdot\alpha_{m} + \sum_{r=\frac{1}{2}}^{\infty} r b_{-r}\cdot b_{r} \tag{42}$$

where the a/b stands for the bosonic/fermionic (in the NS-sector) part. The number operator is quantized in half integer steps.

Because X^{-}, Ψ^{-} are expressed in terms of the transverse coordinates (light-cone gauge), the $\alpha^{-}_{m}, b^{-}_{r}$ can be expressed in terms of the $\alpha^{i}_{m}, b^{i}_{r}$. They they read (for the NS-sector for example):

$$\alpha^{-}_{n} = \frac{1}{p^{+}} \left(\sum_{m=-\infty}^{\infty} :\alpha^{i}_{n-m}\alpha^{i}_{m}: + \sum_{r\in\mathbb{Z}+\frac{1}{2}} r : b^{i}_{m-r} b^{i}_{r} : -2\, a_{NS}\,\delta_{n} \right) \tag{43}$$

$$b^{-}_{r} = \frac{1}{p^{+}} \sum_{s=-\infty}^{\infty} \alpha^{i}_{r-s} b^{i}_{s} \tag{44}$$

where normal ordering is introduced in order to avoid infinities. The p^μ are given in terms of the zero modes ($p^\mu = 2\alpha_0^\mu$) and thus the formula of α_m^- becomes (in the NS-sector)

$$p^+\alpha_0^- = 2\sum_{m>0}\alpha_{-m}^i\alpha_m^i + (\alpha_0^i)^2 + 2\sum_{r=\frac{1}{2}} r b_r^i b_r^i - 2\,a_{NS}$$

$$\frac{1}{2}p^+p^- = \frac{1}{4}(p^i)^2 = 2(N - a_{NS}).$$

Now we define the mass-operator as usual

$$m^2 := -p^\mu p_\mu = 2p^+p^- - p^ip^i = 8(N_{NS} - a_{NS}). \tag{45}$$

a_{NS} is a normal ordering constant which we have mentioned already. For the R-sector an analogous formula can be derived (just replace NS by R).

In the R-sector the zero modes form a target space Clifford Algebra

$$\{d_0^\mu, d_0^\nu\} = \eta^{\mu\nu}. \tag{46}$$

Therefore after the definition $\Gamma^\mu := \sqrt{2}d_0^\mu$ the Γ^μ satisfy the usual Clifford Algebra $\{\Gamma^\mu, \Gamma^\nu\} = 2\eta^{\mu\nu}$.

By pairing the left and right movers (solutions corresponding to σ^+, σ^-) together it becomes obvious that the solutions can lie in either one of the **four sectors: NSNS, NSR, RNS, RR**.

4.1 The Critical Dimension

To interprete the states as particles they must be elements of the representation space of an irreducible representation of the little group of the target space Lorentz group $SO(d-1,1)$. We just consider the NS-sector and show that there arises a condition for the dimension of the target space (the argumentation for the R-sector is the same and leads to the same result).
The first excited state is

$$b_{-\frac{1}{2}}^i \tilde{b}_{-\frac{1}{2}}^j |k\rangle.$$

Its target space index structure (i runs from $1, ..., d-1$) shows that it must be a vector under representations of $SO(d-2)$. $SO(d-2)$ is the little group of $SO(d-1,1)$ for massless particles, hence the first excited state must be massless:

$$0 = m^2 = 8(N_{NS} - a_{NS}) = 8(\frac{1}{2} - a_{NS}).$$

Thus

$$a_{NS} = \frac{1}{2}. \tag{47}$$

Now we demand that naturally the ordering in quantum expressions would be symmetric and write for the number operator, instead of of the above given formula

$$N_{NS} - a_{NS} = \frac{1}{2}\left(\sum_{n=-\infty, n\neq 0}^{\infty} \alpha^i_{-n}\alpha^i_n + \sum_{r\in\mathbb{Z}+\frac{1}{2}} r b^i_{-r} b^i_r\right).$$

By comparing this with the old result

$$N_{NS} - a_{NS} = \left(\sum_{n=1}^{\infty} \alpha^i_{-n}\alpha^i_n + \sum_{r=\frac{1}{2}}^{\infty} r b^i_{-r} b^i_r\right)$$

$$= \frac{1}{2}\left(\sum_{n=1}^{\infty} \alpha^i_{-n}\alpha^i_n + \sum_{r=\frac{1}{2}}^{\infty} r b^i_{-r} b^i_r\right) + \frac{1}{2}\left(\sum_{n=1}^{\infty} \alpha^i_{n}\alpha^i_{-n} - \sum_{r=\frac{1}{2}}^{\infty} r b^i_{r} b^i_{-r}\right)$$

$$= N_{NS} + \frac{1}{2}\sum_{i=2}^{d-1}\left(\sum_{n=1}^{\infty} n - \sum_{r=\frac{1}{2}}^{\infty} r\right)\eta^{ii} = N_{NS} + \frac{1}{2}(d-2)\left(\sum_{n=1}^{\infty} n - \sum_{r=\frac{1}{2}}^{\infty} r\right),$$

and by using a tricky procedure, namely $\sum_{n=0}^{\infty} n = -\frac{1}{12}$ (called *zeta-function regularization*) we can assign a simple number to the infinite sum

$$\alpha_{NS} = -\frac{d-2}{2}\left(-\frac{1}{12} - \sum_{n=0}^{\infty}(n - \frac{1}{2})\right)$$

we find the condition for α_{NS}:

$$\alpha_{NS} = -\frac{d-2}{16}. \tag{48}$$

In the last step we performed again the zeta-function regularization $\sum_{n=0}^{\infty}(n+c) = \zeta(-1,c) = -\frac{1}{2}(6c^2 - 6c + 1)$ with. Therefore, due to (47) we find

$$d = 10. \tag{49}$$

Here we just showed that in the case of $d = 10$ the assignment state - particle is consistent. There are more rigid calculations due to the Lorentz Algebra of the target space that give the same result.

5 The Spectrum of the Supersymmetric String

Now we have to discuss the spectra of the R- and the NS-sector separately and afterwards combine the states to get the general solutions. In the NS-sector we will find a unique ground-state $|k\rangle$, while in the R-sector it will be degenerate, a fact that will have further consequences. We also will perform a certain projection to get rid of the unwanted states.

First we turn to the **NS-sector**:

In the NS-sector there are no fermionic zero modes, the vacuum is always taken to be an eigenstate of the bosonic zero modes (the eigenvalues are the target space momenta of the state k). Thus it is denoted by $|k\rangle$. The mass of the vacuum is found to be $m^2 = -8$, thus there is again a tachyon in the spectrum. Because acting with the oscillator modes won‘t change the target space tensor structure of the states the NS-spectrum is like the spectrum of the bosonic string and the states can be used to describe target space bosons. The first excited state $b^i_{-\frac{1}{2}}|k\rangle$ is a massless one, $m^2 = 0$, it corresponds to the vector representation of $SO(8)$ (because i runs from $1, ...d-1$). The next states are given by $b^i_{\frac{1}{2}} b^j_{\frac{1}{2}}|k\rangle$ with mass $m^2 = 8$. It can be shown that these and the following, which are all tensors of $SO(8)$ combines uniquely to tensors of $SO(9)$, the little group of massive states in 10 dimensions.

To project the tachyon out we use the *GSO-projection*, a method which was first proposed by Gliozzi, Scherk and Olive. This method can be motivated for different reasons we will not discuss here. For this purpose we introduce a fermionic number operator $F(\tilde{F})$ which counts the number of the right- (left-) handed worldsheet fermionic creation operators $b^i_r(\tilde{b}^i_r)$ of each state. For the vacuum we set $F(\tilde{F}) = 1$, so that F can be written as $F = 1 + \sum_{r>0} b^i_{-r} b^i_r$. Then the GSO-projection operator for the NS-sector is defined to be

$$P_{GSO} = \frac{1 + (-1)^F}{2} \cdot \frac{1 + (-1)^{\tilde{F}}}{2}. \tag{50}$$

The projection now is performed by multiply each state with this operator. From the formula it is obvious that the states with odd fermionic numbers are removed, only those with integer $F(\tilde{F})$ are left. Luckily we find that the tachyon ($F = 1$) is removed. The first excited state ($F = 2$) stays, and it is easy to continue this projection. In the end half of the states are removed.

We carry on with the discussion of the **Ramond-sector**:

From the spectrum of the Ramond-sector finally we will receive the target space spinors, one of the main motivations for the Superstring. It will turn out most important to discuss the zero modes d^i_0 which form a target space Clifford Algebra and thus must be proportional to the target space Γ-matrices (we have showed this already). Because we are in light-cone gauge there are 8 d^i_r-operators for the right- and 8 for the left-moving sector each. The d^μ_0 do not change the mass of a state (they do not appear in the mass-operator), thus all the $d^\mu_0|k\rangle$ have the same mass. Now we rearrange them:

$$D_1 = d_0^2 + i d_0^3 \tag{51}$$
$$D_2 = d_0^4 + i d_0^5 \tag{52}$$
$$D_3 = d_0^6 + i d_0^7 \tag{53}$$
$$D_4 = d_0^8 + i d_0^9. \tag{54}$$

Then the only non-vanishing anticommutator of the D_I´s is

$$\{D_I, D_I^\dagger\} = 2, \tag{55}$$

which can easily be seen:

$$\{D_1, D_1^\dagger\} = \{d_0^2 + i d_0^3, d_0^2 + i d_0^3\} = \{d_0^2, d_0^2\} - i(\{d_0^2, d_0^3\} - \{d_0^3, d_0^2\}) + \{d_0^3, d_0^3\}$$
$$= 1 - i(0-0) + 1 = 2.$$

Thus the D_I are nilpotent, i.e. $(D_I^2) = 0$. By the above definition we changed the 8 real into 4 complex modes. We now choose the R-vacuum $|----\rangle$ such as

$$D_I|----\rangle = 0 \qquad \forall I = 1, ..., 4. \tag{56}$$

Due to the above anticommutation relation the $D_I^\dagger$ act as creation operators

$$D_3^\dagger|----\rangle = |--+-\rangle. \tag{57}$$

Because of the nilpotency one finds $D_3^\dagger|--+-\rangle = D_3^\dagger D_3^\dagger|----\rangle = 0$. It is for that reason that the vacuum is 16-fold degenerated. Therefore it is an on-shell *Majorana spinor* in ten dimensions (which has $\frac{1}{2}\cdot 2^{\frac{10}{2}} = 16$ components). For the left movers the calculation is just the same, only with d replaced by $\tilde{d}$. In general the described proceeding is an option to construct (massless) spinor representations when the d_0^i can be identified with the target space Γ-matrices.

Now we will perform the GSO-projection in the R-sector. Therefore we define the operator

$$(-1)^F := 2^4 d_0^2 d_0^3 d_0^4 d_0^5 d_0^6 d_0^7 d_0^8 d_0^9 (-1)^{\tilde{N}} \tag{58}$$

with $\tilde{N} = \sum_{n>0} d_{-n}^i d_n^i$ as the product of all 8-dim. target space Γ-matrices. This is the usual definition of the chirality operator for the ground states. The projection operator is now defined as

$$P_{GSO}^{\pm} := \frac{1 \pm (-1)^F}{2} \tag{59}$$

and to carry out the GSO-projection in the R-sector each state is multiplied by it. For the left-movers the proceeding is just the same.

Now the left and right movers are combined. Obviously there are two possibilities of performing the above projection according to the different signs $\pm$. The projection removes either all states with even or those with odd chirality and the 16-component Majorana spinor becomes an 8-component *Weyl-spinor*. The two different options of projection lead to two different kinds of strings dependent on whether one choses an equal or an opposite sign for the left and right movers :

- type IIA strings: opposite sign
- type IIB strings: same sign.

To get the complete states of the Superstring Theory the states of the different sectors (NS,R) have to be tensored together. Then they can be interpreted as particles. In the following we discuss only the lowest excitation level, i.e. the massless case. There are 4 options to do so:

NSNS The states describe bosons.

NSR The lowest allowed state is $b^i_{-\frac{1}{2}}|k\rangle u_\alpha$, where u_α is an 8 component Majorana-Weyl spinor. Thus there are $8 \times 8 = 64$ possible combinations of the vector- and spinor-components. Those 64 different states decompose into an eight- and a 56-dimensional representation of the target space little group SO(8). The spinors, corresponding to the 56 dimensional representation can be seen to describe a gravitino of fixed chirality, those corresponding to the eight-dimensional one a dilatino of fixed chirality.

RNS An analogous discussion as in the previous paragraph yields again a gravitino and a dilatino for the lowest states.

RR The ground state here is constructed by combining the left and the right moving vacuum. It therefore has 64 components (analogous to the above case). The decomposition into irreducible representations of SO(8) belongs to whether we chose type 2A or type 2B strings.

In the case of *type IIA* we have an eight-dimensional representation which corresponds to an U(1) one-form gauge potential and a 56-dimensional representation, a three-form gauge potential $C_{\mu\nu\rho}$.

By considering the *type IIB* string we find an one-dimensional representation, a zero-form Φ', a 28-dimensional (a two-form $B'_{\mu\nu}$) and a 35-dimensional one, which can be seen as a four-form gauge potential with self-dual field strength $C^*_{\mu\nu\rho\sigma}$.

6 Appendix A: Results of the Bosonic String Theory

This Appendix shall not be more than a brief summary of the main results of the Bosonic String Theory which are used in the previous article. Therefore the reader should apologize any shortcomings.

The *bosonic string action* in light cone gauge is given by

$$S = \frac{1}{2\pi\alpha'} \int \mathrm{d}\sigma^+ \mathrm{d}\sigma^- \partial_- X^\mu \partial_+ X_\mu \tag{60}$$

with α', the Regge slope parameter being of dimension (lenght)^2 or (mass)^2. The equations of motion follow from a variational principle: $\delta S = 0$ by performing the variation $X^\mu(\sigma^-, \sigma^+) \longrightarrow X^\mu(\sigma^-, \sigma^+) + \delta X^\mu(\sigma^-, \sigma^+)$.

$$\begin{aligned}
0 &= \delta S \\
&= \frac{1}{2\pi\alpha'} \int \mathrm{d}\sigma^+ \mathrm{d}\sigma^- (\partial_- X^\mu \delta(\partial_+ X^\mu) + \partial_+ X^\mu \delta(\partial_- X^\mu)) \\
&= -\frac{1}{\pi\alpha'} \int \mathrm{d}\sigma^+ \mathrm{d}\sigma^- (\partial_+ \partial_- X^\mu) \delta X^\mu + (\text{boundary terms})
\end{aligned}$$

The boundary terms, caused by the partial integration vanish because of the boundary conditions of the variation and the equations of motion are:

$$\partial_+ \partial_- X^\mu = 0 \tag{61}$$

By varying the action with respect to the world-sheet metric $h^{\alpha\beta}$ one yields the *energy-momentum tensor* which then can be transformed into light-cone coordinates. The result for its components in light-cone coordinates is

$$\begin{aligned}
T_{++} &= \frac{1}{2} \partial_+ X^\mu \partial_+ X_\mu \\
T_{--} &= \frac{1}{2} \partial_- X^\mu \partial_- X_\mu \\
T_{+-} &= 0 \\
T_{-+} &= 0.
\end{aligned}$$

Now one supplements the equations of motion by the costrained equations $T_{++} = T_{--} = 0$. There is still a residual gauge freedom left, i.e. the earlier mentioned equivalent form of the action is invariant under the transformation $\tau \longrightarrow \frac{1}{2}(\tilde{\sigma}^+(\sigma^+) + \tilde{\sigma}^-(\sigma^-))$. Thus τ can be chosen to be a solution of the equation $\partial_+ \partial_- \tau = 0$. Because it is a solution of the same equation, $X^+ = \frac{1}{\sqrt{2}}(X^0 + X^1)$ can be fixed:

$$X^+ = x^+ + p^+ \tau, \tag{62}$$

where x^+ and p^+ are just integration constants, denoting the center of mass position and momentum of the string in the (+) - direction.

With help of the constraint equations X^- can be expressed in terms of the X^i, $(i = 2, ..., d-1)$, with d the number of bosonic fields. Thus finally we are left with only $d-2$ physical degrees of freedom, $d-2$ free fields.

The general form of the solutions of the equations of motion is

$$X^\mu = X^\mu_R(\sigma^-) + X^\mu_L(\sigma^+), \tag{63}$$

with the left and right moving solutions

$$X^\mu_R = \frac{1}{2}x^\mu + \frac{1}{2}p^\mu\sigma^- + \frac{i}{2}\sum_{n\neq 0}\frac{1}{n}\alpha^\mu_n \exp^{-2in\sigma^-} \tag{64}$$

$$X^\mu_L = \frac{1}{2}x^\mu + \frac{1}{2}p^\mu\sigma^+ + \frac{i}{2}\sum_{n\neq 0}\frac{1}{n}\tilde{\alpha}^\mu_n \exp^{-2in\sigma^+}. \tag{65}$$

With the definition $P^\mu_\tau := \frac{\partial L}{\partial(\partial_\tau X_\mu)} = T\partial_\tau X^\mu$ the Poisson Brackets are found to be

$$[X^\mu(\sigma), X^\nu(\sigma')]_{P.B.} = -i\pi\delta(\sigma-\sigma')\eta^{\mu\nu} \tag{66}$$

$$[X^\mu(\sigma), X^\nu(\sigma')]_{P.B.} = [X^\mu(\sigma), X^\nu(\sigma')]_{P.B.} = 0. \tag{67}$$

The bosonic string is now quantized by considering the functions X^μ as operators and performing the replacement

$$[\quad,\quad]_{P.B.} \longrightarrow \frac{1}{i}[\quad,\quad] \tag{68}$$

From the above Poisson Brackets one finds the commutators of the Fourier modes:

$$[\mathrm{p}^\mu, x^\mu] = -i\eta^{\mu\nu}, \quad [\alpha^\mu_n, \alpha^\nu_k] = n\delta_{n+k,0}\eta^{\mu\nu}, \quad [\tilde{\alpha}^\mu_n, \tilde{\alpha}^\nu_k] = n\delta_{n+k,0}\eta^{\mu\nu}. \tag{69}$$

If one defines $\alpha^\mu_m := \frac{1}{\sqrt{m}}\alpha^\mu_m$ and $\alpha^{\dagger\,\mu}_m := \frac{1}{\sqrt{m}}$ the α^μ satisfy the usual harmonic oszillator commutation relations

$$[\alpha^\mu_m, \alpha^{\dagger\,\nu}_n] = \delta_{m,n}\eta^{\mu\nu} \tag{70}$$

and therefore can be interpreted as creation ($m < 0$) and annihilation operators ($m > 0$) which are acting on a vacuum state and creates all states of the spectrum.

When considering the first excited state of the spectrum one finds that it must be massless otherwise it can´t be interpreted as a particle. By calculating the mass one finds that the dimension of the target space, i.e. the space-time in which the string is propagating, must be 26:

$$d = 26 \tag{71}$$

It follows that the lowest mass in the spectrum, the mass square of the ground state is negative. States with negative mass square are called tachyons, they are bad news because they indicates that the vacuum is incorrectly identified.

In addition there is a graviton in the spectrum, a massless particle with spin 2, a hint that the theory could give meaning to the concept of quantum gravity.

6.1 Appendix B: Poisson Brackets and Dirac Brackets

In general the *Poisson Brackets* of two functions $A(\Psi,\Pi), B(\Psi,\Pi)$ are defined as

$$[A,B]_{P.B.} = -(-1)^{\epsilon_A\epsilon_B}\left(\frac{\partial A}{\partial\Psi}\cdot\frac{\partial B}{\partial\Pi} + (-1)^{\epsilon_A\epsilon_B}\frac{\partial B}{\partial\Psi}\cdot\frac{\partial A}{\partial\Pi}\right), \tag{72}$$

where $\epsilon_A = 1$ for commuting variable A and $\epsilon = 0$ for anticommuting variables.

If there are *second class constraints* Φ_i, Φ_j whose Poisson Brackets do not vanish on the constraint hypersurface of phase space, the Poisson Brackets have to be replaced by *Dirac Brackets*. If the Φ_i form a complete set of second class constraints we define $[\Phi_i,\Phi_j]_{P.B.} := C_{ij}$ and the Dirac Brackets

$$[A,B]_{D.B.} := [A,B]_{P.B.} - [A,\Phi_i]_{P.B.}C^{-1}_{ij}[\Phi_j,B]_{P.B.}. \tag{73}$$