Physikalisches Institut
University Bonn
Theoretische Physik

Exercise 4
Nov 11,13 2003
WS 03/04

Exercises on Elementary Particle Physics

Prof. Dr. H.-P. Nilles

1. *Electron-Muon Scattering*

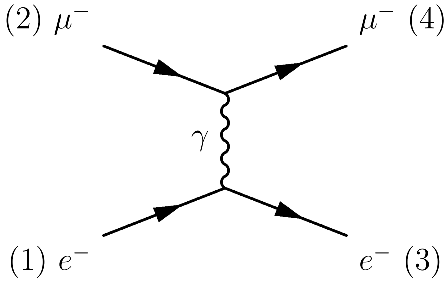

Figure 1: Feynman graph. Time goes from left to right.

We present the Feynman rules to calculate the amplitude $-\mathrm{i}\mathcal{M}$ in QED.

i) An arrow in the direction of time denotes a particle, an arrow in the opposite direction denotes an antiparticle. Assign a label i to each external particle. Assign momenta to each particle (including the internal lines) and indicate them by arrows along the particle lines.

ii) Proceed "backward" along each fermion line. For a particle, proceeding backward means "opposite to the direction of the arrow". For an antiparticle, proceeding backward means "in the direction of the arrow".

iii) Write a factor (Dirac spinor) $u(i)$ for every line which goes into the vertex, and a factor $\bar{u}(i)$ for every line which points away from the vertex.

iv) Vertices and internal lines (propagators) contribute as follows:

$$\text{Vertices:} \quad \mathrm{i}g_e\gamma^\mu$$

$$\text{Fermions:} \quad \mathrm{i}\frac{\not{q}+m}{q^2-m^2}$$

$$\text{Photons:} \quad -\mathrm{i}\frac{g_{\mu\nu}}{q^2}$$

The indices of the γ's are contracted with the $g_{\mu\nu}$ of the photon propagator. The coupling constant is $g_e = \sqrt{4\pi\alpha}$. In Heaviside-Lorentz units, $g_e = e$.

v) Use 4-momentum conservation at the vertices to eliminate the internal momenta.

In the lab frame where the particle B is initially at rest and is assumed to be so heavy that recoil effects are negligible, the **differential cross section** for the process $AB \to AB$ is given by

$$\frac{d\sigma}{d\Omega} = \frac{1}{64\pi^2 m_B^2}|\mathcal{M}|^2. \tag{1}$$

(a) Using the Feynman rules for QED, derive

$$\mathcal{M} = -\frac{g_e^2}{(p_1 - p_3)^2}\left[\bar{u}(3)\gamma^\mu u(1)\right]\left[\bar{u}(4)\gamma_\mu u(2)\right]$$

for the electron-muon scattering amplitude.

(b) To calculate the cross section, we need to know $|\mathcal{M}|^2 = \mathcal{M}\mathcal{M}^*$. Using the identities

$$\bar{u} = u^\dagger\gamma^0, \quad \gamma^{0\dagger} = \gamma^0, \quad \gamma^0\gamma^\mu = \gamma^{\mu\dagger}\gamma^0, \quad (\gamma^0)^2 = \mathbb{1},$$

show that

$$|\mathcal{M}|^2 = \frac{g_e^4}{(p_1 - p_3)^4}\big[\bar{u}(3)\gamma^\mu u(1)\bar{u}(1)\gamma^\nu u(3)\big]\big[\bar{u}(4)\gamma_\mu u(2)\bar{u}(2)\gamma_\nu u(4)\big]. \tag{2}$$

(c) In a typical experiment, the particle beam is unpolarized and the detectors simply count the number of particles scattered in a given direction. Therefore, we have to *avarage* over initial spins and *sum* over final spins.

The avaraging over the initial spins is easy: It contributes a factor of 1/2 for each sum.

Using the completeness relation for Dirac spinors,

$$\sum_s u_a(p,s)\bar{u}_b(p,s) = (\not{p} + m\mathbb{1})_{ab},$$

(a, b spinor indices) show that the summation over the final spins for the first factor in eq. (2) can be written as

$$\sum_{s_1,s_3}\bar{u}(p_3,s_3)\gamma^\mu u(p_1,s_1)\bar{u}(p_1,s_1)\gamma^\mu u(p_3,s_3) = \mathrm{Tr}\big[(\not{p}_3 + m_e)\gamma^\mu\big(\not{p}_1 + m_e\big)\gamma^\nu\big].$$

By relabeling, derive the analogous result for the second factor in eq. (2). The final result is

$$\frac{1}{4}\sum_{s_1,s_2,s_3,s_4}|\mathcal{M}|^2 = \frac{1}{4}\frac{g_e^4}{(p_1 - p_3)^4}\mathrm{Tr}\big[(\not{p}_3+m_e)\gamma^\mu\big(\not{p}_1+m_e\big)\gamma^\nu\big]\ \mathrm{Tr}\big[(\not{p}_4+m_\mu)\gamma_\mu\big(\not{p}_2+m_\mu\big)\gamma_\nu\big]. \tag{3}$$

Note that the problem of calculating the cross section has been reduced to matrix multiplication and taking the trace!

(d) To deal with the above expression, we need some efficient techniques to calculate the trace of products of gamma matrices. Prove the following identities:

i) $\mathrm{Tr}(\gamma^\mu) = 0$.

ii) $\mathrm{Tr}(\gamma^\mu \gamma^\nu) = 4g^{\mu\nu}$
Hint: Use $\{\gamma^\mu, \gamma^\mu\} = 2g^{\mu\nu}\mathbb{1}$ and the fact that the trace is cyclic.

iii) $\mathrm{Tr}(\gamma^\mu \gamma^\nu \gamma^\rho) = 0$
Hint: Use $(\gamma_5)^2 = \mathbb{1}$, $\{\gamma_5, \gamma^\mu\} = 0$ and the fact that the trace is cyclic.

iv) $\mathrm{Tr}(\gamma^\mu \gamma^\nu \gamma^\lambda \gamma^\sigma) = 4(g^{\mu\nu} g^{\lambda\sigma} - g^{\mu\lambda} g^{\nu\sigma} + g^{\mu\sigma} g^{\nu\lambda})$.

(e) Consider the first trace in eq. (3). Using the identities proved in (1d), derive

$$\mathrm{Tr}\big[(\not{p}_3 + m_e)\gamma^\mu \big(\not{p}_1 + m_e)\gamma^\nu\big] = 4(p_1^\mu p_3^\nu + p_1^\nu p_3^\mu - g^{\mu\nu} p_1 \cdot p_3 + g^{\mu\nu} m_e^2).$$

(f) By relabeling, derive the result for the second trace in eq. (3). Substitute your results in eq. (3) and show that it takes the following form:

$$\langle |\mathcal{M}|^2 \rangle = \frac{4g_e^4}{(p_1 - p_3)^4}(p_1^\mu p_3^\nu + p_1^\nu p_3^\mu - g^{\mu\nu} p_1 \cdot p_3 + g^{\mu\nu} m_e^2)(p_{2\mu} p_{4\nu} + p_{2\nu} p_{4\mu} - g_{\mu\nu} p_2 \cdot p_4 + g_{\mu\nu} m_\mu^2).$$

Now expand the brackets and contract the indices. Show that the result is:

$$\langle |\mathcal{M}|^2 \rangle = \frac{8g_e^4}{(p_1 - p_3)^4}\big((p_1 \cdot p_2)(p_3 \cdot p_4) + (p_1 \cdot p_4)(p_2 \cdot p_3) - m_\mu^2 (p_1 \cdot p_3) - m_e^2 (p_2 \cdot p_4) + 2m_e^2 m_\mu^2\big)$$

(g) So far everything is written covariantly and is independent of a special coordinate frame. To make contact with measurements, we specialize to the rest system of the muon and make an approximation as $m_\mu \gg m_e$. Denote by $p := |\vec{p_1}|$ the absolute value of the initial electron momentum. Denote by θ the angle between $\vec{p_1}$ and $\vec{p_3}$.

Draw 2 diagrams, one before the scattering process and one after the scattering process. Write the 4-momenta under the respective diagrams, taking into account the approximation we have made. Show that in the approximation $m_\mu \gg m_e$, conservation of energy-momentum gives $|\vec{p_3}| = |\vec{p_1}| = p$. Prove the following identities:

$$(p_1 - p_3)^2 = -4p^2 \sin^2 \frac{\theta}{2}, \quad p_1 \cdot p_3 = m_e^2 + 2p^2 \sin^2 \frac{\theta}{2},$$
$$(p_1 \cdot p_2)(p_3 \cdot p_4) = E^2 m_\mu^2, \quad p_2 \cdot p_4 = m_\mu^2.$$

(h) Insert the above expressions into eq. (1) for the cross section to obtain the **Mott formula**

$$\frac{d\sigma}{d\Omega} = \frac{1}{64\pi^2}\frac{g_e^4}{p^4 \sin^4 \theta/2}\left[m_e^2 + p^2 \cos^2 \theta/2\right]$$

which reduces to the **Rutherford formula** in the non-relativistic limit $p \ll m_e$.

2. *Spontaneous Symmmetry Breaking in the Linear Sigma Model*

As an application of spontaneous symmetry breaking, we want to have a look at the linear sigma model which consists of N real scalar fields with the Lagrangian

$$\mathcal{L} = \sum_i \left(\frac{1}{2} \partial_\mu \phi^i \partial^\mu \phi^i + \frac{1}{2} \mu^2 \phi^i \phi^i - \frac{\lambda}{4} \left(\phi^i \phi^i \right)^2 \right), \quad i = 1, .., N.$$

(a) Let us find the symmetry group of the Lagrangian: We transform the fields $\phi^i \mapsto R^{ij} \phi^j$. What kind of matrices R are allowed such that $\mathcal{L}$ remains invariant?

(b) Find the minimum ϕ_0^i of the potential. You will find that the minimum is any ϕ_0^i that fulfills

$$\sum_i \phi_0^i \phi_0^i = \frac{\mu^2}{\lambda}.$$

This condition determines only the length of the "vector" ϕ_0^i. We choose coordinates such that ϕ_0^i points into the N-th direction:

$$\phi_0^i(x) = (0, 0, \ldots, 0, v), \quad v = \frac{\mu}{\sqrt{\lambda}}.$$

(c) Now we break the symmetry by defining a set of shifted fields

$$\phi^i(x) := \left(\pi^k(x), v + \sigma(x) \right), \quad k = 1, .., N-1.$$

Rewrite the Lagrangian in terms of the π and σ fields. The result is

$$\begin{aligned} L &= \frac{1}{2} \partial_\mu \sigma \partial^\mu \sigma - \mu^2 \sigma^2 - \sqrt{\lambda} \mu \sigma^3 - \frac{\lambda}{4} \sigma^4 \\ &\quad + \sum_k \left(\frac{1}{2} \partial_\mu \pi^k \partial^\mu \pi^k - \sqrt{\lambda} \mu \sigma \pi^k \pi^k - \frac{\lambda}{2} \sigma^2 \pi^k \pi^k - \frac{\lambda}{4} \left(\pi^k \pi^k \right)^2 \right). \end{aligned}$$

(d) Have a look at the system after spontaneous symmetry breaking. How many massive and massless fields are there now? What is the symmetry of the new Lagrangian? Compare your result to *Goldstone's Theorem* which says that for every spontaneously broken continuous symmetry, the theory must contain a massless particle.