Physikalisches Institut
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Theoretische Physik

Exercise 7
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WS 03/04

Exercises on Elementary Particle Physics

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1. *Renormalization of φ^4 Theory*

   The Lagrangian for φ^4 theory is

$$\mathcal{L} = \frac{1}{2}\,\partial_\mu\varphi_0\,\partial^\mu\varphi_0 - \frac{1}{2}\,m_0^2\,\varphi_0^2 - \frac{1}{4!}\,\lambda_0\,\varphi_0^4. \tag{1}$$

   We write ϕ_0, m_0, λ_0 to emphasize that these are the "bare" wave function, mass, and coupling, respectively, and not the values measured in experiments.

   Define a renormalized field φ by

$$\varphi_0 = Z^{1/2}\varphi.$$

   Substituting the renormalized field into eq. (1) for the bare Lagrangian gives

$$\mathcal{L} = \frac{1}{2}\,Z\,\partial_\mu\varphi\,\partial^\mu\varphi - \frac{1}{2}\,m_0^2\,Z\,\varphi^2 - \frac{1}{4!}\,\lambda_0\,Z^2\,\varphi^4. \tag{2}$$

   (a) Show that by introducing the so-called *counterterms*

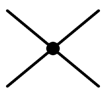
$$\delta_Z = Z - 1, \qquad \delta_m = m_0^2 Z - m^2, \qquad \delta_\lambda = \lambda_0 Z^2 - \lambda,$$

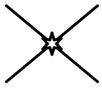
   the Lagrangian in eq. (2) can be rewritten as

$$\mathcal{L} = \frac{1}{2}\,\partial_\mu\varphi\,\partial^\mu\varphi - \frac{1}{2}\,m^2\,\varphi^2 - \frac{1}{4!}\,\lambda\,\varphi^4 + \frac{1}{2}\delta_Z\partial_\mu\varphi\,\partial^\mu\varphi - \frac{1}{2}\delta_m\varphi^2 - \frac{\delta_\lambda}{4!}\varphi^4. \tag{3}$$

   Note that the bare mass m_0 and the bare coupling λ_0 have been "eliminated".

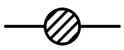
   The new terms which appear in the Lagrangian eq. (3) lead to new Feynman rules:

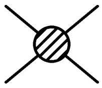
 $\sim \quad \frac{i}{p^2-m^2+i\epsilon}$  $\sim \quad -i\lambda$

 $\sim \quad i(p^2\delta_Z - \delta_m)$  $\sim \quad -i\delta_\lambda$

At the cost of introducing counterterms (which modify the Feynman rules of the original theory), we have rewritten the Lagrangian in terms of renormalized quantities which are the physically measured parameters of the theory. The counterterms have absorbed the infinite, but unobservable shifts between the bare and physical parameters.

Now we have to give a precise definition of the physical mass and the physical coupling constant. Define m^2 to be the pole of the propagator, and λ to be the magnitude of the scattering amplitude at zero momentum:

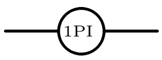
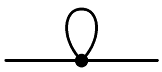
 $$\stackrel{!}{=} \frac{i}{p^2 - m^2} + \text{ terms regular at } p^2 = m^2 \tag{4}$$

 $$\stackrel{!}{=} -i\lambda \text{ at } s = 4m^2, t = u = 0 \tag{5}$$

These equations are called *renormalization conditions.*

Using the first renormalization condition, we will now determine δ_Z and δ_m.

(b) The "basic" 1-loop correction to the propagator will now involve 2 graphs, since we have new Feynman rules:

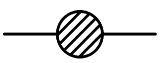
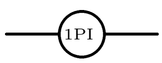
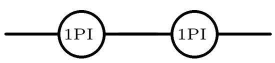
$-iM(p^2) \equiv$  $=$  $+$  (6)

Show that the 1PI is given by

$$-iM^2(p^2) = -\frac{i\lambda}{2}\, I(d, m) + i(p^2\delta_Z - \delta_m), \tag{7}$$

where $I(d, m)$ is the integral calculated in exercise (1g) on the previous exercise sheet. The 1/2 is a *symmetry factor* as will be explained in class.

(c) Show that the full 2-point function at 1-loop is given by

 $\equiv$ $+$  $+$  $+ \cdots$

$$= \frac{i}{p^2 - m^2 - M(p^2)}.$$

Hint: Write down the amplitudes of the graphs on the right hand side, factor out $i/(p^2 - m^2)$, then use the formula for the geometric series.

(d) Use the renormalization condition eq. (4) to derive

$$M^2(p^2) = 0 \text{ at } p^2 = m^2, \qquad \frac{d}{dp^2} M^2(p^2) = 0 \text{ at } p^2 = m^2. \tag{8}$$

Hint: The first equation is trivial to prove. The second one follows from the fact that the residue of the expression in eq. (4) is 1.

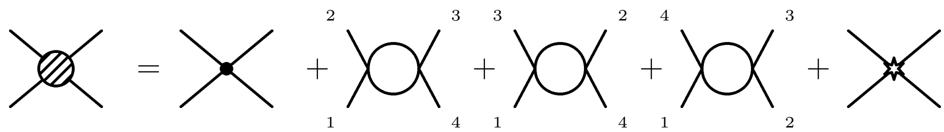
(e) Using the explicit expression for $M^2(p^2)$ which you calculated in eq. (7), verify that

$$\delta_Z = 0, \qquad \delta_m = -\frac{\lambda}{2}\, I(d,m)$$

satisfy the renormalization conditions as given in eq. (8).

Using the second renormalization condition, we will now determine δ_λ.

(f) Consider the basic 2-particle scattering amplitude $i\Lambda(p_1p_2 \to p_3p_4)$:



(9)

By convention, take all momenta as flowing into the vertex. Indicate this convention by drawing the arrows corresponding to momentum flow in the above diagrams. Explain why these are the graphs to be included. As to the loop graphs, consider a permutation of 1,2,3,4 not included in eq. (9), and show that it turns out to be equal to one of the 3 graphs given above.

(g) Introduce the Mandelstam variables

$$s = (p_1 + p_2)^2, \quad t = (p_1 - p_3)^2, \quad u = (p_1 - p_4)^2,$$

where p_1, p_2 flow into the vertex and p_3, p_4 flow out of the vertex. Show that the sum of the incoming momenta of the 3 loop graphs in eq. (9) is s, t, u, respectively.

(h) Show that the scattering amplitude eq. (9) is given by

$$i\Lambda(p_1p_2 \to p_3p_4) = -i\lambda - \frac{1}{2}(-i\lambda)^2 \left[J(s) + J(t) + J(u)\right] - i\delta_\lambda, \tag{10}$$

where $J(p^2)$ is the integral calculated in exercise (1ℓ) on the previous exercise sheet. The 1/2 in front of the J's is a *symmetry factor* as will be explained in class.

(i) Use the renormalization condition eq. (5) to derive

$$\delta_\lambda = -\frac{i\lambda^2}{2}\left[J(4m^2) + 2J(0)\right].$$

(j) *Crosscheck*

Insert the expression for δ_λ into eq. (10) for the physical scattering amplitude to see that it is indeed finite.

2. *Charge Conjugation, Parity & Time Reversal on Spinors*

Under parity, charge conjugation and time reversal a Dirac field ψ transforms as

$$\begin{aligned}
\mathbf{P}\,\psi(x)\,\mathbf{P}^{-1} &= (\eta_P\,P)\;\psi(x_P) \qquad & x_P^\mu = (x_0, -\mathbf{x}) \\
\mathbf{C}\,\psi(x)\,\mathbf{C}^{-1} &= (\eta_C\,C)\;\bar{\psi}(x)^T & \\
\mathbf{T}\,\psi(x)\,\mathbf{T}^{-1} &= (\eta_T\,T)\;\psi(x_T) \qquad & x_T^\mu = (-x_0, \mathbf{x}),
\end{aligned}$$

where $\mathbf{P}$, $\mathbf{C}$, $\mathbf{T}$ are the linear, linear and antilinear operators that implement these operations on Dirac spinors.

(a) To find the parity transformation P, we remember that a Lorentz transformation acts on Dirac spinors as $\psi \mapsto D(\Lambda)\,\psi$ (exercises 2) where $D(\Lambda)$ fulfills

$$D(\Lambda)^{-1}\gamma^\mu D(\Lambda) = \Lambda^\mu{}_\nu\gamma^\nu$$

in order to be consistent with the Dirac equation. Show that an explicit representation of the parity operation in terms of γ matrices is $D = \eta_P\,\gamma^0$, where η_P is a complex phase of modulus one.
(Remark: In the chiral representation the operator γ^0 exchanges the left- and righthanded Weyl spinors ψ_L, ψ_R in ψ in agreement with the definition of parity in exercise 2.3.a)

(b) For the charge conjugation C take the Dirac equations for a particle ψ and its antiparticle ψ^c

$$\begin{aligned}
\big(i\partial\!\!\!/ - eA\!\!\!/ - m\big)\,\psi &= 0 \quad \text{particle} \\
\big(i\partial\!\!\!/ + eA\!\!\!/ - m\big)\,\psi^c &= 0 \quad \text{antiparticle}
\end{aligned}$$

To find the operator that transforms $\psi \mapsto \psi^c$, conjugate ("bar") and transpose the first equation to arrive at

$$\big[\gamma^{\mu T}(-i\partial_\mu - eA_\mu) - m\big]\,\bar{\psi}^T = 0.$$

Show that if we find an operator C such that $C\gamma^{\mu T}C^{-1} = -\gamma^\mu$, we can identify the charge conjugate field as $\psi^c = (\eta_C\,C)\;\bar{\psi}^T$. Here η_C is again a complex phase of modulus one. Show that an explicit realisation of C in terms of γ matrices is possible: $C = i\gamma^0\gamma^2$. Note that the effect of the charge transformation is to reverse the internal quantum numbers (here only the $U(1)_Q$ charge).

(c) To find T, start again with the Dirac equation, perform a time reversal. Show that we need a matrix T that fulfills $T\gamma^{\mu *}T^{-1} = \gamma^{\mu\dagger}$ and that then the correctly transformed spinor is $\psi \mapsto \psi^t = (\eta_T\,T)\;\psi^*$ Show that an explicit realisation in terms of γ matrices is given by $T = i\gamma^5\gamma^0\gamma^2$.

(d) Show that, if X is a matrix acting on Dirac spinors,

$$\begin{aligned}
\mathbf{C}\,\bar{\psi}(x)X\psi(x)\,\mathbf{C}^{-1} &= \bar{\psi}(x)X_C\psi(x) \qquad X_C = CX^tC^{-1}, \\
\mathbf{T}\,\bar{\psi}(x)X\psi(x)\,\mathbf{T}^{-1} &= \bar{\psi}(x_T)X_T\psi(x_T) \quad X_T = T^{-1}X^*T, \\
\mathbf{P}\,\bar{\psi}(x)X\psi(x)\,\mathbf{P}^{-1} &= \bar{\psi}(x_P)X_P\psi(x_P) \quad X_P = \gamma^0X\gamma^0.
\end{aligned}$$

Hint: ψ and $\bar{\psi}$ anti-commute.

(e) Use the preceding exercise to determine the transformation properties of the bilinear covariants under parity, charge conjugation and time reversal .

$$(i)\ \bar{\psi}\psi \quad (ii)\ \bar{\psi}\,i\gamma_5\psi \quad (iii)\ \bar{\psi}\gamma^\mu\psi \quad (iv)\ \bar{\psi}\gamma^\mu\gamma_5\psi \quad (v)\ \bar{\psi}\,i[\gamma^\mu,\gamma^\nu]\psi$$