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Exercise 3
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Exercises on Elementary Particle Physics

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1. *Representations of su(2)*

A Lie algebra $\mathfrak{g}$ is a vector space together with a mapping

$$[\cdot,\cdot]\ :\ \mathfrak{g}\times\mathfrak{g}\rightarrow\mathfrak{g}$$

satisfying the following conditions:

(a) The mapping is bilinear (i.e. linear in both entries).

(b) The mapping is skew-symmetric: $[a,b]=-[b,a]$ for $a,b\in\mathfrak{g}$

(c) It fulfills the Jacoby identity: $[a,[b,c]]+[b,[c,a]]+[c,[a,b]]=0$ for $a,b,c\in\mathfrak{g}$

A representation ρ of a Lie algebra $\mathfrak{g}$ on a vector space V is a linear mapping

$$\rho:\mathfrak{g}\rightarrow\mathrm{End}(V)$$

which is an algebra homomorphism, i.e. $\rho([a,b])=[\rho(a),\rho(b)]$. The dimension of V is called the dimension of the representation: $\dim(\rho):=\dim(V)$.

If there is a vector spce $W\subset V$ so that $\rho(W)\subset W$, then the representation is called reducible and V is called the invariant subspace. If such a W does not exist, the representation is called irreducible.

In other words: a representation is irreducible, iff the only invariant subspace is V itself.

As an example, we will concentrate on the algebra $su(2)$ in the following.

(a) The group SU(2) is the set of all 2-dimensional unitary matrices with determinant 1. Show that the corresponding Lie algebra $su(2)$ is the set of all traceless hermitian matrices. Hint: $\det A=\exp\ \mathrm{Tr}\ \log A$.

(b) Choose the basis

$$\sigma_1=\begin{pmatrix}0&1\\1&0\end{pmatrix}\ ,\qquad\sigma_2=\begin{pmatrix}0&-i\\i&0\end{pmatrix}\ ,\qquad\sigma_3=\begin{pmatrix}1&0\\0&-1\end{pmatrix}$$

for the traceless hermitian matrices. Define

$$J_3 = \frac{1}{2}\sigma_3, \qquad J_+ = \frac{1}{2}\left(\sigma_1 + i\sigma_2\right), \qquad J_- = \frac{1}{2}\left(\sigma_1 - i\sigma_2\right),$$

and verify the commutation relations

$$[J_3, J_+] = J_+, \qquad [J_3, J_-] = -J_-, \qquad [J_+, J_-] = 2J_3 \ .$$

Next, we want to consider **all irreducible, finite-dimensional representations** of $su(2)$ on a vector space V, $\rho(J_i) \in \mathrm{End}(V)$, $i = 3, +, -$. We want to find out, how to classify these representations and which dimensions $\dim(V)$ are allowed.

(c) Since J_3 is diagonal, $\rho(J_3)$ can also be choosen to be diagonal. Therefore V can be decomposed into eigenspaces of $\rho(J_3)$,

$$V = \bigoplus V_\alpha,$$

where α labels the eigenvalue of $\rho(J_3)$, i.e.

$$(\rho(J_3))v = \alpha v, \qquad v \in V_\alpha, \quad \alpha \in \mathbb{C}$$

(shorthand: write J_i for $\rho(J_i)$). Show that $J_+(v) \in V_{\alpha+1}$ and $J_-(v) \in V_{\alpha-1}$

(d) Prove that all complex eigenvalues α which appear in the above decomposition differ from one another by 1.

Hint: Choose an arbitrary $\alpha_0 \in \mathbb{C}$ from the decomposition and prove that

$$\bigoplus_{k\in\mathbb{Z}} V_{\alpha_0+k} \subset V$$

is indeed equal to V using the irreducibility of the representation.

(e) Argue that there is $k \in \mathbb{N}$ for which $V_{\alpha_0+k} \neq 0$ and $V_{\alpha_0+k+1} = 0$. Define $n := \alpha_0 + k$. Note that up to now, we only know that $n \in \mathbb{C}$.

Draw a diagram. Write the vector spaces V_{n-2}, V_{n-1} and V_n in a row and indicate the action of J_3, J_+ and J_- on these vector spaces by arrows.

The eigenvalue n is called highest weight and a vector $v \in V_n$ is called highest weight vector. Is it clear why?

(f) Choose an arbitrary vector $v \in V_n$ (highest weight vector). Prove that the vectors $v, J_-v, J_-^2v, \ldots$ span V.

Hint: Show that the vector space spanned by these vectors is invariant under the action of J_3, J_+ and J_- and use the irreducibility of the representation.

(g) Argue that all eigenspaces V_α are 1-dimensional.

(h) Prove that n is a non-negative integer or half-integer and that

$$V = V_{-n} \oplus \ldots \oplus V_n \ .$$

Complement your diagram drawn in part (e). What is the dimension of the representation?

Hint: The representation is finite dimensional, so there exists $m \in \mathbb{N}$ for which $J_-^{m-1} v \neq 0$ and $J_-^m v = 0$. Evaluate the product $J_+ J_-^m v$.

(i) Consider the tensor product of a 2-dimensional and a 3-dimensional irreducible representation of $su(2)$:

$$V = V^{(2)} \otimes V^{(3)}$$

Show that the resulting representation V is reducible and that it can be decomposed into a 2-dim. and a 4-dim. irreducible representation. Shorthand: $\mathbf{2} \otimes \mathbf{3} = \mathbf{2} \oplus \mathbf{4}$.

Hint: The first thing to note is that the action of a Lie algebra on the tensor product of 2 representations is given by: $X(v \otimes w) = Xv \otimes w + v \otimes Xw$, i.e. the eigenvalue of J_3 on V is the sum of the eigenvalues of J_3 on $V^{(2)}$ and $V^{(3)}$. Draw the diagrams of the eigenvalues (with multiplicities). Then use the fact that the eigenspaces of irreducible representations are 1-dimensional.

2. *The Lorentz group*

The Lorentz group is defined as the set of transformations

$$x^\mu \longrightarrow \Lambda^\mu{}_\nu x^\nu$$

which leave the scalar product $\langle x, x \rangle := \eta_{\mu\nu} x^\mu x^\nu$ invariant.

(a) Show that an element λ of the Lie algebra of the Lorentz group satisfies:

$$\lambda^T = -\eta \lambda \eta$$

Hint: Reformulate the statement about the invariance of the scalar product in $\eta_{\mu\nu} = \eta_{\rho\sigma} \Lambda^\rho{}_\mu \Lambda^\sigma{}_\nu$ and write an element of the Lorentz group as $\Lambda^\mu{}_\nu \simeq \delta^\mu_\nu - i\lambda^\mu{}_\nu$.

(b) Choose

$$(M^{\mu\nu})^\rho{}_\sigma = i(\eta^{\mu\rho}\delta^\nu_\sigma - \eta^{\nu\rho}\delta^\mu_\sigma)$$

as a basis for the Lie algebra. What do these matrices look like? Describe the form of the matrices in words. Verify the commutation relations

$$[M^{\mu\nu}, M^{\rho\sigma}] = -i\left(\eta^{\mu\rho}M^{\nu\sigma} - \eta^{\mu\sigma}M^{\nu\rho} - \eta^{\nu\rho}M^{\mu\sigma} + \eta^{\nu\sigma}M^{\mu\rho}\right). \qquad (1)$$

(c) We split the generators into 2 groups:

$$J^i = \frac{1}{2}\epsilon^{ijk} M^{jk}, \quad K^i = M^{0i}.$$

The J's have only spatial indices, the K's have spatial and timelike indices. Verify the commutation relations

$$\left[J^i, J^j\right] = i\epsilon^{ijk} J^k, \quad \left[J^i, K^j\right] = i\epsilon^{ijk} K^k, \quad \left[K^i, K^j\right] = -i\epsilon^{ijk} J^k,$$

and describe the meaning of each relation in words. What kind of transformations do the J's and K's correspond to?

(d) The form of the commutation relations for the Lorentz algebra can still be simplified. Define

$$T^i_{L/R} = \frac{1}{2}\left(J^i \pm iK^i\right)$$

and verify the commutation relations

$$\left[T^i_L, T^j_L\right] = i\epsilon^{ijk} T^k_L, \quad \left[T^i_R, T^j_R\right] = i\epsilon^{ijk} T^k_R, \quad \left[T^i_L, T^j_R\right] = 0.$$

(e) Classify the representations of the Lorentz algebra using what you learned about $su(2)$.

Conclusion: Every representation of the Lorentz algebra can be characterized by 2 non-negative integers or half-integers (j_L, j_R).

3. *Weyl spinors - part I*

Summarizing our foregoing considerations, the Lorentz transformation on Minkowski space is given by

$$\Lambda = \exp\left(-\frac{i}{2}\omega_{\mu\nu} M^{\mu\nu}\right). \tag{2}$$

Now we take eq. (1) as the definition of the Lorentz algebra and investigate its representations. To make this point clear, we write $D(\Lambda)$ instead of Λ.

(a) Define α, β by the equations $\omega_{ij} = \epsilon_{ijk}\alpha_k$ and $\beta_i = \omega_{0i}$ to show

$$D(\Lambda) = \exp\left(-i\left[\vec{\alpha}\cdot\vec{J} + \vec{\beta}\cdot\vec{K}\right]\right) = \exp\left(-i(\vec{\alpha} - i\vec{\beta})\cdot\vec{T}_L\right)\exp\left(-i(\vec{\alpha} + i\vec{\beta})\cdot\vec{T}_R\right).$$

Note that T^i_R, T^i_L are still unspecified, we only know their algebra. For a particular representation, we have to make a choice!

(b) Specialize to a representation: Choose the T^i_R, T^i_L to be the Pauli matrices σ_i. The simplest representations of the Lorentz group are $(1/2, 0)$ and $(0, 1/2)$. An object transforming in the $(1/2, 0)$ representation is called a **left-chiral Weyl spinor**. The definition of a right-chiral Weyl spinor is obvious. How many entries does a Weyl spinor have? Write down the transformation laws for the 2 types of Weyl spinors.

Let D_L, D_R denote the transformation matrices for the left- and right-chiral Weyl spinors.

To be continued ...