Physikalisches Institut
Universität Bonn
Theoretische Physik

Exercise 6
28. Nov. 2005
WS 05/06

Exercises on Elementary Particle Physics

Prof. Dr. H.-P. Nilles

1. *Spin 1/2 fermions coupled to an electromagnetic field*

   Remember the minimal coupling prescription

$$i\partial_\mu \to i\partial_\mu - QA_\mu \ ,$$

   where Q is the charge of the particle, i.e. $Q_{e^-} = -e = -|e|$ and $Q_{e^+} = +e = +|e|$, and $A_\mu = (\phi, \vec{A})_\mu$ is the electromagnetic four-vector potential.

   (a) What are the Dirac equations for an electron field ψ and a positron field ψ_C, each of them coupled to an electromagnetic field A_μ.

   (b) Assume that a local relation between ψ_C and ψ exists. Show that for a C with

$$-(C\gamma^0)\gamma^{\mu *} = \gamma^\mu (C\gamma^0)$$

   this local relation reads $\psi_C = C\bar{\psi}^T$. C is called the charge conjugation operator and ψ_C the charge conjugated spinor.

   (c) Show that in the Dirac-Pauli representation of the γ-matrices a possible choice for C is

$$C\gamma^0 = i\gamma^2 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

   and

$$\left(C\gamma^0\right)\left(C\gamma^0\right) = \mathbb{1}$$

   (d) Show that $\left(\psi^C\right)^C = \psi$.

   (e) Now let $\psi^{(1)} = u^{(1)}(p)e^{-ip\cdot x}$ and $\phi^{(1)} = v^{(1)}(p)e^{ip\cdot x}$ be two of the 'standard' solutions we found in exercises 1 and 2. Show that $\psi_C^{(1)} = \phi^{(1)}$, i.e. that the charge conjugation operator C changes a positive-energy electron into a positive-energy positron.

Abbildung 1: Feynman graph. Time goes from left to right.

2. *Electron-Muon Scattering*

We present the Feynman rules to calculate the amplitude $-\mathrm{i}\mathcal{M}$ in QED.

(a) An arrow in the direction of time denotes a particle, an arrow in the opposite direction denotes an antiparticle. Assign a label i to each external particle. Assign momenta to each particle (including the internal lines) and indicate them by momentum-arrows beside the particle lines.

(b) For the following rules, proceed "backwards" with respect to the particle arrow for each fermion line. I.e. for a particle, proceeding backward means "opposite to the direction of time". For an antiparticle, proceeding backward means "in the direction of time".

(c) Write a factor $u(p_i)$, $v(p_i)$ for every external particle, respectively anti-particle, line which arrow points towards a vertex and $\bar{u}(p_i)$, $\bar{v}(p_i)$ for lines that point away from a vertex.

(d) Vertices and internal lines (propagators) contribute as follows:

$$\text{Vertices:} \quad \mathrm{i}e\gamma^\mu$$

$$\text{Fermions:} \quad \mathrm{i}\frac{\not{q} + m}{q^2 - m^2}$$

$$\text{Photons:} \quad -\mathrm{i}\frac{g_{\mu\nu}}{q^2}$$

The indices of the γ's are contracted with the $g_{\mu\nu}$ of the photon propagator.

(e) Use 4-momentum conservation at the vertices to eliminate the internal momenta.

In the lab frame where the particle B is initially at rest and is assumed to be so heavy that recoil effects are negligible, the **differential cross section** for the process $AB \to AB$ is given by

$$\frac{d\sigma}{d\Omega} = \frac{1}{64\pi^2 m_B^2}|\mathcal{M}|^2. \tag{1}$$

(a) Using the Feynman rules for QED, derive

$$\mathcal{M} = -\frac{e^2}{(p_1 - p_3)^2} \left[\bar{u}(p_3)\gamma^\mu u(p_1)\right] \left[\bar{u}(p_4)\gamma_\mu u(p_2)\right]$$

for the electron-muon scattering amplitude.

(b) To calculate the cross section, we need to know $|\mathcal{M}|^2$. Show that

$$\begin{aligned} |\mathcal{M}|^2 &= \mathcal{M}\mathcal{M}^* = \mathcal{M}\mathcal{M}^\dagger \\ &= \frac{e^4}{(p_1 - p_3)^4} \left[\bar{u}(p_3)\gamma^\mu u(p_1)\bar{u}(p_1)\gamma^\nu u(p_3)\right] \left[\bar{u}(p_4)\gamma_\mu u(p_2)\bar{u}(p_2)\gamma_\nu u(p_4)\right] \end{aligned} \quad (2)$$

(c) In a typical experiment, the particle beam is unpolarized and the detectors simply count the number of particles scattered in a given direction. Therefore, we have to *average* over initial spins and *sum* over final spins.

The avareging over the initial spins is easy: It contributes a factor of 1/2 for each sum.

Using the completeness relation for Dirac spinors (Exercise 2 - 2(e)),

$$\sum_{s=1,2} u^{(s)}(p)\, \bar{u}^{(s)}(p) = \not{p} + m$$

show that the summation over the final spins for the first factor in eq. (2) can be written as

$$\sum_{s_1,s_3} \bar{u}^{(s_3)}(p_3)\gamma^\mu u^{(s_1)}(p_1)\bar{u}^{(s_1)}(p_1)\gamma^\nu u^{(s_3)}(p_3) = \mathrm{Tr}\left[(\not{p}_3 + m_e)\gamma^\mu(\not{p}_1 + m_e)\gamma^\nu\right].$$

By relabeling, derive the analogous result for the second factor in eq. (2). The final result is

$$\frac{1}{4}\sum_{s_1,s_2,s_3,s_4} |\mathcal{M}|^2 = \frac{1}{4}\frac{e^4}{(p_1 - p_3)^4}\mathrm{Tr}\left[(\not{p}_3+m_e)\gamma^\mu(\not{p}_1+m_e)\gamma^\nu\right]\ \mathrm{Tr}\left[(\not{p}_4+m_\mu)\gamma_\mu(\not{p}_2+m_\mu)\gamma_\nu\right]. \quad (3)$$

Note that the problem of calculating the cross section has been reduced to matrix multiplication and taking the trace!

(d) To deal with the above expression, we need some efficient techniques to calculate the trace of products of gamma matrices (see Exercise 2 - 3). Prove that

$$\mathrm{Tr}(\text{odd number of } \gamma\text{-matrices}) = 0.$$

(e) Consider the first trace in eq. (3). Using the identities proved in (2d), derive

$$\mathrm{Tr}\left[(\not{p}_3 + m_e)\gamma^\mu(\not{p}_1 + m_e)\gamma^\nu\right] = 4(p_1^\mu p_3^\nu + p_1^\nu p_3^\mu - g^{\mu\nu} p_1 \cdot p_3 + g^{\mu\nu} m_e^2).$$

(f) By relabeling, derive the result for the second trace in eq. (3). Substitute your results in eq. (3) and show that it takes the following form:

$$\langle|\mathcal{M}|^2\rangle = \frac{4e^4}{(p_1-p_3)^4}(p_1^\mu p_3^\nu + p_1^\nu p_3^\mu - g^{\mu\nu} p_1\cdot p_3 + g^{\mu\nu} m_e^2)(p_{2\mu}p_{4\nu} + p_{2\nu}p_{4\mu} - g_{\mu\nu} p_2\cdot p_4 + g_{\mu\nu} m_\mu^2).$$

Now expand the brackets and contract the indices. Show that the result is:

$$\langle|\mathcal{M}|^2\rangle = \frac{8e^4}{(p_1-p_3)^4}\big((p_1\cdot p_2)(p_3\cdot p_4) + (p_1\cdot p_4)(p_2\cdot p_3) - m_\mu^2(p_1\cdot p_3) - m_e^2(p_2\cdot p_4) + 2m_e^2 m_\mu^2\big)$$

(g) So far everything is written covariantly and is independent of a special coordinate frame. To make contact with measurements, we specialize to the rest system of the muon and make an approximation as $m_\mu \gg m_e$. Denote by $p := |\vec{p_1}|$ the absolute value of the initial electron momentum. Denote by θ the angle between $\vec{p_1}$ and $\vec{p_3}$.

Draw 2 diagrams, one before the scattering process and one after the scattering process. Write the 4-momenta under the respective diagrams, taking into account the approximation we have made. Show that in the approximation $m_\mu \gg m_e$, conservation of energy-momentum gives $|\vec{p_3}| = |\vec{p_1}| = p$. Prove the following identities:

$$(p_1 - p_3)^2 = -4p^2 \sin^2\frac{\theta}{2}, \quad p_1\cdot p_3 = m_e^2 + 2p^2\sin^2\frac{\theta}{2},$$
$$(p_1\cdot p_2)(p_3\cdot p_4) = E^2 m_\mu^2, \quad p_2\cdot p_4 = m_\mu^2.$$

(h) Insert the above expressions into eq. (1) for the cross section to obtain the **Mott formula**

$$\frac{d\sigma}{d\Omega} = \frac{1}{64\pi^2}\frac{e^4}{p^4\sin^4\theta/2}\left[m_e^2 + p^2\cos^2\theta/2\right]$$

which reduces to the **Rutherford formula** in the non-relativistic limit $p \ll m_e$.