Physikalisches Institut
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Theoretische Physik

Exercise 12
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Exercises on Elementary Particle Physics

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1. *Renormalization of the Electric Charge in QED - part II*

   In the last exercise we computed the matrix element $i\Pi^{\mu\nu}(q)$ corresponding to the vacuum polarization diagram. The intermediate result was:

$$i\Pi^{\mu\nu}(q) = -4ie^2 \int \frac{d^4\ell}{(2\pi)^4} \int_0^1 dx \frac{\frac{1}{2}g^{\mu\nu}\ell^2 + g^{\mu\nu}x(1-x)q^2 + g^{\mu\nu}m^2}{(\ell^2+\Delta)^2} \tag{1}$$

   Now, we will solve the integral and interprete the resulting correction of the photon propagator as a renormalization of the electric charge.

   (a) Prove that

$$\int d\Omega_4 = 2\pi^2$$

   Hint: Multiply the known integrals

$$\int_{-\infty}^{+\infty} d\ell_i e^{-\ell_i^2} = \sqrt{\pi}$$

   for $i = 0, \ldots, 3$ and change from Cartesian coordinates to 4-dim. spherical coordinates $d^4\ell = |\ell|^3 \, d|\ell| d\Omega_4$. Then substitue $z = |\ell|^2$ and solve the remaining integral using partial integration.

   (b) In Euclidean space we can now change eqn. (1) to polar coordinates. Perform the substitution $z = |\ell|^2$.

   (c) Next, we want to solve the integrals over z. Therefore, perform the following integrations:

$$\int_a^b \frac{z^2}{(z+\Delta)^2} = \left(z - 2\Delta\ln z - \frac{\Delta^2}{z}\right)_{a+\Delta}^{b+\Delta} \qquad \int_a^b \frac{z}{(z+\Delta)^2} = \left(\ln z + \frac{\Delta}{z}\right)_{a+\Delta}^{b+\Delta}$$

   Using the boundaries from 0 to $+\infty$, we see that they are divergent. We regularize them by a energy cutoff, i.e. we integrate from 0 to Λ^2.

   Note: $z = |\ell|^2 = |k + xq|^2$, so the momentum k in the loop only runs up to an upper limit.

(d) Verify that in the limit of large Λ the following approximations hold

$$\int_0^{\Lambda^2} \frac{z^2}{(z+\Delta)^2} dz \to \Lambda^2 - 2\Delta \ln\frac{\Lambda^2}{\Delta} + \Delta, \qquad \int_0^{\Lambda^2} \frac{z}{(z+\Delta)^2} dz \to \ln\frac{\Lambda^2}{\Delta} - 1$$

in order to obtain

$$i\Pi^{\mu\nu}(q) = -\frac{ie^2}{4\pi^2} \int_0^1 dx \left\{ \frac{1}{2} g^{\mu\nu} \left(\Lambda^2 - 2\,\Delta \ln\frac{\Lambda^2}{\Delta} + \Delta \right) + \right.$$
$$\left. + g^{\mu\nu} \left[x(1-x)\, q^2 + m^2 \right] \left(\ln\frac{\Lambda^2}{\Delta} - 1 \right) \right\}.$$

(e) This result is not gauge invariant, because the cutoff regularisation does not respect the QED symmetry. We can, however, restore the symmetry by discarding all terms that are not proportional to q^2. (The terms not proportional to q^2 would give rise to a photon mass which is not allowed by the gauge symmetry.)

(f) We choose the cutoff to be extremly large (of the order of the GUT scale), so we can assume that the cutoff is much larger than the external momentum q, i.e. $\Lambda^2 \gg q^2$.

(g) Next, we consider two limits: (i) q^2 small and (ii) q^2 large.

i. q^2 small - In this limit, we define the measurable value of the electric charge.

Use $m^2 \gg x(1-x)q^2$ to prove the final result for the matrix element:

$$i\Pi^{\mu\nu}(q) = \frac{ie^2}{12\pi^2} g^{\mu\nu} q^2 \ln\frac{m^2}{\Lambda^2}.$$

We can now use this result to calculate the loop corrected photon propagator. Calculate the correction at one loop and follow that the propagator is given by

$$-\frac{ig^{\mu\nu}}{q^2} \left[1 + \frac{e^2}{12\pi^2} \ln\frac{m^2}{\Lambda^2} \right].$$

Now calculate the correction to all orders (several one-loop diagrams one after another). Using the geometric series

$$\frac{1}{1-x} = 1 + x + x^2 + \dots$$

you will obtain

$$-\frac{ig^{\mu\nu}}{q^2} \left[\frac{1}{1 - \frac{e^2}{12\pi^2} \ln\frac{m^2}{\Lambda^2}} \right] =: -\frac{ig^{\mu\nu}}{q^2} Z_3.$$

As every propagator ends in two vertices, we can also use our original propagator and multiply $\sqrt{Z_3}$ to each vertex $ie\gamma^\mu$ (see 6.2) instead. Thus, we can regard $\sqrt{Z_3}$ as a factor multiplying the electromagnetic charge which gives the *renormalized charge* or *renormalized coupling constant*:

$$e_R := \sqrt{Z_3}\, e$$

Note that it is the renormalized charge that is measured in experiments. In order to distinguish the renormalized (physical) charge from the original parameter e in the Lagrangian, we speak of e as the *bare charge* or *bare coupling constant*.

ii. q large - In this limit, we can calculate the dependence of the charge e on the momentum q.

First, write the logarithm as:

$$\ln\left(\frac{\Lambda^2}{m^2 - x(1-x)q^2}\right) = -\ln\left(-\frac{q^2}{\Lambda^2}\right) - \ln\left(x(1-x)\right) - \ln\left(1 - \frac{m^2}{q^2 x(1-x)}\right)$$

The last term vanishes for $q^2 \gg m^2$. For the x-integration, you need (without prove):

$$\int_0^1 dx\; x(1-x)\, \ln(x(1-x)) = -\frac{5}{18}$$

Show that the final result for the matrix element reads:

$$i\Pi^{\mu\nu}(q) = \frac{ie^2}{12\pi^2} g^{\mu\nu} q^2 \left(\ln\left(-\frac{q^2}{\Lambda^2}\right) - \frac{5}{3}\right).$$

Following the discussion of part (i) you find:

$$\alpha_R(q) = \frac{\alpha}{1 - \frac{\alpha}{3\pi}\left(\ln\left(-\frac{q^2}{\Lambda^2}\right) - \frac{5}{3}\right)}$$

2. *Dynkin Diagram of SO(2n) - part I*

The orthogonal groups are given by matrices which satisfy $A^t A = \mathbb{1}$.

(a) Using the correspondence between elements of the group and elements of the Lie algebra, $A = \exp \mathcal{A} \approx \mathbb{1} + \mathcal{A}$, show that the requirement is:

$$\mathcal{A} + \mathcal{A}^t = 0$$

Clearly these matrices have only off-diagonal elements. As a result, it would be hard to find the Cartan subalgebra as we did for $SU(n)$ by using diagonal matrices. To avoid this problem, we perform a unitary transformation on the matrices A.

(b) Use the ansatz

$$A = UBU^\dagger$$

with U unitary, define $K = U^t U$ to show that

$$B^\dagger K B = K.$$

Furthermore, expand B in the usual way $B = \exp \mathcal{B} \approx \mathbb{1} + \mathcal{B}$ and follow the condition:

$$\mathcal{B}^t K + K\mathcal{B} = 0 \tag{2}$$

(c) A convenient choice for U in the case of $SO(2n)$ is

$$U = \frac{1}{\sqrt{2}} \begin{pmatrix} i\ \mathbb{1} & -i\ \mathbb{1} \\ -1\ \mathbb{1} & -1\ \mathbb{1} \end{pmatrix}$$

with $\mathbb{1}$ being the $n \times n$ identity matrix.
What is the form of K?

(d) We represent $\mathcal{B}$ in terms of $n \times n$ matrices $\mathcal{B}_i$:

$$\mathcal{B} = \begin{pmatrix} \mathcal{B}_1 & \mathcal{B}_2 \\ \mathcal{B}_3 & \mathcal{B}_4 \end{pmatrix}$$

Show that from eqn. (2) follows:

$$\mathcal{B}_1 = -\mathcal{B}_4^t \quad \mathcal{B}_2 = -\mathcal{B}_2^t \quad \mathcal{B}_3 = -\mathcal{B}_3^t$$

A basis of $2n \times 2n$ matrices fulfilling these conditions is given by $(j, k \leq n)$:

$$\begin{aligned} e^1_{jk} &= e_{j,k} - e_{k+n,j+n} \\ e^2_{jk} &= e_{j,k+n} - e_{k,j+n} \qquad j < k \\ e^3_{jk} &= e_{j+n,k} - e_{k+n,j} \qquad j < k \end{aligned}$$

A basis for the Cartan subalgebra is given by $h_j = e^1_{jj}$.