Physikalisches Institut
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Theoretische Physik

Exercise 13
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WS 05/06

Exercises on Elementary Particle Physics

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1. *Dynkin Diagram of SO(2n) - part II*

Last time, we showed that a basis of $2n \times 2n$ matrices fulfilling the condition

$$\mathcal{B}^t K + K\mathcal{B} = 0$$

is given by $(j, k \leq n)$:

$$\begin{aligned} e^1_{jk} &= e_{j,k} - e_{k+n,j+n} & j \neq k \\ e^2_{jk} &= e_{j,k+n} - e_{k,j+n} & j < k \\ e^3_{jk} &= e_{j+n,k} - e_{k+n,j} & j < k \end{aligned}$$

and by the elements of the Cartan subalgebra $h_j = e^1_{jj}$. So, a general element of the Cartan subalgebra can be written as:

$$h = \sum_i \lambda_i h_i$$

(a) Determine the eigenvalues of the adjoint of h, i.e.

$$\text{ad}(h)\ e^a_{jk} = [h, e^a_{jk}] = \alpha_{e^a_{jk}}(h)\ e^a_{jk} \qquad a = 1, 2, 3$$

Solution:

$$\begin{aligned} [h, e^1_{jk}] &= (\lambda_j - \lambda_k) e^1_{jk} & j \neq k \\ [h, e^2_{jk}] &= (\lambda_j + \lambda_k) e^2_{jk} & j < k \\ [h, e^3_{jk}] &= -(\lambda_j + \lambda_k) e^3_{jk} & j < k \end{aligned}$$

Therefore, all roots are given by:

$$\begin{aligned} \alpha_{e^1_{jk}}(h) &= (\lambda_j - \lambda_k) & j \neq k \\ \alpha_{e^2_{jk}}(h) &= (\lambda_j + \lambda_k) & j < k \\ \alpha_{e^3_{jk}}(h) &= -(\lambda_j + \lambda_k) & j < k \end{aligned}$$

(b) Convince yourself that the following roots form a basis of all roots and are furthermore positive and simple:

$$\begin{aligned} \alpha_1(h) &= \lambda_1 - \lambda_2 \\ \alpha_2(h) &= \lambda_2 - \lambda_3 \\ & \dots \\ \alpha_{n-1}(h) &= \lambda_{n-1} - \lambda_n \\ \alpha_n(h) &= \lambda_{n-1} + \lambda_n \end{aligned}$$

Hint: Use Ex.8.2(d).

(c) Show that the Killing form of two elements h and h' of the Cartan subalgebra can be written in general as

$$\mathcal{K}(h, h') = 4(n-1) \sum_j \lambda'_j \lambda_j.$$

Hint: Use Ex.8.2(f).

(d) Use the theorem of Ex.8.2 and the result of the last part to obtain from

$$\mathcal{K}(h_{\alpha_i}, h') \overset{\text{theorem}}{=} \alpha_i(h') \overset{\text{part (b)}}{=} \begin{cases} \lambda'_i - \lambda'_{i+1} & i < n \\ \lambda'_{n-1} + \lambda'_n & i = n \end{cases}$$

the coefficients λ_j of h_{α_i}.

(e) Calculate the Cartan matrix and draw the Dynkin diagram of $SO(2n)$.