Universität Bonn
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Theoretische Physik

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Example sheet 1/2
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General Relativity

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1. *Lorentz transformations*

Let us consider the *Minkowski space* $\mathbb{R}^{1,3}$, the $\mathbb{R}^4$ together with the metric $-\eta = \mathrm{diag}(1, -1, -1, -1)$. Elements x of this space are called *four-vectors* and denoted

$$x = \begin{pmatrix} x^0 \\ \overrightarrow{x} \end{pmatrix} = \begin{pmatrix} ct \\ \overrightarrow{x} \end{pmatrix} \equiv x^\mu \ .$$

Thus the distance squared of two four-vectors x, y is written

$$(x-y)^2 \equiv -\eta_{\mu\nu}(x-y)^\mu(x-y)^\nu = (x^0-y^0)^2 - |\overrightarrow{x} - \overrightarrow{y}|^2$$

and the infinitesimal distance squared, the length element,

$$\mathrm{d}s^2 = -\eta_{\mu\nu}\mathrm{d}x^\mu \mathrm{d}x^\nu \ .$$

(a) For a Lorentzian metric the four-vectors can be divided into three classes: *space-like*, *lightlike* and *timelight* vectors. Explain these terms. What is the condition for light beams?

(b) The *principle of relativity* states that physics laws must be the same in all inertial systems. Therefore we now consider transformations Λ between inertial systems. What is the most general form of such transformation?

(c) Show that the requirement that the speed of light is the same in all inertial frames leads to a constraint equation for Λ

$$(x-y)^2 = (\Lambda(x-y))^2 \ .$$

(d) Write this constraint in components, as $\Lambda^\rho{}_\mu \Lambda^\sigma{}_\nu \eta_{\rho\sigma} = \eta_{\mu\nu}$. Compare with the condition for orthogonal matrices.

(e) Convince yourself that the set of transformations Λ forms a group, the *Lorentz group*

$$\mathcal{L} = \mathrm{O}(3,1) \equiv \left\{ \Lambda \in \mathrm{Mat}(4, \mathbb{R}) \,|\, \Lambda^t\,\eta\,\Lambda = \eta \right\} \ .$$

(f) Deduce from the defining condition that $|\Lambda^0{}_0| \geq 1$ and $\det\Lambda = \pm 1$. Explain how the Lorentz group is divided into four branches. Which branches constitute the *proper Lorentz group* SO(3,1)?

(g) What form does Λ take for parity, time reversal and purely spatial rotations?

(h) Now we consider Lorentz transformations which connect space and time components, the *boosts*. Consider a boost along the z-axis (i.e. the origin of the x'^μ-system moves along the z-axis with constant speed v). Use condition (d) to derive Λ for this situation. Define $\beta \equiv \frac{v}{c}$ and $\gamma \equiv (1-\beta)^{-\frac{1}{2}}$.
(Hint: the only interesting components are $\Lambda^0{}_0$, $\Lambda^0{}_3$, $\Lambda^3{}_0$ *and* $\Lambda^3{}_3$.
Also, $\cosh x = \frac{1}{\sqrt{1-\tanh^2 x}}$, $\sinh x = \frac{\tanh x}{\sqrt{1-\tanh^2 x}}$ *and* $\cosh^2 x - \sinh^2 = 1$*)*

(i) Multiply the matrices of two parallel boosts (with velocities v_1, v_2) along the z-axis and calculate how the velocities have to be summed. Consider the limit $c \gg v_i$ (for $i = 1, 2$) and the case that one of the velocities is c.
(Use $\sinh(x \pm y) = \sinh x \cosh y \pm \cosh x \sinh x$ *and* $\cosh(x+y) = \cosh x \cosh y \pm \sinh x \sinh y$ *.)*

2. *Contra- and covariant vectors and tensors*

The x^μ, which we used in 1., are called *contravariant* components of a four-vector x. They are defined through their transformation behaviour under Lorentz transformations

$$x'^\mu = \Lambda^\mu{}_\nu x^\nu \ .$$

Now we use the metric η to define *covariant* components x_μ of x by

$$x_\mu \equiv \eta_{\mu\nu}\, x^\nu \ .$$

In general η is used to raise and lower indices.
(*This means switching between the tangent and cotangent space of the manifold.*)

(a) Show for the components of the inverse matrix of a Lorentz transformation $(\Lambda^{-1})^\mu{}_\nu = \Lambda_\nu{}^\mu$. (*Use* $(\eta^{-1})_{\mu\nu} \equiv \eta^{\mu\nu} = \eta_{\mu\nu}$ *.*)

(b) How do the covariant components transform under Λ?

(c) Show that the derivative $\partial_\mu \equiv \frac{\partial}{\partial x_\mu}$ transforms as a covariant and $\partial^\mu \equiv \frac{\partial}{\partial x_\mu}$ as a contravariant vector under Lorentz transformations.

(d) Why is the d'Alembert operator $\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \triangle$ lorentzinvariant?

(e) In analogy we can define tensors with more contra- and covariant indices. E.g. define a tensor $T^{\mu\nu}$ with two contravariant indices through its transformation property under Lorentz transformations

$$T^{\mu\nu} \mapsto \Lambda^{\mu}{}_{\rho}\Lambda^{\nu}{}_{\sigma}T^{\rho\sigma} \ .$$

Deduce the transformation properties of tensors with two covariant and tensors with one covariant and one contravariant index.

It should be clear that this can be repeated for tensors with arbitrary many indices.

(f) How is the trace of a tensor defined? Calculate $\mathrm{tr}\,\eta$.

(g) Why is $\varepsilon^{\alpha\beta\gamma\delta}$ *pseudo-tensor*?

(h) Explain the statement 'an equation is covariant'. Translate the principle of relativity into our new language.

3. Electric current

The electrical charge and current densities of a collection of charged point particles with positions $\vec{x}_n(t)$ and charges e_n are

$$\vec{J}(\vec{x},t) \equiv \sum_n e_n\,\delta^3(\vec{x}-\vec{x}_n(t))\,\frac{d\vec{x}_n(t)}{dt} \tag{1}$$

and

$$\varepsilon(\vec{x},t) \equiv \sum_n e_n\,\delta^3(\vec{x}-\vec{x}_n(t))\,. \tag{2}$$

(a) Write $J^\alpha \equiv \binom{\varepsilon}{\vec{J}}$ as a single expression. Show that J^α transforms correctly as a spacetime four-vector.

(b) Show that J^α is a conserved four-current:

$$\partial_\alpha J^\alpha(x) = 0\,, \tag{3}$$

where $\partial_\alpha \equiv \partial/\partial x^\alpha \equiv (\partial/\partial t, \vec{\nabla})$.

(c) Verify that $Q = \int d^3x J^0(x)$ is time-independent.

4. Electromagnetism

Maxwell's equations are

$$\vec{\nabla}\cdot\vec{E} = \varepsilon\,, \qquad \vec{\nabla}\times\vec{B} = \frac{\partial\vec{E}}{\partial t} + \vec{J}\,, \qquad \vec{\nabla}\cdot\vec{B} = 0\,, \qquad \vec{\nabla}\times\vec{E} = -\frac{\partial\vec{B}}{\partial t}\,. \tag{4}$$

To make their properties under Lorentz transformations explicit, we can choose an antisymmetric tensor $F^{\mu\nu} = -F^{\nu\mu}$ such that $F^{12} = B_3\,, F^{23} = B_1\,, F^{31} = B_2\,,$ and $F^{01} = E_1\,, F^{02} = E_2\,, F^{03} = E_3\,.$

(a) Show that

$$\partial_\mu F^{\mu\nu} = -J^\nu \quad \text{and} \quad \epsilon^{\mu\nu\rho\sigma}\partial_\nu F_{\rho\sigma} = 0 \tag{5}$$

reproduce Maxwell's equations. ($\epsilon^{0123} \equiv +1$)

(b) Verify in the rest frame that

$$f^\mu \equiv \frac{dp^\mu}{d\tau} = eF^\mu{}_\nu \frac{dx^\nu}{d\tau} \tag{6}$$

is the correct equation for the electromagnetic four-force f^μ acting on a charged particle. ($p^\mu = m\,dx^\mu/d\tau$)

5. *Energy-momentum tensor*

In analogy to the electrical charge and current densities in equations (1) and (2), we can define a charge and current density for the four-momentum p^μ, the *energy-momentum tensor*

$$T^{\mu\nu}(\vec{x}, t) \equiv \sum_n p_n^\mu(t)\,\frac{dx_n^\nu(t)}{dt}\,\delta^3(\vec{x} - \vec{x}_n(t)) \tag{7}$$

(a) Show that the energy-momentum tensor is only conserved up to a *force density* G^μ which vanishes for free particles:

$$\partial_\nu T^{\mu\nu} = G^\mu\,. \tag{8}$$

(b) Check that for the electromagnetic forces given in (6), we get $G^\mu = F^\mu{}_\nu\,J^\nu$.

(c) To obtain a conserved energy-momentum tensor, we have forgotten to include the contribution of the electromagnetic field itself:

$$T_{em}^{\mu\nu} \equiv F^\mu{}_\rho F^{\nu\rho} - \frac{1}{4}\eta^{\mu\nu}F_{\rho\sigma}F^{\rho\sigma}\,. \tag{9}$$

Write T_{em}^{00} and T_{em}^{i0} in terms of $\vec{E}$ and $\vec{B}$. Do you recognize the expressions?

(d) Show that $\partial_\nu T_{em}^{\mu\nu}$ cancels G^μ from (b) exactly.
(Use (5); the second equation is equivalent to $\partial_\mu F^{\nu\rho} + \partial_\nu F^{\rho\mu} + \partial_\rho F^{\mu\nu} = 0$.)

(e) Show that the total momentum $p^\mu = \int d^3x\,T^{\mu 0}(\vec{x}, t)$ is a conserved quantity.
(This is completely analogous to 3.(c))