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General Relativity

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Exercises available on: *www.th.physik.uni-bonn.de/nilles/* - Exercises/StudentSeminars

1. ***Non-orthogonal coordinates***

Let us take a Cartesian coordinate system spanned by the unit vectors $\vec{\epsilon}_1$ and $\vec{\epsilon}_2$, and a non-orthogonal system spanned by $\vec{e}_1$ and $\vec{e}_2$. The tilted system's basis vectors are given by

$$\vec{e}_1 = \vec{\epsilon}_1 \,, \qquad \vec{e}_2 = \vec{\epsilon}_1 + 2\vec{\epsilon}_2 \,. \tag{1}$$

We can write any point X in the Cartesian system as

$$X = \xi^1 \cdot \vec{\epsilon}_1 + \xi^2 \cdot \vec{\epsilon}_2 \equiv \xi^a \vec{\epsilon}_a \equiv \begin{pmatrix} \xi^1 \\ \xi^2 \end{pmatrix}_\epsilon \,. \tag{2}$$

The coefficients ξ^a $(a = 1, 2)$ are called *contravariant* coordinates.

(a) Rewrite X in the tilted system to obtain the contravariant coordinates x^i of $\vec{\xi}$ in terms of the ξ^a.

(b) Show that the distance s from point X to the origin can be written in the Cartesian system as $s^2 = \eta_{ab}\,\xi^a\xi^b$. What does the *metric* η_{ab} of the Cartesian system look like?

(c) If we require that distances should not depend on the choice of coordinate system, we can write s^2 as

$$s^2 = \eta_{ab}\,\xi^a\xi^b = g_{ij}\,x^i x^j \,, \tag{3}$$

where we now denote the metric of the tilted system by g_{ij}.
Compute the metric components g_{ij}.

(d) Write the g_{ij} in terms of the η_{ab} generally. Verify the result of (c).

2. ***Free movement in a gravitational field***

We will use the *principle of equivalence* (PE),

> *...at any spacetime point in an arbitrary gravitational field it is possible to choose a 'locally inertial coordinate system' such that, within a sufficiently small region of the point in question, the laws of nature take the same form as in unaccelerated Cartesian coordinate systems in the absence of gravity...*

to derive the equations of motion (EOM) for a freely moving particle in a gravitational field. Then we show that this is equivalent to demanding that the particle moves on geodesics in the curved spacetime, which is described by the metric tensor $g_{\mu\nu}$.

(a) Consider the EOM of a particle in the gravitational field of the earth. Use the PE to find a coordinate system in which gravitational forces vanish.

(b) Write down the EOM for a freely falling particle in a gravitational field.

(c) Use the PE to derive the EOM in any other coordinate system,

$$\frac{\mathrm{d}^2 x^\lambda}{\mathrm{d}\tau^2} + \Gamma^\lambda_{\ \mu\nu} \frac{\mathrm{d}x^\mu}{\mathrm{d}\tau} \frac{\mathrm{d}x^\nu}{\mathrm{d}\tau}, \qquad \text{where} \quad \Gamma^\lambda_{\ \mu\nu} = \frac{\partial x^\lambda}{\partial \xi^\alpha} \frac{\partial^2 \xi^\alpha}{\partial x^\mu \partial x^\nu}. \tag{4}$$

(d) Are there any differences in the derivation of Eq. (4), if we consider light (or massless particles)?
How many of the $\frac{\mathrm{d}x^\mu}{\mathrm{d}\tau}$ are independent?

(e) Show that the *Christoffel symbols* $\Gamma^\lambda_{\ \mu\nu}$ can be written as

$$\Gamma^\lambda_{\ \mu\nu} = \frac{1}{2} g^{\lambda\kappa} \left(\frac{\partial g_{\kappa\mu}}{\partial x^\nu} + \frac{\partial g_{\kappa\nu}}{\partial x^\mu} - \frac{\partial g_{\mu\nu}}{\partial x^\kappa} \right). \tag{5}$$

(f) Write down the variational principle for geodesics in a space with metric $g_{\mu\nu}(x)$. Why does it lead to the EOM for a freely falling particle?

(g) Calculate the EOM from the variational principle of (f).

(h) How do the EOM change, if we take into account non-gravitational forces?

3. ***Locally varying coordinates***

One familiar example of location-dependent coordinates are spherical coordinates

$$\xi^1 = r\cos\theta\,, \quad \xi^2 = r\sin\theta\cos\phi\,, \quad \xi^3 = r\sin\theta\sin\phi\,. \tag{6}$$

Let us write $(r, \theta, \phi) \equiv (x^1, x^2, x^3)$.

(a) Use the invariance of $\mathrm{d}s^2$ under general coordinate transformations to determine $g_{ij}(x)$ for the spherical coordinates.
What are the covariant coordinates x_i?

(b) Calculate all Christoffel symbols $\Gamma^l_{\ mn}$ for the spherical coordinates.

(c) What are the EOM for r, $\theta\,\phi$?