Universität Bonn
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Theoretische Physik

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Example sheet 4
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General Relativity

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1. *Calculus and covariant differentiation*

On the four dimensional Lorentz manifold, which underlies the Theory of General Relativity, vectors and tensors are defined by their transformation properties under general coordinate transformations (GL(4,$\mathbb{R}$)). We define the components of a (p, q)-tensor t by

$$t^{\mu_1 \ldots \mu_p}_{\nu_1 \ldots \nu_q} \mapsto \frac{\partial x'^{\mu_1}}{\partial x^{\rho_1}} \cdots \frac{\partial x'^{\mu_p}}{\partial x^{\rho_p}} \frac{\partial x^{\sigma_1}}{\partial x'^{\nu_1}} \cdots \frac{\partial x^{\sigma_q}}{\partial x'^{\nu_q}} \, t^{\rho_1 \ldots \rho_p}_{\sigma_1 \ldots \sigma_q} \,. \tag{1}$$

(a) To which type of tensors do contra- and covariant vestors belong? How do they transform?

(b) Explain, why the ordinary derivative $\frac{\partial V^\mu}{\partial x^\nu}$ of a component of a vector is not the component of a $(1, 1)$-tensor.

Now we will construct the *covariant derivative* D_μ in such a way that for the components of a (p, q)-tensor t the components of the covariant derivative $\mathrm{D}_\mu t$ belong to a $(p, q + 1)$-tensor (therefore the name *covariant derivative*).

Consider components of a vector V^μ. We define the covariant derivative in x^ν-direction by

$$\mathrm{D}_\nu V^\mu \equiv \lim_{\Delta x^\nu \to 0} \frac{V^\mu(x + \Delta x) - \tilde{V}^\mu(x + \Delta x)}{\Delta x^\nu} \,, \quad \text{with } \tilde{V}^\mu(x+\Delta x) \equiv V^\mu(x) - \Gamma^\mu{}_{\nu\lambda} V^\lambda(x) \Delta x^\nu \,. \tag{2}$$

Think of the $\Gamma^\mu{}_{\nu\lambda}$ as being defined by Eq. (2). For the next hour, forget about the definition of them on the previous exercise sheet.

(c) Motivate the above definition and calculate $\mathrm{D}_\nu V^\mu$.

(d) How does $\Gamma^\mu{}_{\nu\lambda}$ transform, in order for the $\mathrm{D}_\nu V^\mu$ being components of a $(1, 1)$-tensor? Is it a tensor?

(e) Find $\mathrm{D}_\nu V_\mu$.
(Hint: In order for Eq. (2) to define a derivative, D_ν additionally has to obey the product rule.)

(f) Generalize the covariant derivative to components of tensors of arbitrary type.

(g) Let $\Gamma^{\mu}{}_{\nu\lambda}$ be coefficients of an arbitrary connection. Show that $\Gamma^{\mu}{}_{\nu\lambda} + t^{\mu}{}_{\nu\lambda}$ are coefficients of another connection, provided that $t^{\mu}{}_{\nu\lambda}$ is a tensor.

(h) A vector is said to be *parallel transported* along a curve $x^{\nu}(t)$, if

$$\frac{\mathrm{d}x^{\nu}(t)}{\mathrm{d}t}\mathrm{D}_{\nu}X^{\mu}(x(t)) = 0\,. \tag{3}$$

Write this equation in components and take X^{μ} as a tangent vector to the curve. Do you recognize this equation.
Thus the today's $\Gamma^{\mu}{}_{\nu\lambda}$ are exactly the same as last week's!

(i) From now on we demand that the metric is *covariantly constant*, that is: The inner product of two vectors remains constant, if we parallel transport them along any curve. What are the conditions for the $g_{\mu\nu}$? Use these conditions to calculate the $\Gamma^{\mu}{}_{\nu\lambda}$.
(Hint: Demand the symmetry property $\Gamma^{\mu}{}_{\nu\lambda} = \Gamma^{\mu}{}_{\lambda\nu}$. A connection with this property is called a Levi-Civita connection.)
How many independent components does a *Levi-Civita connection* have?

2. *Gradient, Rotation and Divergence*

Let's recapitulate some calculations from the lecture!

(a) Do the covariant expressions for the gradient and the rotation operator change?

(b) Calculate $\Gamma^{\mu}{}_{\mu\nu}$ and derive the expression for the divergence operator

$$\mathrm{D}_{\mu}V^{\mu} = \frac{1}{\sqrt{-g}}\partial_{\mu}\left(\sqrt{-g}V^{\mu}\right)\,.$$

(c) Show that $\sqrt{-g}\,\mathrm{d}^{N}x$ is the invariant volume element. Write *Gauss' theorem*, using the invariant volume element and (b).