General Relativity and Cosmology

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Example sheet 1

Lorentz Transformations.

1. A Lorentz transformation is defined such that it leaves the "proper time"[1]

$$ds^2 = \eta_{\mu\nu} dx^\mu dx^\nu = -dt^2 + d\bar{x}^2 = -d\tau^2 \tag{1}$$

invariant. (Here $\eta_{\mu\nu} = diag(-1, 1, 1, 1)$ is the *Minkowski metric*). It transforms one system of spacetime coordinates x^α to another x'^α as

$$x'^\alpha = \Lambda^\alpha{}_\beta x^\beta + a^\alpha \tag{2}$$

or in matrix form

$$x' = \Lambda x + a$$

where a^α and $\Lambda^\alpha{}_\beta$ are constants.

   (a) How does the coordinate differential transform under a Lorentz transformation?

   (b) Requiring that (1) be invariant under a Lorentz transformation (2) implies a condition on $\Lambda^\alpha{}_\beta$. What is this condition? (Write it in matrix form as well).

   (c) Use this to show that the speed of light is the same in all inertial frames (write down also your definition of inertial frame).

2. The Lorentz group contains a familiar subgroup (specially for gymnasts, divers, skaters...) when restricted to spatial dimensions (spatial components of Λ only).

   (a) Which group is this? (concentrate on the spatial components of Λ and use the condition you found in previous question to find the properties that the subgroup should obey. Thus identify it.)

   (b) Compare this subgroup to the full Lorentz group. Using this information, what can you say about the Lorentz group? (which group is it?)

   (c) Using your knowledge on the explicit form of the Lorentz transformations, write down Λ in matrix form for a boost along the y direction.

   (d) Do Lorentz transformations commute?

   (e) Can you find a different parametrisation of Λ such that its form resembles closely that of its subgroup discussed above? (Hint: define $v = \tanh\phi$)

3. Two important implications of the Lorentz transformations are the so called Lorentz contraction and time dilation. From the Lorentz transformations derive (explain your steps)

   (a) The relation for the Lorentz contraction $L' = \gamma L$

   (b) The relation for the time dilation $T = \gamma T'$

(here $\gamma = (1 - v^2)^{-1/2}$).

[1] In units where $c = 1$. Units can always be recovered when needed by dimensional analysis.

4. A high speed train of length l moves along the x direction and goes through a tunnel of the same length at constant velocity v. In a spacetime diagram, show the world lines of the train and the tunnel from their respective reference frames. Show clearly

   (a) the point at which the front of the train emerge form the tunnel

   (b) the point at which the end of the train enters the tunnel

   (c) the point where the end of the train is, when its front emerges from the tunnel, in the *train frame*

   (d) the point where the front of the train is when its end enters the tunnel in the *tunnel frame*

   (e) Does the train fit inside the tunnel from the tunnel frame? How about the train frame? Explain.

(Four)Vectors and Tensors.

5. Given the tensor $T^{\mu\nu}$ and the vector V^{μ}, with components

$$T^{\mu\nu} = \begin{pmatrix} 2 & 0 & 1 & -1 \\ -1 & 0 & 3 & 2 \\ -1 & 1 & 0 & 0 \\ -2 & 1 & 1 & -2 \end{pmatrix}; \qquad V^{\mu} = (-1, 2, 0, -2)$$

   Find the components of (write explicitly all the steps on how you calculate them)

   (a) $T^{\mu}{}_{\nu}$

   (b) $T_{\mu}{}^{\nu}$

   (c) $T^{\lambda}{}_{\lambda}$

   (d) $V^{\mu}V_{\mu}$

   (e) $V_{\mu}T^{\mu\nu}$