General Relativity and Cosmology

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Example sheet 11

1. **Kruskal spacetime**

   Consider the Schwarzschild spacetime with metric given by

   $$ds^2 = -hdt^2 + h^{-1}dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2) \tag{1}$$

   where $h = 1 - 2m/r$, and $m = GM$ $(c = 1)$.

   (a) Show that (1) can be written as

   $$ds^2 = h(-dt^2 + dr^{*2}) + r^2(d\theta^2 + \sin^2\theta d\phi^2) \tag{2}$$

   where r^* is the *Regge-Wheeler radial coordinate*.

   (b) Define further the *ingoing Eddington-Finkelstein coordinates*

   $$v = t + r^*, \qquad -\infty < v < \infty \tag{3}$$

   Rewrite the metric (2) in these new coordinates.

   (c) Repeat the steps above for the *outgoing Eddington-Finkelstein coordinates*

   $$u = t - r^*, \qquad -\infty < u < \infty \tag{4}$$

   (d) Show that the Schwarzschild metric in the region $r > 2m$ can be written in terms of the ingoing *and* outgoing coordinates as

   $$ds^2 = -h\,dudv + r^2(d\theta^2 + \sin^2\theta d\phi^2) \tag{5}$$

   Introduce the *Kruskal-Szekeres coordinates* (U, V) defined (for $r > 2m$) by

   $$U = -e^{-u/4m}\,, \qquad V = e^{v/4m} \tag{6}$$

   and show that the metric becomes

   $$ds^2 = -\frac{32m^3}{r}e^{r/2m}\,dUdV + r^2(d\theta^2 + \sin^2\theta d\phi^2) \tag{7}$$

   Write down UV in terms of r^* and r.

2. **Physics in the vicinity of a massive object: Spectral shift**

   Consider the Schwarschild spacetime (1). In this problem, we study some of the physics involved in the vicinity of such curved spacetime.

(a) Suppose that a signal is sent from an emitter at a fixed point (r_E, θ_E, ϕ_E) and travels along a null geodesic and is received by a receiver at a fixed point (r_R, θ_R, ϕ_R). If t_E is the coordinate time of emission and t_R the coordinate time of reception, then the signal passes from the event with coordinates $(t_E, r_E, \theta_E, \phi_E)$ to the event with coordinates $(t_R, r_R, \theta_R, \phi_R)$. Draw a spacetime diagram illustrating these events.

(b) Let u be an affine parameter along the null geodesic with $u = u_E$ at the event of emission and $u = u_R$ at the event of reception. Show that

$$\frac{dt}{du} = \left[\left(1 - \frac{2m}{r}\right)^{-1} g_{ij} \frac{dx^i}{du} \frac{dx^j}{du} \right]^{1/2} \tag{8}$$

What does $g_{ij} dx^i dx^j$ corresponds to?

(c) On integrating the expression above we obtain

$$t_R - t_E = \int_{u_E}^{u_R} \left[\left(1 - \frac{2m}{r}\right)^{-1} g_{ij} \frac{dx^i}{du} \frac{dx^j}{du} \right]^{1/2} du \tag{9}$$

Argue that

$$\Delta t_R = \Delta t_E \tag{10}$$

where $\Delta t = t^{(2)} - t^{(1)}$ for two signals 1 and 2. That is, the coordinate time difference at the point of emission equals the coordinate time difference at the point of reception.

(d) The clock of an observer situated at the point of emission records proper time and not coordinate time. Find the relation between these two times and use it to show that

$$\Delta \tau_E = (1 - 2m/r_E)^{1/2} \Delta t_E \tag{11}$$

and similarly for $\Delta \tau_R$ and thus

$$\frac{\Delta \tau_R}{\Delta \tau_E} = \left[\frac{1 - 2m/r_R}{1 - 2m/r_E} \right]^{1/2} \tag{12}$$

(e) Suppose that the emitter is a pulsating atom, and that in the proper time interval $\Delta \tau_E$ it emits n pulses. An observer situated at the emitter will assign to the atom a frequency of pulsation $\nu_E \equiv n/\Delta \tau_E$, and this is the proper frequency of the pulsating atom. An observer situated at the receiver will see these n pulses in a proper time interval $\Delta \tau_R$ and thus assigna a frequency $\nu_R \equiv n/\Delta \tau_R$ to the pulsating atom. Find the relation between the two frequencies ν_R/ν_E.

(f) Expand the relation obtained above for $r_E \ll 2GM$ and $r_R \ll 2GM$ to show that the fractional shift can be written as

$$\frac{\Delta \nu}{\nu_E} \equiv \frac{\nu_R - \nu_E}{\nu_E} \simeq GM \left(\frac{1}{r_R} - \frac{1}{r_E} \right) \tag{13}$$

Discuss this relation when the emitter (receiver) is nearer the massive object that the receiver (emitter).